The motive of a Severi-Brauer variety
and K-theory of central simple algebras

Marc Levine, joint w. Bruno Kahn

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Outline

- The motivic Postnikov tower in $\text{SH}_{S^1}(k)$ and $\text{DM}^{\text{eff}}(k)$
- Slices of twisted $K$-theory
- Slices of Severi-Brauer varieties
- The reduced norm map
The motivic Postnikov tower in $\text{SH}_{S^1}(k)$ and $\text{DM}^{\text{eff}}(k)$
An overview

$E \in \text{SH}$ has its $n-1$-connected cover $E\{n\} \to E$ and its Postnikov tower

$$\ldots \to E\{n+1\} \to E\{n\} \to \ldots \to E$$

with layers $\Sigma^n\text{EM}(\pi_nE)$.

For $\mathcal{F} \in D(\text{Ab})$, the canonical homological truncation does the same

$$\ldots \to \tau_{\geq n+1}\mathcal{F} \to \tau_{\geq n}\mathcal{F} \to \ldots \to \mathcal{F}$$

with layers the homology $H_n(\mathcal{F})[n]$. 
Voevodsky tells us how to form the same picture for $\text{SH}(k)$ and $\text{DM}(k)$, replacing suspension $\Sigma$ with $T$-suspension $\Sigma_t$ or Tate twist $\otimes \mathbb{Z}(1)[2]$.

For $\mathcal{E} \in \text{SH}(k)$, we get the homotopy motives $\pi_n^\mu \mathcal{E}$, and for $\mathcal{F} \in \text{DM}(k)$ the homology motives $\mathbb{H}_n^\mu \mathcal{F}$.

Instead of being abelian groups, the $\pi_n^\mu \mathcal{E}$ and $\mathbb{H}_n^\mu \mathcal{F}$ are birational motives.
Cast of characters

\( \text{Sm}/k := \text{smooth varieties over } k \)

\( \text{Spc}_*(k) := \text{presheaves of pointed simplicial sets on } \text{Sm}/k \)

\( T := (\mathbb{P}^1, \infty) \in \text{Spc}_*(k) \)

\( \text{Spt}_{S^1}(k) := \text{the category of presheaves of spectra on } \text{Sm}/k \)

\( \text{SH}_{S^1}(k) := \text{the } \mathbb{A}^1\text{-homotopy category of } S^1 \text{ spectra} \)
\( := \text{Spt}_{S^1}(k)[W_{\text{Nis}}^{-1}, W_{\mathbb{A}^1}^{-1}] \)

\( \Sigma_t, \Omega_t: \text{the } T\text{-suspension and } T\text{-loops functors on } \text{SH}_{S^1}(k) \)

\( \text{SH}(k) := \text{the homotopy category of } T \text{ spectra (of } S^1 \text{ spectra)} \)

\( \Sigma_t \text{ on } \text{SH}_{S^1}(k) \text{ becomes invertible on } \text{SH}(k). \)
$\Sigma^\infty_t : \text{SH}_{S^1}(k) \leftrightarrow \text{SH}(k) : \Omega^\infty_t$

the infinite $T$-suspension and 0-spectrum functors

$S_k, \mathbb{H}\mathbb{Z} \in \text{SH}(k)$ the motivic sphere spectrum and motivic cohomology spectrum

$\text{SmCor}(k) :=$ the category of finite correspondences

$\text{DM}^{\text{eff}}(k) :=$ the category of effective motives

$\subset D(\text{Sh}_{\text{Nis}}(\text{SmCor}(k)))$ as $\mathbb{A}^1$-local objects.

$\text{EM} : \text{DM}^{\text{eff}}(k) \to \text{SH}_{S^1}(k)$ the Eilenberg-Maclane spectrum functor
Voevodsky has defined the “Tate” analog of the classical Postnikov tower in $\text{SH}$. We give the version for $\text{SH}_{S^1}(k)$:

Taking $T$-suspensions of $\text{SH}_{S^1}(k)$ gives the tower of full triangulated localizing subcategories of $\text{SH}_{S^1}(k)$:

$$
\ldots \subset \Sigma^{n+1}_{t} \text{SH}_{S^1}(k) \subset \Sigma^{n}_{t} \text{SH}_{S^1}(k) \subset \ldots \subset \Sigma_{t} \text{SH}_{S^1}(k) \subset \text{SH}_{S^1}(k)
$$

**Lemma**

*The inclusion functor $i_{n} : \Sigma^{n}_{t} \text{SH}_{S^1}(k) \rightarrow \text{SH}(k)$ admits an exact right adjoint $r_{n} : \text{SH}(k) \rightarrow \Sigma^{n}_{t} \text{SH}_{S^1}(k)$.***
The motivic Postnikov tower

\[ \ldots \subset \Sigma_{t}^{n+1}SH_{S1}(k) \subset \Sigma_{t}^{n}SH_{S1}(k) \subset \ldots \subset \Sigma_{t}SH_{S1}(k) \subset SH_{S1}(k) \]

**Lemma**

*The inclusion functor* $i_{n} : \Sigma_{t}^{n}SH_{S1}(k) \to SH(k)$ *admits an exact right adjoint* $r_{n} : SH(k) \to \Sigma_{t}^{n}SH_{S1}(k)$.

