The motive of a Severi-Brauer variety and K-theory of central simple algebras

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Outline

- The motivic Postnikov tower in $\mathrm{SH}_{S^1}(k)$ and $\mathrm{DM}^{\mathrm{eff}}(k)$
- Slices of twisted K-theory
- Slices of Severi-Brauer varieties
- The reduced norm map

The motivic Postnikov tower in $\mathrm{SH}_{S^1}(k)$ and $\mathrm{DM}^{\mathrm{eff}}(k)$

An overview

$E \in \mathrm{SH}$ has its *$n-1$-connected cover* $E\{n\} \to E$ and its *Postnikov tower*

$$\ldots \to E\{n+1\} \to E\{n\} \to \ldots \to E$$

with layers $\Sigma^n \mathrm{EM}(\pi_n E)$.

For $\mathcal{F} \in D(\mathbf{Ab})$, the *canonical homological truncation* does the same

$$\ldots \to \tau_{\geq n+1}\mathcal{F} \to \tau_{\geq n}\mathcal{F} \to \ldots \to \mathcal{F}$$

with layers the homology $H_n(\mathcal{F})[n]$.

An overview

Voevodsky tells us how to form the same picture for $\mathrm{SH}(k)$ and $\mathrm{DM}(k)$, replacing suspension Σ with T-suspension Σ_t or Tate twist $\otimes\mathbb{Z}(1)[2]$.

For $\mathcal{E} \in \mathrm{SH}(k)$, we get the *homotopy motives* $\pi_n^\mu\mathcal{E}$, and for $\mathcal{F} \in \mathrm{DM}(k)$ the *homology motives* $\mathbb{H}_n^\mu\mathcal{F}$.

Instead of being abelian groups, the $\pi_n^\mu\mathcal{E}$ and $\mathbb{H}_n^\mu\mathcal{F}$ are *birational motives*.

Cast of characters

$\mathbf{Sm}/k :=$ smooth varieties over k

$\mathbf{Spc}_*(k) :=$ presheaves of pointed simplicial sets on $\mathbf{Sm}/k$

$T := (\mathbb{P}^1, \infty) \in \mathbf{Spc}_*(k)$

$\mathbf{Spt}_{S^1}(k) :=$ the category of presheaves of spectra on $\mathbf{Sm}/k$

$\mathrm{SH}_{S^1}(k) :=$ the $\mathbb{A}^1$-homotopy category of S^1 spectra
$:= \mathbf{Spt}_{S^1}(k)[W_{\mathrm{Nis}}^{-1}, W_{\mathbb{A}^1}^{-1}]$

Σ_t, Ω_t: the T-suspension and T-loops functors on $\mathrm{SH}_{S^1}(k)$

$\mathrm{SH}(k) :=$ the homotopy category of T spectra (of S^1 spectra)
Σ_t on $\mathrm{SH}_{S^1}(k)$ becomes invertible on $\mathrm{SH}(k)$.

Cast of characters

$\Sigma^\infty_t : \mathrm{SH}_{S^1}(k) \leftrightarrow \mathrm{SH}(k) : \Omega^\infty_t$
the infinite T-suspension and 0-spectrum functors

$S_k, \mathcal{HZ} \in \mathrm{SH}(k)$ the motivic sphere spectrum and motivic cohomology spectrum

$\mathrm{SmCor}(k) :=$ the category of finite correspondences

$\mathrm{DM}^{\mathrm{eff}}(k) :=$ the category of effective motives
$\subset D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k)))$ as $\mathbb{A}^1$-local objects.

$\mathrm{EM} : \mathrm{DM}^{\mathrm{eff}}(k) \to \mathrm{SH}_{S^1}(k)$ the Eilenberg-Maclane spectrum functor

The motivic Postnikov tower

Voevodsky has defined the "Tate" analog of the classical Postnikov tower in SH. We give the version for $\mathrm{SH}_{S^1}(k)$:

Taking T-suspensions of $SH_{S^1}(k)$ gives the tower of full triangulated localizing subcategories of $\mathrm{SH}_{S^1}(k)$:

$$\ldots \subset \Sigma_t^{n+1}\mathrm{SH}_{S^1}(k) \subset \Sigma_t^{n}\mathrm{SH}_{S^1}(k) \subset \ldots \subset \Sigma_t\mathrm{SH}_{S^1}(k) \subset \mathrm{SH}_{S^1}(k)$$

Lemma

The inclusion functor $i_n : \Sigma_t^n\mathrm{SH}_{S^1}(k) \to \mathrm{SH}(k)$ admits an exact right adjoint $r_n : \mathrm{SH}(k) \to \Sigma_t^n\mathrm{SH}_{S^1}(k)$.

