Morse theory, Floer theory, and String Topology

Ralph Cohen

Stanford University

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Let M be a closed, oriented, smooth n-manifold, $T^*M \xrightarrow{p} M$ cotangent bundle. Canonical symplectic form $\omega = d\theta$, where for $x \in M$, $u \in T^*_x M$,

$$\theta(x,u) : T_{(x,u)}(T^*M) \xrightarrow{dp} T_x M \xrightarrow{u} \mathbb{R}$$

ω is nondegenerate. ω^n defines a volume form.

Question: How can one use techniques of algebraic topology to study the symplectic topology of T^*M (and other symplectic manifolds)?

Floer Theory: Let

$$H : \mathbb{R}/\mathbb{Z} \times T^*M \to \mathbb{R}$$

be a 1-periodic Hamiltonian, "quadratic near ∞". Corresponding (time dependent) Hamiltonian vector field X_H defined by

$$\omega(X_H(t;x,u),v) = -dH_{(t;x,u)}(v)$$

for all $t \in \mathbb{R}/\mathbb{Z}$, $(x,u) \in T^*M$, and $v \in T_{(x,u)}(T^*M)$.

Let

$$\mathcal{P}(H) = \{\alpha : \mathbb{R}/\mathbb{Z} \to T^*M \;:\; \frac{d\alpha}{dt} = X_H(t,\alpha(t)).\}$$

$\mathcal{P}(H)$ = set of critical points of perturbed symplectic action functional

$$\begin{aligned} \mathcal{A}_H : L(T^*M) &\to \mathbb{R} \\ \gamma &\to \int_{\mathbb{R}/\mathbb{Z}} \gamma^*(\theta) - H(t, \gamma(t))dt \end{aligned} \tag{1}$$

Assume $\mathcal{P}(H)$ is nondegenerate.

Let J be a compatible almost $\mathbb{C}$-structure on T^*M. Yields metric,

$$\langle v, w \rangle = \omega(v, Jw).$$

Let $u : \mathbb{R} \to L(T^*M)$ be a gradient trajectory of $\mathcal{A}_H$ connecting critical points α and β. View as cylinder

$$u : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \longrightarrow T^*M$$
$$t, s$$

Satisfies (perturbed) Cauchy-Riemann equation

$$\partial_s u - J(\partial_t u - X_H(t, u(t, s))) = 0.$$

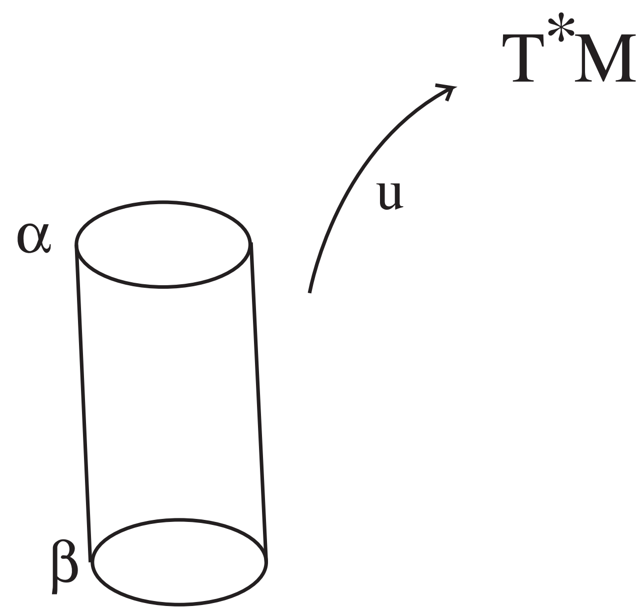


Figure 1: J-holomorphic cylinder spanning periodic orbits α and β

Theorem 1 *(Abbondandolo-Schwarz + Gluing folk theorem) With respect to a generic almost complex structure J, the moduli spaces of piecewise flow lines $\bar{\mathcal{M}}(\alpha, \beta)$ are compact, oriented manifold with corners of dimensions $\mu(\alpha) - \mu(\beta) - 1$, where $\mu(\alpha) =$ Conley-Zehnder index.* □

Floer chain complex

$$\to \cdots \to CF_p(T^*M) \xrightarrow{\partial} CF_{p-1}(T^*M) \xrightarrow{\partial} \cdots$$

generated by $\mathcal{P}_H$, where if $\mu(\alpha) = p$, then

$$\partial[\alpha] = \sum_{\mu(\beta)=p-1} \#\mathcal{M}(\alpha,\beta)[\beta]$$

Question. Is there a $C.W$-spectrum $Z(T^*M)$ with one cell of dimension k for each $\alpha \in \mathcal{P}_H$ with $\mu(\alpha) = k$, and the cellular chain complex = Floer complex?

More general question: When is a given chain complex

$$\to \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots \to C_0$$

is isomorphic to the cellular chain complex of a $C.W$-complex or spectrum?

Assume each C_i is a finitely generated free abelian group with basis $\mathcal{B}_i$

First, assume the complex is finite: $C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_0$.