Define $f_{n} := i_{n}r_{n} : SH_{S1}(k) \to SH_{S1}(k)$, giving the natural tower

\[ \ldots \to f_{n+1}E \to f_{n}E \to \ldots \to E \]

called the *motivic Postnikov tower*.

The layers

\[ s_{n}E := \text{cofib}(f_{n+1}E \to f_{n}E) \]

are Voevodsky’s *slices*. 
Properties of the motivic Postnikov tower

1. $f_n \circ \Omega_t = \Omega_t \circ f_{n+1}$.

2. $K :=$ the $K$-theory presheaf $Y \mapsto K(Y)$. We have:

$$s_n(K) = \text{EM}(\mathbb{Z}(n)[2n]).$$

This yields the Atiyah-Hirzebruch spectral sequence for $K$-theory:

$$E_2^{p,q} := H^{p-q}(X, \mathbb{Z}(-q)) \implies K_{-p-q}(X).$$

This is the same one as constructed by Bloch-Lichtenbaum (for fields) and extended to arbitrary $X$ by Friedlander-Suslin. We’ll see the proof later on.

3. $s_0(S_k) = \mathcal{H} \mathbb{Z}$ (Voevodsky). This is taking place in $\text{SH}(k)$. 

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The motivic Postnikov tower in $\text{DM}^\text{eff}(k)$

We have a motivic Postnikov tower in $\text{DM}^\text{eff}(k)$ as well, replacing $\Sigma_t$ with $\mathbb{Z}(1) \otimes -$.

This gives us the tower of localizing subcategories

$$\ldots \subset \mathbb{Z}(n) \otimes \text{DM}^\text{eff}(k) \subset \ldots \subset \mathbb{Z}(1) \otimes \text{DM}^\text{eff}(k) \subset \text{DM}^\text{eff}(k).$$

Write the resulting tower of truncation functors as

$$\ldots \rightarrow f_{n+1}^\text{mot} \rightarrow f_n^\text{mot} \rightarrow \ldots \rightarrow f_1^\text{mot} \rightarrow f_0^\text{mot} = \text{id}$$

and the slices as $s_n^\text{mot}$:

$$f_{n+1}^\text{mot} \rightarrow f_n^\text{mot} \rightarrow s_n^\text{mot} \rightarrow f_{n+1}[1]$$
The motivic Postnikov tower in $\text{DM}^{\text{eff}}(k)$

The Postnikov towers for $\text{SH}_{S^1}(k)$ and $\text{DM}^{\text{eff}}(k)$ are related by the Eilenberg-Maclane functor $EM : \text{DM}^{\text{eff}}(k) \to \text{SH}_{S^1}(k)$:

$$f_n \circ EM = EM \circ f_n^{\text{mot}}$$

We have the same compatibility with $\Omega_t := \mathcal{H}om(\mathbb{Z}(1)[2], -)$ as in $\text{SH}_{S^1}(k)$:

$$f_n^{\text{mot}} \circ \Omega_t = \Omega_t \circ f_{n+1}^{\text{mot}},$$

**Warning:** $\cap_n \mathbb{Z}(n) \otimes \text{DM}^{\text{eff}}(k) \neq \{0\}$. 
Let $\Delta^n := \text{Spec } k[t_0, \ldots, t_n]/\sum_i t_i - 1$, $\Delta^* := \text{the cosimplicial scheme } n \mapsto \Delta^n$.

For a field $F$, let $\Delta^n_F, 0$ be the semi-local scheme of the vertices in $\Delta^n_F$.
This gives the cosimplicial subscheme $\Delta^*_F, 0, n \mapsto \Delta^n_F, 0$ of $\Delta^*_F$.

**Theorem**

*[0th slice]* For $X \in \text{Sm}/k$, $E \in \text{SH}_{S^1}(k)$ there is a natural isomorphism in $\text{SH}$:

$s_0(E)(X) \cong E(\Delta^*_k(X), 0)$. 

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The motive of a Severi-Brauer variety
The homotopy motives

Theorem

Take \( E \in \text{SH}_{S^1}(k) \) which is the 0-spectrum of some \( \mathcal{E} \in \text{SH}(k) \).

1. There is a motive \( \pi^\mu_0(E) \in \text{DM}^\text{eff}(k) \) with

\[
s_0 E = \text{EM}(\pi^\mu_0(E)).
\]

2. \( s_n E = \sum_t \text{EM}(\pi^\mu_0(\Omega^n_t E)) = \text{EM}(\pi^\mu_0(\Omega^n_t E)(n)[2n]) \).