The motivic Postnikov tower

$$\ldots \subset \Sigma_t^{n+1}\mathrm{SH}_{S^1}(k) \subset \Sigma_t^n\mathrm{SH}_{S^1}(k) \subset \ldots \subset \Sigma_t\mathrm{SH}_{S^1}(k) \subset \mathrm{SH}_{S^1}(k)$$

Lemma

The inclusion functor $i_n : \Sigma_t^n\mathrm{SH}_{S^1}(k) \to \mathrm{SH}(k)$ admits an exact right adjoint $r_n : \mathrm{SH}(k) \to \Sigma_t^n\mathrm{SH}_{S^1}(k)$.

Define $f_n := i_n r_n : \mathrm{SH}_{S^1}(k) \to \mathrm{SH}_{S^1}(k)$, giving the natural tower

$$\ldots \to f_{n+1}E \to f_nE \to \ldots \to E$$

called the *motivic Postnikov tower*.

The layers

$$s_nE := \mathrm{cofib}(f_{n+1}E \to f_nE)$$

are Voevodsky's *slices*.

Properties of the motivic Postnikov tower

1. $f_n \circ \Omega_t = \Omega_t \circ f_{n+1}$.

2. $K :=$ the K-theory presheaf $Y \mapsto K(Y)$. We have:

$$s_n(K) = \mathrm{EM}(\mathbb{Z}(n)[2n]).$$

This yields the Atiyah-Hirzebruch spectral sequence for K-theory:

$$\boxed{E_2^{p,q} := H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X).}$$

This is the same one as constructed by Bloch-Lichtenbaum (for fields) and extended to arbitrary X by Friedlander-Suslin. We'll see the proof later on.

3. $s_0(S_k) = \mathcal{H}\mathbb{Z}$ (Voevodsky). This is taking place in $\mathrm{SH}(k)$.

The motivic Postnikov tower in $\mathrm{DM}^{\mathrm{eff}}(k)$

We have a motivic Postnikov tower in $\mathrm{DM}^{\mathrm{eff}}(k)$ as well, replacing Σ_t with $\mathbb{Z}(1) \otimes -$.

This gives us the tower of localizing subcategories

$$\ldots \subset \mathbb{Z}(n) \otimes \mathrm{DM}^{\mathrm{eff}}(k) \subset \ldots \subset \mathbb{Z}(1) \otimes \mathrm{DM}^{\mathrm{eff}}(k) \subset \mathrm{DM}^{\mathrm{eff}}(k).$$

Write the resulting tower of truncation functors as

$$\ldots \to f_{n+1}^{\mathrm{mot}} \to f_n^{\mathrm{mot}} \to \ldots \to f_1^{\mathrm{mot}} \to f_0^{\mathrm{mot}} = \mathrm{id}$$

and the slices as s_n^{mot}:

$$f_{n+1}^{\mathrm{mot}} \to f_n^{\mathrm{mot}} \to s_n^{\mathrm{mot}} \to f_{n+1}^{\mathrm{mot}}[1]$$

The motivic Postnikov tower in $\mathrm{DM}^{\mathrm{eff}}(k)$

The Postnikov towers for $\mathrm{SH}_{S^1}(k)$ and $\mathrm{DM}^{\mathrm{eff}}(k)$ are related by the Eilenberg-Maclane functor $\mathrm{EM} : \mathrm{DM}^{\mathrm{eff}}(k) \to \mathrm{SH}_{S^1}(k)$:

$$f_n \circ \mathrm{EM} = \mathrm{EM} \circ f_n^{\mathrm{mot}}$$

We have the same compatibility with $\Omega_t := \mathcal{H}om(\mathbb{Z}(1)[2], -)$ as in $\mathrm{SH}_{S^1}(k)$:

$$f_n^{\mathrm{mot}} \circ \Omega_t = \Omega_t \circ f_{n+1}^{\mathrm{mot}},$$

Warning: $\cap_n \mathbb{Z}(n) \otimes \mathrm{DM}^{\mathrm{eff}}(k) \neq \{0\}$.

The 0th slice

Let $\Delta^n := \mathrm{Spec}\, k[t_0, \ldots, t_n]/\sum_i t_i - 1$,
$\Delta^* :=$ the cosimplicial scheme $n \mapsto \Delta^n$.

For a field F, let $\Delta^n_{F,0}$ be the semi-local scheme of the vertices in Δ^n_F.
This gives the cosimplicial subscheme $\Delta^*_{F,0}$, $n \mapsto \Delta^n_{F,0}$ of Δ^*_F.

Theorem
[0th slice] For $X \in \mathbf{Sm}/k$, $E \in \mathrm{SH}_{S^1}(k)$ there is a natural isomorphism in SH:

$$s_0(E)(X) \cong E(\Delta^*_{k(X),0}).$$

The homotopy motives

Theorem

Take $E \in \mathrm{SH}_{S^1}(k)$ which is the 0-spectrum of some $\mathcal{E} \in \mathrm{SH}(k)$.
1. There is a motive $\pi_0^\mu(E) \in \mathrm{DM}^{\mathrm{eff}}(k)$ with

$$s_0 E = \mathrm{EM}(\pi_0^\mu(E)).$$

2. $s_n E = \Sigma_t^n \mathrm{EM}(\pi_0^\mu(\Omega_t^n E)) = \mathrm{EM}(\pi_0^\mu(\Omega_t^n E)(n)[2n])$.

