Tate motives and the fundamental group

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- **main property**: $\mathfrak{Real}(k)$ $\mathbb{Q}$-Tannaka (via fiber functors M_{Betti} or M_{DR}); Tannaka group: semi-direct product of $\mathbb{G}_m$ with a pro-unipotent $\mathbb{Q}$-group; independent of choice of fiber functor (as $H^1(\mathbb{Q}, \mathbb{G}_a \text{ or } \mathbb{G}_m) = 0$)

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- **Thm** (Deligne): *For $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ over $\mathbb{Q}$, and $a \in X(\mathbb{Q})$, there is a pro-group scheme object $\pi_1^{\rm mot}(X, a)$ in $\mathfrak{Real}(k)$, such that $(\pi_1^{\rm mot}(X, a)_{\rm Betti}, \pi_1^{\rm mot}(X, a)_{DR})$ is the Malčev completion together with its mixed Hodge structure.*

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  $D_{gm}(k)$: invert $\mathbb{Z}(1) = \mathrm{cone}\big([\mathbb{P}^1] \to \mathrm{Spec}(k)\big)[-3]$, i.e. $\mathrm{Hom}_{DM_{gm}(k)}(X, Y) = \varinjlim_{n \geq 0} \mathrm{Hom}_{DM_{gm}^{eff}(k)}(X \otimes \mathbb{Z}(n), Y \otimes \mathbb{Z}(n))$ $\rightsquigarrow$ *triangulated tensor category of geometric motives*

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- **Conclusion**: for X smooth over k, $a \in X(k)$, $\mathcal{P}_a(X)$ is an ind-object in $DM_{gm}(k)$. **But** we do not have endofunctors on $DM_{gm}(k)$ defining $H^0(\mathcal{P}_a^{\leq N})$.

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  $\mathrm{Hom}_{DMT(k)}(\mathbb{Q}(a)[s], \mathbb{Q}(b)[t]) = CH^{a-b}(k, 2(a-b)-t+s) \otimes \mathbb{Q}$

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- **Thm** (Levine): if $H^p(k, \mathbb{Q}(q)) = 0$ for $p \leq 0$ and $q > 0$ (Beilinson-Soulé conjecture) then $\exists$ t-structure on $DMT(k)$ with heart $MT(k)$, the mixed Tate category over k

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  2) for $a \in \mathfrak{m}/\mathfrak{m}^2 \setminus \{0\}$ at $\bar{a} \in S(k)$, $\exists$ a pro-group scheme object in $MT(k)$ with realization $\pi_1^{\mathrm{mot}}(X, a)$ in $\mathfrak{Real}(k)$
  3) extend $MT(k)$ to $MAT(k)$ (Artin-Tate) to allow finite motives $k' \supset k$ finite $\rightsquigarrow \pi_1^{\mathrm{mot}}(X, a)$ defined in $MAT(k)$ for X rational variety over k, $a \in X(k)$.

- **Question** (Deligne-Goncharov): localization implies $k(t)$ fulfills Beilinson-Soulé's vanishing $\rightsquigarrow$ $MT(k(t)) \subset DM_{gm}(k(t))$ defined ; what is the relation between $\pi_1^{\mathrm{mot}}(X, a), G(MT(k(t)), \mathrm{gr}^W), G(MT(k), \mathrm{gr}^W)$?

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  $X \subset \mathbb{P}^1, (\mathbb{P}^1 \setminus X) \subset \mathbb{P}^1(k) \Longrightarrow$ Levine's construction yields $MT(X) \subset DM_{gm}(X)$ $\mathbb{Q}$ linear abelian rigid tensor category with $\mathrm{gr}^W : MT(X) \to Vec_{\mathbb{Q}}$ neutral fiber functor $\rightsquigarrow$ $G(MT(X), \mathrm{gr}^W)$

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- **Thm** (E-Levine): *$a \in X(k)$ defines section of ϵ^*, thus K_a in* $\mathrm{Rep}\big(G(MT(k), \mathrm{gr}^W)\big) \cong MT(k)$. *Then $K_a \cong \pi_1^{\mathrm{mot}}(X, a)$.*

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  Kriz-May: bounded derived category $\mathcal{D}^f_{\mathcal{N}_k}$ of Adams graded dg-$\mathcal{N}_k$-modules; t-structure via $H^q(M \otimes^L \mathbb{Q}) = 0$ for $q > 0$ or $q < 0$, defining heart $\mathcal{H}^f_{\mathcal{N}_k} = \{M, H^q(M \otimes^L \mathbb{Q}) = 0 \ q \neq 0\}$

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- Following Beilinson's construction yields identification Ker_a with it.

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- Motivic π_1 defined only for X rational, so π_1^{top} of compactification $= 0$. What about pro-algebraic completion: Simpson's non-abelian Hodge theory, Toën?