This relies on Voevodsky’s theorem \( s_0(S_k) = \mathcal{H}\mathbb{Z} \), the result of Østvær-Röndigs identifying \( \text{DM}(k) \) with \( \mathcal{H}\mathbb{Z} \)-modules, and a lifting of the slice filtration to the model category level due to P. Pelaez-Menaldo.

**Definition** \( \pi^\mu_n E := \pi^\mu_0(\Omega^n_t E) \) is the \( n \)th homotopy motive of \( E \).

\[\implies s_n E = \sum_t \text{EM}(\pi^\mu_n E).\]
The Atiyah-Hirzebruch spectral sequence

The motivic Postnikov tower for $E \in \text{SH}_{S^1}(k)$

$$\ldots \rightarrow f_{n+1}E \rightarrow f_nE \rightarrow \ldots \rightarrow f_1E \rightarrow E$$

gives the spectral sequence for $X \in \text{Sm}/k$,

$$E_2^{p,q} = H^{p-q}(X,(\pi^\mu_{-q}E)(-q)) \Rightarrow \pi_{-p-q}(E(X))$$

(assuming $E = \Omega_\infty^t \mathcal{E}$ for some $\mathcal{E} \in \text{SH}(k)$).
**Definition**  Call $\mathcal{F} \in \text{DM}^{\text{eff}}(k)$ a **birational motive** if for all dense open immersions $j : U \to X$ in $\text{Sm}/k$, the pullback

$$j^* : \mathbb{H}^n_{\text{Nis}}(X, \mathcal{F}) \to \mathbb{H}^n_{\text{Nis}}(U, \mathcal{F})$$

is an isomorphism for all $n$.

**Example.** The constant sheaf with transfers $\mathbb{Z}$ is a birational motive.

**Proposition**

*Let $\mathcal{F}$ be a birational motive. Then for every $n \geq 0$,

$$f^\text{mot}_m(\mathcal{F}(n)) = s^\text{mot}_m(\mathcal{F}(n)) = \begin{cases} \mathcal{F}(n) & \text{for } m \leq n \\ 0 & \text{for } m > n. \end{cases}$$*
In fact:

**Proposition**

*For $E$ the 0-spectrum of some $\xi \in \text{SH}(k)$, the homotopy motive $\pi^n_\mu E$ is a birational motive for all $n$."

Thus the birational motives play the role in motivic homotopy theory that abelian groups do in classical homotopy theory:

*All objects in $\text{SH}(k)$ are built out of the Eilenberg-Maclane spectra of Tate twists of birational motives:

$$s_n E = \text{EM}(\pi^n_\mu E(n)[2n]).$$
The simplest birational motives are the \textit{birational motivic sheaves}: a birational motive $\mathcal{F}$ such that $\mathcal{H}^m_{\text{Nis}}(\mathcal{F}) = 0$ for $m \neq 0$.

The simplest $E \in \text{SH}_{S^1}(k)$ are those such that the (birational) homotopy motives $\pi_n^\mu E$ are birational motivic sheaves.

\textbf{Definition} Call $E \in \text{SH}_{S^1}(k)$ \textit{well-connected} if

1. $E(X)$ is $-1$ connected for all $X \in \text{Sm}/k$.
2. The homotopy sheaf $\pi_m^{\text{Nis}} s_0(\Omega^*_t E)$ is zero for all $m \neq 0$.

\textbf{Proposition}

If $E$ is well-connected, then $\pi_0^{\text{Nis}} s_0(\Omega^*_t E) = \pi_0^{\text{Nis}} \Omega^*_t E$. If $E = \Omega^*_t \mathcal{E}$ then

$$\pi_n^\mu E = \pi_0^{\text{Nis}}(\Omega^*_t E).$$
Slices of twisted $K$-theory
The slices of $K$-theory

**Proposition**

The algebraic $K$-theory presheaf $X \mapsto K(X)$ is well-connected.

**Proof.** $K$ is quasi-fibrant by the homotopy property and Quillen localization. $K(X)$ is -1 connected by construction.

To show $\pi_n(K(\Delta^n_F,0)) = 0$ for $n \neq 0$: It comes down to showing that

$$K_0(\Delta^n_F,0, \partial\Delta^n_F,0) = 0$$

for all $n \geq 1$ (cf. Friedlander-Suslin). This follows from the long exact sequence

$$\ldots \to K_1(\Delta^n_F,0) \xrightarrow{i^*} K_1(\partial\Delta^n_F,0) \to K_0(\Delta^n_F,0, \partial\Delta^n_F,0) \to 0$$

Since $\Delta^n_F,0$ and $\partial\Delta^n_F,0$ are semi-local, the $K_1$’s are just units, so surjectivity of $i^*$ is easy to check.
The slices of $K$-theory

Theorem

$$\pi_n^H K = \text{EM}(\mathbb{Z}); \quad s_n(K) = \text{EM}(\mathbb{Z}(n)[2n])$$

for all $n \geq 0$.