This relies on Voevodsky's theorem $s_0(S_k) = \mathcal{HZ}$, the result of Østvær-Röndigs identifying $\mathrm{DM}(k)$ with $\mathcal{HZ}$-modules, and a lifting of the slice filtration to the model category level due to P. Pelaez-Menaldo.

Definition $\pi_n^\mu E := \pi_0^\mu(\Omega_t^n E)$ is the *nth homotopy motive* of E.

$$\implies s_n E = \Sigma_t^n \mathrm{EM}(\pi_n^\mu E).$$

The Atiyah-Hirzebruch spectral sequence

The motivic Postnikov tower for $E \in \mathrm{SH}_{S^1}(k)$

$$\ldots \to f_{n+1}E \to f_nE \to \ldots \to f_1E \to E$$

gives the spectral sequence for $X \in \mathbf{Sm}/k$,

$$\boxed{E_2^{p,q} = H^{p-q}(X, (\pi^{\mu}_{-q}E)(-q)) \Longrightarrow \pi_{-p-q}(E(X))}$$

(assuming $E = \Omega^{\infty}_t \mathcal{E}$ for some $\mathcal{E} \in \mathrm{SH}(k)$).

Birational motives

Definition Call $\mathcal{F} \in \mathrm{DM}^{\mathrm{eff}}(k)$ a *birational motive* if for all dense open immersions $j : U \to X$ in $\mathbf{Sm}/k$, the pullback

$$j^* : \mathbb{H}^n_{\mathrm{Nis}}(X, \mathcal{F}) \to \mathbb{H}^n_{\mathrm{Nis}}(U, \mathcal{F})$$

is an isomorphism for all n.

Example. The constant sheaf with transfers $\mathbb{Z}$ is a birational motive.

Proposition

Let $\mathcal{F}$ be a birational motive. Then for every $n \geq 0$,

$$f_m^{\mathrm{mot}}(\mathcal{F}(n)) = s_m^{\mathrm{mot}}(\mathcal{F}(n)) = \begin{cases} \mathcal{F}(n) & \text{for } m \leq n \\ 0 & \text{for } m > n. \end{cases}$$

Birational motives

In fact:

Proposition

For E the 0-spectrum of some $\mathcal{E} \in \mathrm{SH}(k)$, the homotopy motive $\pi^\mu_n E$ is a birational motive for all n.

Thus the birational motives play the role in motivic homotopy theory that abelian groups do in classical homotopy theory:

All objects in $\mathrm{SH}(k)$ are built out of the Eilenberg-Maclane spectra of Tate twists of birational motives:

$$s_n E = \mathrm{EM}(\pi^\mu_n E(n)[2n]).$$

Well-connected theories

The simplest birational motives are the *birational motivic sheaves*: a birational motive $\mathcal{F}$ such that $\mathcal{H}^m_{\mathrm{Nis}}(\mathcal{F}) = 0$ for $m \neq 0$.

The simplest $E \in \mathrm{SH}_{S^1}(k)$ are those such that the (birational) homotopy motives $\pi^\mu_n E$ are birational motivic sheaves.

Definition Call $E \in \mathrm{SH}_{S^1}(k)$ *well-connected* if

1. $E(X)$ is -1 connected for all $X \in \mathbf{Sm}/k$.
2. The homotopy sheaf $\pi^{\mathrm{Nis}}_m s_0(\Omega^n_t E)$ is zero for all $m \neq 0$.

Proposition

If E is well-connected, then $\pi^{\mathrm{Nis}}_0 s_0(\Omega^n_t E) = \pi^{\mathrm{Nis}}_0 \Omega^n_t E$. If $E = \Omega^\infty_t \mathcal{E}$ then

$$\pi^\mu_n E = \pi^{\mathrm{Nis}}_0(\Omega^n_t E).$$

Slices of twisted K-theory

Proposition

The algebraic K-theory presheaf $X \mapsto K(X)$ is well-connected.

Proof. K is quasi-fibrant by the homotopy property and Quillen localization. $K(X)$ is -1 connected by construction.

To show $\pi_n(K(\Delta^*_{F,0})) = 0$ for $n \neq 0$: It comes down to showing that

$$K_0(\Delta^n_{F,0}, \partial\Delta^n_{F,0}) = 0$$

for all $n \geq 1$ (*cf.* Friedlander-Suslin). This follows from the long exact sequence

$$\ldots \to K_1(\Delta^n_{F,0}) \xrightarrow{i^*} K_1(\partial\Delta^n_{F,0}) \to K_0(\Delta^n_{F,0}, \partial\Delta^n_{F,0}) \to 0$$

Since $\Delta^n_{F,0}$ and $\partial\Delta^n_{F,0}$ are semi-local, the K_1's are just units, so surjectivity of i^* is easy to check.