Let X be a filtered space

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n = X,$$

where each $X_{i-1} \hookrightarrow X_i$ is a cofibration with cofiber, $K_i = X_i \cup c(X_{i-1}) \simeq X_i/X_{i-1}$. When this is skeletal filtration, have

$$K_i \simeq \bigvee_{\alpha \in \mathcal{B}_i} S_\alpha^{\ell_i}$$

so $\tilde{H}_*(K_i) = C_i$.

Using homotopy theory, one one can "rebuild" the homotopy type of the n-fold suspension, $\Sigma^n X$ as the union of iterated cones and suspensions of the K_i's.

$$\Sigma^n X \simeq \Sigma^n K_0 \cup c(\Sigma^{n-1} K_1) \cup \cdots \cup c^i(\Sigma^{n-i} K_i) \cup \cdots \cup c^n K_n.$$

This decomposition is created using iterated Puppe extensions of the cofibration sequences, $X_{i-1} \xrightarrow{u_{i-1}} X_i \xrightarrow{p_i} K_i$.

Have attaching maps

$$\phi_{i,j} : c^{i-j-1}\Sigma^{n-i}K_i \to \Sigma^{n-j}K_j$$

where for $j = i - 1$ and $K_q = \bigvee_{\alpha\in\mathcal{B}_q} S^q_\alpha$, have

$$\begin{array}{ccc} H_n(\Sigma^{n-i}K_i) & \xrightarrow{(\phi_{i,i-1})_*} & H_n(\Sigma^{n-i+1}K_{i-1}) \\ \cong\big\downarrow & & \big\downarrow\cong \\ C_i & \xrightarrow[\partial_i]{} & C_{i-1} \end{array}$$

Describe functorially. Notice that

$$c^p \Sigma^j X = \Big((\mathbb{R}_+)^p \times \mathbb{R}^j \Big) \cup \infty \wedge X.$$

For any two integers $n > m$, define the space

$$\mathbb{R}_+^{n,m} = \{ t_i,\ i \in \mathbb{Z}, \quad \text{where each } t_i \text{ is a nonnegative real number,} \\ \text{and} \quad t_i = 0, \text{ unless}\, m < i < n. \}$$

$\mathbb{R}_+^{n,m} \cong (\mathbb{R}_+)^{n-m-1}$. Have inclusions,

$$\iota : \mathbb{R}_+^{n,m} \times \mathbb{R}_+^{m,p} \hookrightarrow \mathbb{R}_+^{n,p}.$$

Let

$$J(n,m) = \begin{cases} \emptyset \quad \text{if} \quad n < m \\ * \quad \text{if} \quad n = m \\ S^0 \quad \text{if} \quad n = m+1 \\ \mathbb{R}_+^{n,m} \cup \infty \quad \text{if} \quad n > m+1, \end{cases}$$

Define a category $\mathcal{J}$ to have objects $= \mathbb{Z}$.

$Mor(n,m) = J(n,m)$.

Theorem 2 *(C., Jones, Segal) A finite chain complex of free abelian groups C_* can be realized by a finite C.W-spectrum, iff the assignment*

$$m \longrightarrow Z(m) = \bigvee_{b \in \mathcal{B}_m} S_b^L$$

extends to a functor $Z : \mathcal{J} \to Sp_*$ = finite spectra. □

Note on morphisms, have $J(m,n) \wedge Z(m) \longrightarrow Z(n)$ given by maps

$$\phi_{m,n} : \bigvee_{\alpha \in \mathcal{B}_m} c^{m-n-1} S_\alpha^L \longrightarrow \bigvee_{\beta \in \mathcal{B}_n} S_\beta^L$$

defining attaching maps of spectrum.

More geometrically: Assume attaching map

$$\phi_{\alpha,\beta} : c^{n-m-1}S^L_\alpha \to S^L_\beta$$
$$((\mathbb{R}_+)^{n-m-1} \times \mathbb{R}^L) \cup \infty \to \mathbb{R}^L \cup \infty$$

is smooth. Pull back regular point, get a *framed manifold with corners*, embedded in $(\mathbb{R}_+)^{n-m-1} \times \mathbb{R}^L$,

$$M_{\alpha,\beta} \xrightarrow{\subset} M_{\alpha,\beta} \times \mathbb{R}^L \xrightarrow[\textit{open set}]{\subset} (\mathbb{R}_+)^{n-m-1} \times \mathbb{R}^L$$

"Framed embedding" When $m = n+1$,
$\# M_{\alpha,\beta} = \textit{degree}\, \phi_{\alpha,\beta} : S^L_\alpha \to S^L_\beta$ determines (and is determined by) the boundary homomorphism,

$$\partial : C_{n+1} \to C_n.$$

Recall: When $f : N \to \mathbb{R}$ is a Morse function on a closed Riemannian manifold (satisfying Palais-Smale transversality), there is a Morse $C.W$ complex $Z_f(N) \simeq N$ with one k-cell for each critical point of index k.