Proof.

$$s_n(K) = \sum_t^n s_0(\Omega_t^n K) = \sum_t^n s_0 K.$$

Since $K$ is well-connected,

$$s_0 K = \text{EM}(\pi_0^{\text{Nis}}(K)) = \text{EM}(\mathbb{Z})$$
Twisted $K$-theory

$A :=$ a central simple algebra over $k$.
$K^A$ the presheaf $X \mapsto K(X; A)$.

Proposition

$K^A$ is well-connected.

The proof is the same as for $K$.

What are the slices $s_n(K^A)$?
The sheaf $\mathcal{Z}_A$

The Nisnevich sheaf $\pi^\text{Nis}_0(K^A)$ is the sheaf associated to the presheaf

$$X \mapsto K_0(X, A)$$

on $\text{Sm}/k$. $\pi^\text{Nis}_0(K^A)$ has transfers induced by the pushforward for finite maps in twisted $G$-theory $G(–, A)$.

**Definition**  Denote the sheaf with transfers $\pi^\text{Nis}_0(K^A)$ by $\mathcal{Z}_A$.

**Remarks**  1. $\mathcal{Z}_A$ is a homotopy invariant sheaf with transfers.

2. For $X \in \text{Sm}/k$ irreducible, $\mathcal{Z}_A(X) \subset \mathcal{Z}(X) = \mathbb{Z}$ is the subgroup of index $= \text{index}(A_{k(X)}) \implies \mathcal{Z}_A$ is a birational motivic sheaf.
The sheaf $\mathcal{Z}_A$

**Theorem**

Let $A$ be a c.s.a over $k$. Then

$$\pi_n^\mu(K^A) = \text{EM}(\mathcal{Z}_A); \quad s_n(K^A) = \text{EM}(\mathcal{Z}_A(n)[2n])$$

for all $n \geq 0$.

The proof is the same as for $K$, noting $\pi_0^\text{Nis} K^A = \mathcal{Z}_A$.

**Corollary**

The Atiyah-Hirzebruch spectral sequence for $K(X; A)$ is

$$E_2^{p,q} = H^{p-q}(X, \mathcal{Z}_A(-q)) \implies K_{-p-q}(X; A).$$
Computations in low degree

- $H^1(k, \mathbb{Z}_A(1)) = K_1(A)$
- We have an exact sequence

$$0 \to H^{-1}(k, \mathbb{Z}_A(1)) \to H^2(k, \mathbb{Z}_A(2)) \to K_2(A) \to H^0(k, \mathbb{Z}_A(1)) \to 0.$$

- There is an edge homomorphism $H^n(k, \mathbb{Z}_A(n)) \xrightarrow{e_n} K_n(A)$.
- $H^n(X, \mathbb{Z}_A(0)) = 0$ for $n \neq 0$,
- $H^0(X, \mathbb{Z}_A(0)) = \mathbb{Z}_A(k(X)) = K_0(A_{k(X)})$. 

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The motive of a Severi-Brauer variety
**Remark:** Bloch showed that

\[ H^n(k, \mathbb{Z}(1)) = 0 \]

for \( n \neq 1 \).

We do not know if \( H^n(k, \mathbb{Z}_A(1)) = 0 \) for \( n \neq 1 \) in general, but this is true if \( A \) has square-free index.

Thus, for \( A \) of square-free index, we have

\[ H^2(k, \mathbb{Z}_A(2)) = K_2(A). \]
Slices of Severi-Brauer varieties
**Definition** Let $X$ be a smooth projective variety over $k$. Let $\mathcal{CH}^n(X)$ be the Nisnevich sheaf associated to the presheaf

$$Y \mapsto \text{CH}^n(X \times_k Y).$$

Set $\mathcal{CH}_n(X) := \mathcal{CH}^{\dim_k X - n}(X)$.

**Proposition**

[Kahn-Huber] $\mathcal{CH}^n(X)$ is a birational motivic sheaf.
Definition. For $M \in \text{DM}^{\text{eff}}(k)$, we have the *homology motive* $\mathbb{H}^\mu_n M$ defined by

$$\mathbb{H}^\mu_n M = s^\text{mot}_0 \Omega^n_t M = \Omega^n_t s^\text{mot}_n M$$

i.e., $s^\text{mot}_n M = (\mathbb{H}^\mu_n M)(n)[2n]$. Write $\mathbb{H}^\mu_n(X) := \mathbb{H}^\mu_n M_{\text{gm}}(X)$.

Via the analogy $\text{DM}^{\text{eff}}(k) \leftrightarrow D(\text{Ab})$, $\mathbb{H}^\mu_n M$ corresponds to $H_n C$.