The slices of K-theory

Theorem

$$\pi_n^{\mu} K = \mathrm{EM}(\mathbb{Z}); \quad s_n(K) = \mathrm{EM}(\mathbb{Z}(n)[2n])$$

for all $n \geq 0$.

Proof.

$$s_n(K) = \Sigma_t^n s_0(\Omega_t^n K) = \Sigma_t^n s_0 K.$$

Since K is well-connected,

$$s_0 K = \mathrm{EM}(\pi_0^{\mathrm{Nis}}(K)) = \mathrm{EM}(\mathbb{Z})$$

Twisted K-theory

$A :=$ a central simple algebra over k.
K^A the presheaf $X \mapsto K(X; A)$.

Proposition

K^A is well-connected.

The proof is the same as for K.

What are the slices $s_n(K^A)$?

The sheaf $\mathbb{Z}_A$

The Nisnevich sheaf $\pi_0^{\mathrm{Nis}}(K^A)$ is the sheaf associated to the presheaf

$$X \mapsto K_0(X, A)$$

on $\mathbf{Sm}/k$. $\pi_0^{\mathrm{Nis}}(K^A)$ has transfers induced by the pushforward for finite maps in twisted G-theory $G(-, A)$.

Definition Denote the sheaf with transfers $\pi_0^{\mathrm{Nis}}(K^A)$ by $\mathbb{Z}_A$.

Remarks 1. $\mathbb{Z}_A$ is a homotopy invariant sheaf with transfers.

2. For $X \in \mathbf{Sm}/k$ irreducible, $\mathbb{Z}_A(X) \subset \mathbb{Z}(X) = \mathbb{Z}$ is the subgroup of index $= \mathrm{index}(A_{k(X)}) \Longrightarrow \mathbb{Z}_A$ is a birational motivic sheaf.

The sheaf $\mathbb{Z}_A$

Theorem

Let A be a c.s.a over k. Then

$$\pi_n^{\mu}(K^A) = \mathrm{EM}(\mathbb{Z}_A); \quad s_n(K^A) = \mathrm{EM}(\mathbb{Z}_A(n)[2n])$$

for all $n \geq 0$.

The proof is the same as for K, noting $\pi_0^{\mathrm{Nis}} K^A = \mathbb{Z}_A$.

Corollary

The Atiyah-Hirzebruch spectral sequence for $K(X; A)$ is

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}_A(-q)) \Longrightarrow K_{-p-q}(X; A).$$

Computations in low degree

- $H^1(k, \mathbb{Z}_A(1)) = K_1(A)$
- We have an exact sequence

$$0 \to H^{-1}(k, \mathbb{Z}_A(1)) \to H^2(k, \mathbb{Z}_A(2)) \\ \to K_2(A) \to H^0(k, \mathbb{Z}_A(1)) \to 0.$$

- There is an edge homomorphism $H^n(k, \mathbb{Z}_A(n)) \xrightarrow{e_n} K_n(A)$.
- $H^n(X, \mathbb{Z}_A(0)) = 0$ for $n \neq 0$,
  $H^0(X, \mathbb{Z}_A(0)) = \mathbb{Z}_A(k(X)) = K_0(A_{k(X)})$.

Computations in low degree

Remark: Bloch showed that

$$H^n(k, \mathbb{Z}(1)) = 0$$

for $n \neq 1$.

We do not know if $H^n(k, \mathbb{Z}_A(1)) = 0$ for $n \neq 1$ in general, but this is true if A has square-free index.

Thus, for A of square-free index, we have

$$H^2(k, \mathbb{Z}_A(2)) = K_2(A).$$

Slices of Severi-Brauer varieties

Definition Let X be a smooth projective variety over k. Let $\mathcal{CH}^n(X)$ be the Nisnevich sheaf associated to the presheaf

$$Y \mapsto \mathrm{CH}^n(X \times_k Y).$$

Set $\mathcal{CH}_n(X) := \mathcal{CH}^{\dim_k X - n}(X)$.

Proposition

[Kahn-Huber] $\mathcal{CH}^n(X)$ is a birational motivic sheaf.

Slices of smooth projective varieties

Definition For $M \in \mathrm{DM}^{\mathrm{eff}}(k)$, we have the *homology motive* $\mathbb{H}_n^{\mu} M$ defined by

$$\mathbb{H}_n^{\mu} M = s_0^{\mathrm{mot}} \Omega_t^n M = \Omega_t^n s_n^{\mathrm{mot}} M$$

i.e., $s_n^{\mathrm{mot}} M = (\mathbb{H}_n^{\mu} M)(n)[2n]$. Write $\mathbb{H}_n^{\mu}(X) := \mathbb{H}_n^{\mu} M_{\mathrm{gm}}(X)$.

Via the analogy $\mathrm{DM}^{\mathrm{eff}}(k) \leftrightarrow D(\mathbf{Ab})$, $\mathbb{H}_n^{\mu} M$ corresponds to $H_n C$.

Just as for the homotopy motives of S^1-spectra, the homology motives of M are always birational motives, but not always birational motivic sheaves.

Slices of smooth projective varieties

We do know the following:

Proposition

[Huber-Kahn] Let X be a smooth projective variety. Then

1. $f_n^{\mathrm{mot}}(M_{\mathrm{gm}}(X)) = 0$ *for* $n > \dim_k X$.
2. *For* $0 \leq n \leq \dim_k X$,

$$\mathcal{H}^0_{\mathrm{Nis}}(\mathbb{H}^\mu_n(X))) = C\mathcal{H}_n(X).$$

3. $\mathcal{H}^m_{\mathrm{Nis}}(\mathbb{H}^\mu_n(X)) = 0$ *for* $m > 0$.

Skip proof

Slices of smooth projective varieties

Idea: $M_{\mathrm{gm}}(X)$ is represented by the complex of presheaves $C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X,0))$:

$$Y \mapsto [\ldots \to z_{\mathrm{equi}}(X,0)(Y \times \Delta^n) \to \ldots \to z_{\mathrm{equi}}(X,0)(Y)],$$

where $z_{\mathrm{equi}}(X,r)$ is the presheaf of cycles of relative dimension r on X.

By the 0th-slice theorem

$$\begin{aligned}\mathbb{H}_0^{\mu}(X)(Y) := s_0^{\mathrm{mot}} M_{\mathrm{gm}}(X)(Y) = C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X,0))(\Delta^*_{k(Y),0}) \\ := [\ldots \to z_{\mathrm{equi}}(X_{k(Y)},0)(\Delta^1_{k(Y),0}) \to z_0(X_{k(Y)})]\end{aligned}$$

Thus $\mathcal{H}^0_{\mathrm{Nis}}(\mathbb{H}_0^{\mu}(X)) = C\mathcal{H}_0(X)$ and there is no higher cohomology.

Slices of smooth projective varieties

For the higher homology motives: The Gysin sequence implies $\Omega^n_t M_{\mathrm{gm}}(X) = C^{\mathrm{Sus}}_*(z_{\mathrm{equi}}(X,n))$.

Thus by the 0th-slice theorem

$$\begin{aligned}\mathbb{H}^\mu_n(X)(Y) &= \Omega^n_t(s^{\mathrm{mot}}_n M_{\mathrm{gm}}(X))(Y)\\ &= s^{\mathrm{mot}}_0(\Omega^n_t M_{\mathrm{gm}}(X))(Y) = C^{\mathrm{Sus}}_*(z_{\mathrm{equi}}(X,n))(\Delta^*_{k(Y),0}).\end{aligned}$$

and hence

$$\mathcal{H}^0_{\mathrm{Nis}}(\mathbb{H}^\mu_n(X)) = C\mathcal{H}_n(X),$$

and no higher cohomology.

Question: Is $\mathbb{H}^{\mu}_n(X)$ a birational motivic *sheaf*, i.e., is

$$\mathcal{H}^m_{\mathrm{Nis}}(\mathbb{H}^{\mu}_n(X)) = 0, \quad \text{for } m \neq 0?$$

Said another way, is

$$s^{\mathrm{mot}}_n(M_{\mathrm{gm}}(X)) = C\mathcal{H}_n(X)(n)[2n]?$$

Example. $M_{\mathrm{gm}}(\mathbb{P}^m) = \oplus_{i=0}^m \mathbb{Z}(i)[2i]$, so

$$s^{\mathrm{mot}}_n(M_{\mathrm{gm}}(\mathbb{P}^m)) = \mathbb{Z}(n)[2n] = C\mathcal{H}_n(\mathbb{P}^m)(n)[2n]$$

for $0 \leq n \leq m$.

For a general X, the negative cohomology of $s^{\mathrm{mot}}_n(M_{\mathrm{gm}}(X))$ is not zero. However:

Slices of Severi-Brauer varieties

Theorem

Let A be a c.s.a. over k of prime rank ℓ, $X = \mathrm{S.B.}(A)$, $d = \ell - 1 = \dim_k X$. Then

$$\mathbb{H}^{\mu}_n(X) = \mathbb{Z}_{A^{\otimes n+1}} = C\mathcal{H}_n(X),$$

i.e.,

$$s_n^{\mathrm{mot}} M_{\mathrm{gm}}(X) = \mathbb{Z}_{A^{\otimes n+1}}(n)[2n] = C\mathcal{H}_n(X)(n)[2n]$$

for $n = 0, \ldots, d$.

The argument uses 3 tricks:

- The duality trick
- The $R\mathcal{H}om$ trick
- The K-theory trick

Slices of Severi-Braurer varieties: The duality trick

We have the operation $R\mathcal{H}om(X, -)$ on $\mathrm{SH}_{S^1}(k)$ and $\mathrm{DM}^{\mathrm{eff}}(k)$:
For fibrant E

$$R\mathcal{H}om(X, E)(Y) = E(X \times Y).$$

Proposition

Let X be smooth and projective of dimension d over k. Then

$$\mathbb{H}^{\mu}_{d-n}(X) \cong s_0^{\mathrm{mot}}[R\mathcal{H}om(X, \mathbb{Z}(n)[2n])]$$

for $0 \leq n \leq d$.