Just as for the homotopy motives of $S^1$-spectra, the homology motives of $M$ are always birational motives, but not always birational motivic sheaves.
We do know the following:

**Proposition**

*Huber-Kahn* Let $X$ be a smooth projective variety. Then

1. $f_n^{\text{mot}}(M_{\text{gm}}(X)) = 0$ for $n > \dim_k X$.

2. For $0 \leq n \leq \dim_k X$,

$$H^0_\text{Nis}(H^\mu_n(X)) = CH_n(X).$$

3. $H^m_\text{Nis}(H^\mu_n(X)) = 0$ for $m > 0$. 

Skip proof

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The motive of a Severi-Brauer variety
Slices of smooth projective varieties

**Idea:** $M_{gm}(X)$ is represented by the complex of presheaves $C^*_{Susz}(z_{equi}(X, 0))$:

$$Y \mapsto [\ldots \rightarrow z_{equi}(X, 0)(Y \times \Delta^n) \rightarrow \ldots \rightarrow z_{equi}(X, 0)(Y)],$$

where $z_{equi}(X, r)$ is the presheaf of cycles of relative dimension $r$ on $X$.

By the 0th-slice theorem

$$H^0_{\mu}(X)(Y) := s^0_{mot} M_{gm}(X)(Y) = C^*_{Susz}(z_{equi}(X, 0)) (\Delta^*_{k(Y), 0})$$

$$:= [\ldots \rightarrow z_{equi}(X_{k(Y)}, 0)(\Delta^1_{k(Y), 0}) \rightarrow z_0(X_{k(Y)})]$$

Thus $\mathcal{H}_{Nis}^0(H^0_{\mu}(X)) = CH_0(X)$ and there is no higher cohomology.
Slices of smooth projective varieties

For the higher homology motives: The Gysin sequence implies
\[ \Omega^n_{t} M_{gm}(X) = C^*_{\text{Sus}}(z_{\text{equi}}(X, n)). \]

Thus by the 0th-slice theorem

\[
\check{H}^n_{\mu}(X)(Y) = \Omega^n_{t}(s^{\text{mot}}_n M_{gm}(X))(Y) \\
= s^{\text{mot}}_0(\Omega^n_{t} M_{gm}(X))(Y) = C^*_{\text{Sus}}(z_{\text{equi}}(X, n))(\Delta^*_k(Y), 0).
\]

and hence

\[
\check{H}^0_{\text{Nis}}(\check{H}^n_{\mu}(X)) = C\check{H}_n(X),
\]

and no higher cohomology.

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The motive of a Severi-Brauer variety
**Question**: Is $H^\mu_n(X)$ a birational motivic sheaf, i.e., is

$$\mathcal{H}^m_{Nis}(H^\mu_n(X)) = 0, \text{ for } m \neq 0?$$

Said another way, is

$$s^\text{mot}_n(M_{gm}(X)) = C\mathcal{H}_n(X)(n)[2n]?$$

**Example.** $M_{gm}(\mathbb{P}^m) = \bigoplus_{i=0}^m \mathbb{Z}(i)[2i]$, so

$$s^\text{mot}_n(M_{gm}(\mathbb{P}^m)) = \mathbb{Z}(n)[2n] = C\mathcal{H}_n(\mathbb{P}^m)(n)[2n]$$

for $0 \leq n \leq m$.

For a general $X$, the negative cohomology of $s^\text{mot}_n(M_{gm}(X))$ is not zero. However:
Theorem

Let $A$ be a c.s.a. over $k$ of prime rank $\ell$, $X = \text{S. B.}(A)$, $d = \ell - 1 = \dim_k X$. Then

$$\mathbb{H}_n^\mu(X) = \mathbb{Z}_{A \otimes n+1} = C\mathcal{H}_n(X),$$

i.e.,

$$s_n^{\text{mot}} M_{\text{gm}}(X) = \mathbb{Z}_{A \otimes n+1}(n)[2n] = C\mathcal{H}_n(X)(n)[2n]$$

for $n = 0, \ldots, d$.

The argument uses 3 tricks:

- The duality trick
- The $R\text{Hom}$ trick
- The $K$-theory trick
We have the operation $R\mathcal{H}om(X, -)$ on $\text{SH}_{S^1}(k)$ and $\text{DM}^{\text{eff}}(k)$: For fibrant $E$

$$R\mathcal{H}om(X, E)(Y) = E(X \times Y).$$

**Proposition**

Let $X$ be smooth and projective of dimension $d$ over $k$. Then

$$\mathbb{H}_{d-n}^{\mu}(X) \cong s_0^{\text{mot}}[R\mathcal{H}om(X, \mathbb{Z}(n)[2n])]$$

for $0 \leq n \leq d$.

This uses duality: $M_{\text{gm}}(X)^\vee = M_{\text{gm}}(X)(-d)[-2d]$, compatibility of slices with $\Omega_t$ and the cancellation theorem.

The advantage: Shifts the computation to the 0th slice.
Slices of Severi-Brauer varieties: The $R\mathcal{H}om$ trick

For $E \in \text{SH}_{S^1}(k)$, apply $s_0R\mathcal{H}om(X, -)$ to the motivic Postnikov tower of $E$. We get the tower

$$\rightarrow s_0R\mathcal{H}om(X, f_nE) \rightarrow .. \rightarrow s_0R\mathcal{H}om(X, f_1E) \rightarrow s_0R\mathcal{H}om(X, E)$$

with layers $s_0R\mathcal{H}om(X, s_nE)$.

Lemma

For $X$ smooth and projective of dimension $d$ over $k$, $s_0R\mathcal{H}om(X, f_{d+1}E) = 0$.

This gives the spectral sequence:

$$E_{a,b}^1 = \pi_{a+b}(s_0R\mathcal{H}om(X, s_aE)(Y)) \Rightarrow \pi_{a+b}(s_0R\mathcal{H}om(X, E)(Y)).$$

for $0 \leq a \leq d$, $E_{a,b}^1 = 0$ else.
Slices of Severi-Brauer varieties: The $K$-theory trick

We take $E = K$:

$$E^1_{a, b} = \pi_{a+b}(s_0 R\text{Hom}(X, s_a K)(Y)) \Rightarrow \pi_{a+b}(s_0 R\text{Hom}(X, K)(Y)).$$

for $0 \leq a \leq d$, $E^1_{a, b} = 0$ else.

**Lemma**

Suppose $X = \text{S. B.}(A)$ for $A$ a c.s.a. over $k$ of prime rank $\ell$. Then the $s_0 R\text{Hom}(X, K)$ spectral sequence degenerates at $E^1$.

**Proof.** For $X = \mathbb{P}^{\ell-1}$, this follows from the projective bundle formula. Thus the differentials are killed by $\ell$.

$s_a K = EM(\mathbb{Z}(a)[2a])$. Thus Adams operations on $K$ act on the spectral sequence and $\psi_k$ acts by $k^a$ on $E^1_{a, b}$. Since $0 \leq a \leq \ell - 1$, the differentials die after inverting $(\ell - 1)!$
Proposition

$X = \text{S. B.}(A)$ for $A$ a c.s.a. over $k$ of prime rank $\ell$. Then

$$H^n_{\text{Nis}}(s_0^{\text{mot}} R\text{Hom}(X, \mathbb{Z}(a)[2a])) = 0$$

for $n \neq 0$: $s_0^{\text{mot}} R\text{Hom}(X, \mathbb{Z}(a)[2a])$ is a birational motivic sheaf.

Proof. By Quillen, $R\text{Hom}(X, K) = \bigoplus_{i=0}^{\ell} K^{A^\otimes i}$. Thus

$$s_0[R\text{Hom}(X, K)] = \bigoplus_{i=0}^{\ell} s_0[K^{A^\otimes i}] = \bigoplus_{i=0}^{\ell} \text{EM}(\mathbb{Z}_{A^\otimes i}).$$

Since $\mathbb{Z}_{A^\otimes i}$ is a sheaf, so is $s_0[R\text{Hom}(X, K)]$. 
Slices of Severi-Braurer varieties

\[
\pi^\text{Nis}_m(s_0[R\text{Hom}(X, K)]) = 0; \quad \text{for } m \neq 0.
\]

Since the \(s_0R\text{Hom}(X, K)\) spectral sequence degenerates, the same vanishing is true for the layers

\[
s_0R\text{Hom}(X, s_aK) = s_0R\text{Hom}(X, \text{EM}(\mathbb{Z}(a)[2a]))
= \text{EM}(s_0^{\text{mot}}R\text{Hom}(X, \mathbb{Z}(a)[2a])).
\]

Thus \(s_0^{\text{mot}}R\text{Hom}(X, \mathbb{Z}(a)[2a])\) is a sheaf.
Slices of Severi-Brauer varieties

Theorem
Let $A$ be a c.s.a. over $k$ of prime rank $\ell$, $X = S.B.(A)$, $d = \ell - 1 = \dim_k X$. Then

$$s^\text{mot}_n M_{\operatorname{gm}}(X) = \mathbb{Z}_{A \otimes n+1}(n)[2n] = C\mathcal{H}_n(X)(n)[2n]$$

for $n = 0, \ldots, d$.