This uses duality: $M_{\mathrm{gm}}(X)^{\vee} = M_{\mathrm{gm}}(X)(-d)[-2d]$, compatibility of slices with Ω_t and the cancellation theorem.
The advantage: Shifts the computation to the 0th slice.

For $E \in \mathrm{SH}_{S^1}(k)$, apply $s_0 R\mathcal{H}om(X, -)$ to the motivic Postnikov tower of E. We get the tower

$$\to s_0 R\mathcal{H}om(X, f_n E) \to .. \to s_0 R\mathcal{H}om(X, f_1 E) \to s_0 R\mathcal{H}om(X, E)$$

with layers $s_0 R\mathcal{H}om(X, s_n E)$.

Lemma

For X smooth and projective of dimension d over k, $s_0 R\mathcal{H}om(X, f_{d+1} E) = 0$.

This gives the spectral sequence:

$$E^1_{a,b} = \pi_{a+b}(s_0 R\mathcal{H}om(X, s_a E)(Y)) \Rightarrow \pi_{a+b}(s_0 R\mathcal{H}om(X, E)(Y)).$$

for $0 \leq a \leq d$, $E^1_{a,b} = 0$ else.

We take $E = K$:

$$E^1_{a,b} = \pi_{a+b}(s_0 R\mathcal{H}om(X, s_a K)(Y)) \Rightarrow \pi_{a+b}(s_0 R\mathcal{H}om(X, K)(Y)).$$

for $0 \le a \le d$, $E^1_{a,b} = 0$ else.

Lemma

Suppose $X = \mathrm{S.B.}(A)$ for A a c.s.a. over k of prime rank ℓ. Then the $s_0 R\mathcal{H}om(X, K)$ spectral sequence degenerates at E^1.

Proof. For $X = \mathbb{P}^{\ell-1}$, this follows from the projective bundle formula. Thus the differentials are killed by ℓ.

$s_a K = \mathrm{EM}(\mathbb{Z}(a)[2a])$. Thus Adams operations on K act on the spectral sequence and ψ_k acts by k^a on $E^1_{a,b}$. Since $0 \le a \le \ell - 1$, the differentials die after inverting $(\ell - 1)!$

Proposition

$X = \mathrm{S.B.}(A)$ for A a c.s.a. over k of prime rank ℓ. Then

$$\mathcal{H}^n_{\mathrm{Nis}}(s_0^{\mathrm{mot}} R\mathcal{H}om(X, \mathbb{Z}(a)[2a])) = 0$$

for $n \neq 0$: $s_0^{\mathrm{mot}} R\mathcal{H}om(X, \mathbb{Z}(a)[2a])$ is a birational motivic sheaf.

Proof. By Quillen, $R\mathcal{H}om(X, K) = \oplus_{i=0}^{\ell} K^{A^{\otimes i}}$. Thus

$$s_0[R\mathcal{H}om(X, K)] = \bigoplus_{i=0}^{\ell} s_0[K^{A^{\otimes i}}] = \bigoplus_{i=0}^{\ell} \mathrm{EM}(\mathbb{Z}_{A^{\otimes i}}).$$

Since $\mathbb{Z}_{A^{\otimes i}}$ is a sheaf, so is $s_0[R\mathcal{H}om(X, K)]$.

$$\pi_m^{\mathrm{Nis}}(s_0[R\mathcal{H}om(X,K)]) = 0;\ \text{for } m \neq 0.$$

Since the $s_0R\mathcal{H}om(X,K)$ spectral sequence degenerates, the same vanishing is true for the layers

$$\begin{aligned} s_0R\mathcal{H}om(X, s_aK) &= s_0R\mathcal{H}om(X, \mathrm{EM}(\mathbb{Z}(a)[2a])) \\ &= \mathrm{EM}(s_0^{\mathrm{mot}}R\mathcal{H}om(X, \mathbb{Z}(a)[2a])). \end{aligned}$$

Thus $s_0^{\mathrm{mot}}R\mathcal{H}om(X, \mathbb{Z}(a)[2a])$ is a sheaf.

Theorem

Let A be a c.s.a. over k of prime rank ℓ, $X = \mathrm{S.B.}(A)$, $d = \ell - 1 = \dim_k X$. Then

$$s_n^{\mathrm{mot}} M_{\mathrm{gm}}(X) = \mathbb{Z}_{A^{\otimes n+1}}(n)[2n] = C\mathcal{H}_n(X)(n)[2n]$$

for $n = 0, \ldots, d$.

Proof. We have already computed

$$\mathcal{H}^0_{\mathrm{Nis}}(\mathbb{H}^\mu_n(X)) = C\mathcal{H}_n(X)$$

We need to see that $\mathbb{H}^\mu_n(X)$ has no other cohomology sheaves.