Proof. We have already computed

$$\mathcal{H}^0_{\operatorname{Nis}}(\mathbb{H}_n^\mu(X)) = C\mathcal{H}_n(X)$$

We need to see that $\mathbb{H}_n^\mu(X)$ has no other cohomology sheaves.
Putting our tricks together shows that

$$\mathbb{H}^\mu_{d-n}(X) = s_0^{\text{mot}} R\text{Hom}(X, \mathbb{Z}(n)[2n])$$

the RHS has no cohomology beside $\mathcal{H}^0$, and that

$$\mathcal{H}^0(\mathbb{H}^\mu_n(X)) = C\mathcal{H}_n(X) = \mathbb{Z}_A \otimes n+1$$
The reduced norm map
A := a c.s.a over a field \( k \).

We have the reduced norm map

\[ \text{Nrd} : K_0(A) \rightarrow K_0(k) : \]

Take \( L/k \) a splitting field, so \( K(A_L) \cong K(L) \) by Morita equivalence. Define Nrd by

\[ K_0(A) \rightarrow K_0(A_L) = K_0(L) \leftarrow K_0(k). \]
A similar construction defines

\[ \text{Nrd} : K_1(A) \to K_1(k). \]

A more difficult construction (Suslin) defines

\[ \text{Nrd} : K_2(A) \to K_2(k). \]

There is no reasonable Nrd on \( K_n(A) \) for \( n \geq 3 \).
**Question:** What is the kernel and image of $\text{Nrd}$ on $K_n$, $n = 0, 1, 2$?

For $n = 0$ it is easy to see that $\text{Nrd}$ is injective and

$$\text{image}(\text{Nrd}) = (\text{index}(A)) \subset \mathbb{Z} = K_0(k).$$

**Theorem**

[ Wang] $\text{Nrd} : K_1(A) \rightarrow K_1(k)$ is injective if $A$ has square-free index.

In general, $\text{Nrd}$ is not injective on $K_1(A)$ (Platonov, Merkurjev).
The reduced norm on $K_0(A)$ gives the injective map of sheaves with transfer

$$Nrd : \mathbb{Z}_A \rightarrow \mathbb{Z}$$

Thus we have

$$Nrd : H^p(X, \mathbb{Z}_A(q)) \rightarrow H^p(X, \mathbb{Z}(q))$$

for all $X \in \text{Sm}/k$, $p$, $q$. 
Proposition

The reduced norm maps are compatible via the edge homomorphisms in the A-H spectral sequence for $K^A$:

The diagram

\[
\begin{array}{ccc}
H^n(k, \mathbb{Z}_A(n)) & \xrightarrow{e_n} & K_n(A) \\
\downarrow \text{Nrd} & & \downarrow \text{Nrd} \\
H^n(k, \mathbb{Z}(n)) & \xrightarrow{e_n} & K_n(k)
\end{array}
\]

commutes for $n = 0, 1, 2$. 
We recall the *Beilinson-Lichtenbaum conjectures*

**Theorem**

[Rost, Voevodsky, …] Let $\alpha$ be the change of topology morphism from $\text{DM}_{\text{eff}}^*(k)$ to $\text{DM}_{\text{ét}}^*\left(k\right)$. Then for $X \in \text{Sm}/k$

$$H^p(X, \mathbb{Z}(q)) \xrightarrow{\alpha^*} H_{\text{ét}}^p(X, \alpha^*\mathbb{Z}(q))$$

is an isomorphism for $p \leq q + 1$ and an injection for $p = q + 2$. 
Using the Beilinson-Lichtenbaum conjectures, one can view $\text{Nrd}$ as a change of topology map.

Note that $\alpha^* \mathbb{Z}_A(q) = \alpha^* \mathbb{Z}(q)$, and the diagram

$$
\begin{array}{ccc}
H^p(X, \mathbb{Z}_A(q)) & \xrightarrow{\text{Nrd}} & H^p(X, \mathbb{Z}(q)) \\
\downarrow \alpha^* & & \downarrow \alpha^* \\
H^p_{\text{ét}}(X, \alpha^* \mathbb{Z}_A(q)) & \equiv & H^p_{\text{ét}}(X, \alpha^* \mathbb{Z}(q))
\end{array}
$$

commutes.
The main result on reduced norms

**Theorem**

Let $A$ be a c.s.a of square-free index over $k$.

1. For $p < q$ the reduced norm map

$$Nrd : H^p(k, \mathbb{Z}_A(q)) \rightarrow H^p(k, \mathbb{Z}(q))$$

is an isomorphism.

2. For every $n \geq 0$, the sequence

$$0 \rightarrow H^n(k, \mathbb{Z}_A(n)) \xrightarrow{Nrd} H^n(k, \mathbb{Z}(n)) \xrightarrow{\cup[A]} H^{n+3}_{\text{ét}}(k, \alpha^*\mathbb{Z}(n + 1))$$

is exact.

$[A] \in Br(k) = H^2_{\text{ét}}(k, \mathbb{G}_m) = H^3_{\text{ét}}(k, \alpha^*\mathbb{Z}(1))$ is the class of $A$ in the Brauer group.
Corollary

Let $A$ be a c.s.a of square-free index over $k$. For $n = 0, 1, 2$:

1. The edge homomorphism

$$e_n : H^n(k, \mathbb{Z}_A(n)) \to K_n(A)$$

is an isomorphism.

2. The sequence

$$0 \to K_n(A) \xrightarrow{\text{Nrd}} K_n(k) = H^n(k, \mathbb{Z}(n)) \xrightarrow{\cup[A]} H_{\text{ét}}^{n+3}(k, \alpha^*\mathbb{Z}(n + 1))$$

is exact. In particular, $\text{Nrd}$ is injective.
The main result on reduced norms: Idea of proof

**Proof.** May assume $A$ has prime rank $\ell$ over $k$. $X := S.B.(A)$, $d = \ell - 1$. Start with the motivic Postnikov tower

$$0 \to f_d^{\text{mot}} M(X) \to \ldots \to f_1^{\text{mot}} M(X) \to M(X)$$

with layers $\mathbb{Z}_{A \otimes n+1}(n)[2n]$.

Apply $R\alpha_*\alpha^*$: This gives the tower

$$0 \to R\alpha_*\alpha^*[f_d^{\text{mot}} M(X)] \to \ldots$$

$$\to R\alpha_*\alpha^*[f_1^{\text{mot}} M(X)] \to R\alpha_*\alpha^*[M(X)] = R\alpha_*\alpha^*[M(\mathbb{P}^d)]$$

with layers $R\alpha_*\alpha^*[\mathbb{Z}(n)[2n]]$. 

Marc Levine, joint w. Bruno Kahn

The motive of a Severi-Brauer variety
The main result on reduced norms: Idea of proof

Define $\overline{f_n^{\text{mot}} M(X)}$ as “cone” of $f_n^{\text{mot}} M(X) \rightarrow R\alpha_*\alpha^*[f_n^{\text{mot}} M(X)]$. This gives the tower

$$0 \rightarrow \overline{f_d^{\text{mot}} M(X)} \rightarrow \ldots \rightarrow \overline{f_1^{\text{mot}} M(X)} \rightarrow \overline{M(X)}$$

with layers

$$\overline{\mathbb{Z}_{A \otimes n+1}(n)[2n]} := \text{“cone”}(\mathbb{Z}_{A \otimes n+1}(n)[2n] \rightarrow R\alpha_*\alpha^*[\mathbb{Z}(n)[2n]])].$$

Use Beilinson-Lichtenbaum, duality and theory of birational motivic sheaves to analyze the $E_2$ spectral sequence associated to $\text{Hom}_{\text{DM}_{\text{eff}}(k)}(\mathbb{Z}(d)[2d - 2n - 2], (-)(n + 1))$ applied to the last tower.
The main result on reduced norms: Idea of proof

Beilinson-Lichtenbaum\(\oplus\) duality \(\Rightarrow\) sequence converges to 0 in some range.

Beilinson-Lichtenbaum \(\Rightarrow \) \(E_2\) terms in some range are computable in terms of \(H^p(k, \mathbb{Z}_A(q)) \xrightarrow{\text{Nrd}} H^p(k, \mathbb{Z}(q))\).

Beilinson-Lichtenbaum\(\oplus\) induction \(\Rightarrow\) most of the \(E_2\) terms die and sequence degenerates soon (at \(E_2\) for part 1, at \(E_3\) for part 2).
The main result on reduced norms: Idea of proof

The map $H^n(k, \mathbb{Z}(n)) \to H_{\text{ét}}^{n+3}(k, \alpha^*\mathbb{Z}(n + 1))$ comes from the boundary map in the distinguished triangle

$$
\alpha^*\mathbb{Z}(d)[2d] \to \alpha^* f_{d-1}^{\text{mot}} M_{gm}(X) \to \alpha^* \mathbb{Z}_A(d - 1)[2d - 2] \xrightarrow{\partial} \alpha^* \mathbb{Z}(d)[2d + 1]
$$

But

$$
\text{Hom}_{DM_{\text{ét}}}^{\text{eff}}(k)(\alpha^*\mathbb{Z}(d - 1)[2d - 2], \alpha^*\mathbb{Z}(d)[2d + 1]) = H_{\text{ét}}^3(k, \alpha^*\mathbb{Z}(1))
$$

A calculation shows

$$
\partial \mapsto [A] \in H^2_{\text{ét}}(k, \mathbb{G}_m) = H_{\text{ét}}^3(k, \alpha^*\mathbb{Z}(1)).
$$
With a similar argument, we recover results of Merkurjev and Suslin (but without using Beilinson-Lichtenbaum)

**Theorem**

Let \( X = S.B.(A) \), \( A \) a c.s.a of prime rank over \( k \). Then

\[
H^0(X, \mathcal{K}_2) = K_2(k)
\]

and

\[
H^1(X, \mathcal{K}_2) \cong K_1(A) \subset k^\times.
\]
Thank you!