Putting our tricks together shows that

$$\mathbb{H}^{\mu}_{d-n}(X) = s_0^{\mathrm{mot}} R\mathcal{H}om(X, \mathbb{Z}(n)[2n]),$$

the RHS has no cohomology beside $\mathcal{H}^0$, and that

$$\mathcal{H}^0(\mathbb{H}^{\mu}_n(X)) = C\mathcal{H}_n(X) = \mathbb{Z}_{A^{\otimes n+1}}$$

The reduced norm map

Reduced norms for Azumaya algebras

$A :=$ a c.s.a over a field k.
We have the reduced norm map

$$\mathrm{Nrd} : K_0(A) \to K_0(k) :$$

Take L/k a splitting field, so $K(A_L) \cong K(L)$ by Morita equivalence. Define Nrd by

$$K_0(A) \to K_0(A_L) = K_0(L) \xleftarrow{\sim} K_0(k).$$

Reduced norms for Azumaya algebras

A similar construction defines

$$\mathrm{Nrd} : K_1(A) \to K_1(k).$$

A more difficult construction (Suslin) defines

$$\mathrm{Nrd} : K_2(A) \to K_2(k).$$

There is no reasonable Nrd on $K_n(A)$ for $n \geq 3$.

Reduced norms for Azumaya algebras

Question: What is the kernel and image of Nrd on K_n, $n = 0, 1, 2$?

For $n = 0$ it is easy to see that Nrd is injective and

$$\mathrm{image}(\mathrm{Nrd}) = (\mathrm{index}(A)) \subset \mathbb{Z} = K_0(k).$$

Theorem

[Wang] $\mathrm{Nrd} : K_1(A) \to K_1(k)$ *is injective if A has square-free index.*

In general, Nrd is not injective on $K_1(A)$ (Platonov, Merkurjev).

Reduced norm for $\mathbb{Z}_A$

The reduced norm on $K_0(A)$ gives the injective map of sheaves with transfer

$$\mathrm{Nrd} : \mathbb{Z}_A \to \mathbb{Z}$$

Thus we have

$$\mathrm{Nrd} : H^p(X, \mathbb{Z}_A(q)) \to H^p(X, \mathbb{Z}(q))$$

for all $X \in \mathbf{Sm}/k$, p, q.

Comparison of reduced norms

Proposition

The reduced norm maps are compatible via the edge homomorphisms in the A-H spectral sequence for K^A: The diagram

$$\begin{array}{ccc} H^n(k, \mathbb{Z}_A(n)) & \xrightarrow{e_n} & K_n(A) \\ \downarrow{\scriptstyle \mathrm{Nrd}} & & \downarrow{\scriptstyle \mathrm{Nrd}} \\ H^n(k, \mathbb{Z}(n)) & \xrightarrow[e_n]{} & K_n(k) \end{array}$$

commutes for $n = 0, 1, 2$.

We recall the *Beilinson-Lichtenbaum conjectures*

Theorem

[Rost, Voevodsky, ...] Let α *be the change of topology morphism from* $\mathrm{DM}^{\mathrm{eff}}(k)$ *to* $\mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(k)$. *Then for* $X \in \mathbf{Sm}/k$

$$H^p(X, \mathbb{Z}(q)) \xrightarrow{\alpha^*} H^p_{\acute{e}t}(X, \alpha^*\mathbb{Z}(q))$$

is an isomorphism for $p \leq q + 1$ *and an injection for* $p = q + 2$.

Reduced norm and étale motivic cohomology

Using the Beilinson-Lichtenbaum conjectures, one can view Nrd as a change of topology map.

Note that $\alpha^*\mathbb{Z}_A(q) = \alpha^*\mathbb{Z}(q)$, and the diagram

$$\begin{array}{ccc} H^p(X, \mathbb{Z}_A(q)) & \xrightarrow{\text{Nrd}} & H^p(X, \mathbb{Z}(q)) \\ \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} \\ H^p_{\text{ét}}(X, \alpha^*\mathbb{Z}_A(q)) & = & H^p_{\text{ét}}(X, \alpha^*\mathbb{Z}(q)) \end{array}$$

commutes.

The main result on reduced norms

Theorem

Let A be a c.s.a of square-free index over k.

1. For $p < q$ the reduced norm map

$$\mathrm{Nrd} : H^p(k, \mathbb{Z}_A(q)) \to H^p(k, \mathbb{Z}(q))$$

is an isomorphism.

2. For every $n \geq 0$, the sequence

$$0 \to H^n(k, \mathbb{Z}_A(n)) \xrightarrow{\mathrm{Nrd}} H^n(k, \mathbb{Z}(n)) \xrightarrow{\cup[A]} H^{n+3}_{\acute{e}t}(k, \alpha^*\mathbb{Z}(n+1))$$

is exact.

$[A] \in Br(k) = H^2_{\text{ét}}(k, \mathbb{G}_m) = H^3_{\text{ét}}(k, \alpha^*\mathbb{Z}(1))$ is the class of A in the Brauer group.

Boundary Map

Corollary

Let A be a c.s.a of square-free index over k. For $n = 0, 1, 2$:

1. The edge homomorphism

$$e_n : H^n(k, \mathbb{Z}_A(n)) \to K_n(A)$$

is an isomorphism.

2. The sequence

$$0 \to K_n(A) \xrightarrow{\mathrm{Nrd}} K_n(k) = H^n(k, \mathbb{Z}(n)) \xrightarrow{\cup[A]} H^{n+3}_{\acute{e}t}(k, \alpha^*\mathbb{Z}(n+1))$$

is exact. In particular, Nrd *is injective.*

Skip Proof The end

The main result on reduced norms: Idea of proof

Proof. May assume A has prime rank ℓ over k. $X := \mathrm{S.B.}(A)$, $d = \ell - 1$. Start with the motivic Postnikov tower

$$0 \to f_d^{\mathrm{mot}} M(X) \to \ldots \to f_1^{\mathrm{mot}} M(X) \to M(X)$$

with layers $\mathbb{Z}_{A^{\otimes n+1}}(n)[2n]$.

Apply $R\alpha_*\alpha^*$: This gives the tower

$$\begin{aligned} 0 \to R\alpha_*\alpha^*[f_d^{\mathrm{mot}} M(X)] \to \ldots \\ \to R\alpha_*\alpha^*[f_1^{\mathrm{mot}} M(X)] \to R\alpha_*\alpha^*[M(X)] = R\alpha_*\alpha^*[M(\mathbb{P}^d)] \end{aligned}$$

with layers $R\alpha_*\alpha^*[\mathbb{Z}(n)[2n]]$.

The main result on reduced norms: Idea of proof

Define $\overline{f_n^{\mathrm{mot}} M(X)}$ as "cone" of $f_n^{\mathrm{mot}} M(X) \to R\alpha_*\alpha^*[f_n^{\mathrm{mot}} M(X)]$. This gives the tower

$$0 \to \overline{f_d^{\mathrm{mot}} M(X)} \to \ldots \to \overline{f_1^{\mathrm{mot}} M(X)} \to \overline{M(X)}$$

with layers

$$\overline{\mathbb{Z}}_{A^{\otimes n+1}}(n)[2n] := \text{"cone"}(\mathbb{Z}_{A^{\otimes n+1}}(n)[2n] \to R\alpha_*\alpha^*[\mathbb{Z}(n)[2n]]).$$

Use Beilinson-Lichtenbaum, duality and theory of birational motivic sheaves to analyze the E_2 spectral sequence associated to $\mathrm{Hom}_{\mathrm{DM^{eff}}(k)}(\mathbb{Z}(d)[2d-2n-2], (-)(n+1))$ applied to the last tower.

The main result on reduced norms: Idea of proof

Beilinson-Lichtenbaum+ duality $\Rightarrow$ sequence converges to 0 in some range.

Beilinson-Lichtenbaum $\Rightarrow$ E_2 terms in some range are computable in terms of $H^p(k, \mathbb{Z}_A(q)) \xrightarrow{\text{Nrd}} H^p(k, \mathbb{Z}(q))$.

Beilinson-Lichtenbaum+ induction $\Rightarrow$ most of the E_2 terms die and sequence degenerates soon (at E_2 for part 1, at E_3 for part 2).

Main Theorem

The main result on reduced norms: Idea of proof

The map $H^n(k,\mathbb{Z}(n)) \to H^{n+3}_{\text{ét}}(k,\alpha^*\mathbb{Z}(n+1))$ comes from the boundary map in the distinguished triangle

$$\alpha^*\mathbb{Z}(d)[2d] \to \alpha^* f^{\text{mot}}_{d-1} M_{\text{gm}}(X) \to \alpha^*\mathbb{Z}_A(d-1)[2d-2] \xrightarrow{\partial} \alpha^*\mathbb{Z}(d)[2d+1]$$

But

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k)}(\alpha^*\mathbb{Z}(d-1)[2d-2], \alpha^*\mathbb{Z}(d)[2d+1]) = H^3_{\text{ét}}(k,\alpha^*\mathbb{Z}(1))$$

A calculation shows

$$\partial \mapsto [A] \in H^2_{\text{ét}}(k,\mathbb{G}_m) = H^3_{\text{ét}}(k,\alpha^*\mathbb{Z}(1)).$$

The end

K-cohomology

With a similar argument, we recover results of Merkurjev and Suslin (but without using Beilinson-Lichtenbaum)

Theorem

Let $X = \mathrm{S.B.}(A)$, A a c.s.a of prime rank over k. Then

$$H^0(X, \mathcal{K}_2) = K_2(k)$$

and

$$H^1(X, \mathcal{K}_2) \cong K_1(A) \subset k^\times.$$

The End

Thank you!