Theorem 3 *(C., Jones, Segal) The spaces of "piecewise flow lines" connecting two critical points, $\bar{\mathcal{M}}(a, b)$ is a compact, framed manifold with corners whose cobordism type determines the (stable) attaching maps of $Z_f(N)$.* □

Theorem 4 *Given a Hamiltonian* $H : \mathbb{R}/\mathbb{Z} \times T^*M \to \mathbb{R}$ *satisfying above properties, then the moduli spaces of* J*-holomorphic cylinders* $\bar{\mathcal{M}}(\alpha, \beta)$ *have framings that are compatible with gluing. Therefore there is a "Floer homotopy type" (a C.W-spectrum)* $Z(T^*M)$ *with one cell of dimension* k *for each periodic orbit* $\alpha \in \mathcal{P}_H$ *with* $\mu(\alpha) = k$.

Furthermore there is a homotopy equivalence,

$$Z(T^*M) \simeq \Sigma^\infty(LM_+).$$

This theorem generalizes a theorem of C. Viterbo, stating that

$$HF_*(T^*M) \cong H_*(LM).$$

String topology: (Chas-Sullivan) Intersection theory in LM defines considerable structure on $H_*(LM)$.

Theorem 5 *(Chas-Sullivan, C.-Jones, Godin, Tradler-Zeinalian, Kaufmann) Let $\mathcal{M}_{g,p+q}$ be the moduli space of bordered Riemann surfaces of genus g, and p- incoming boundary components, and q-outgoing. Then there are operations*

$$\mu_{g,p+q} : H_*(\mathcal{M}_{g,p+q}) \otimes H_*(LM)^{\otimes p} \longrightarrow H_*(LM)^{\otimes q}$$

respecting gluing of surfaces. Works with any generalized h_* admitting an orientation of M. Need $q > 0$. □

"Homological Conformal Field Theory"
(positive boundary)

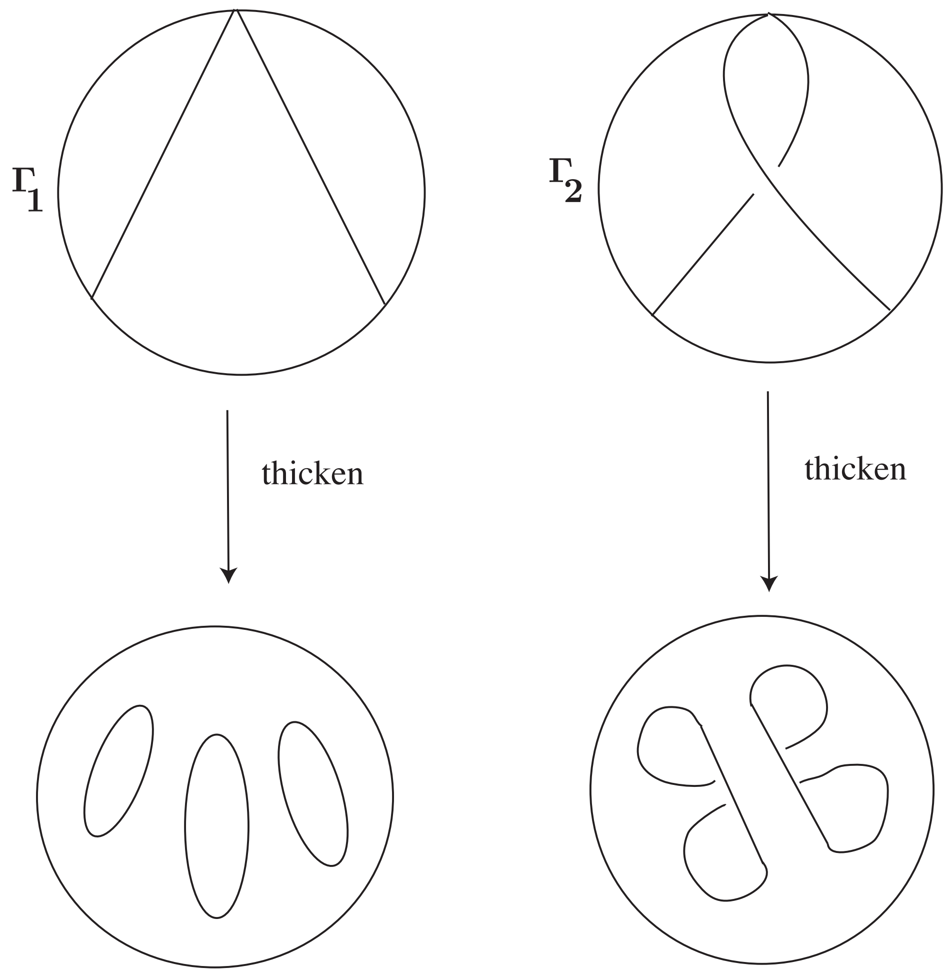
The constructions of these operations use “fat” (ribbon) graphs (Thurston, Harer, Penner, Strebel, Kontsevich)

A *fat graph* is a finite, combinatorial graph (one dimensional CW complex - no “leaves”) such that

1. Vertices are at least trivalent

2. Each vertex has a cyclic order of the (half) edges.

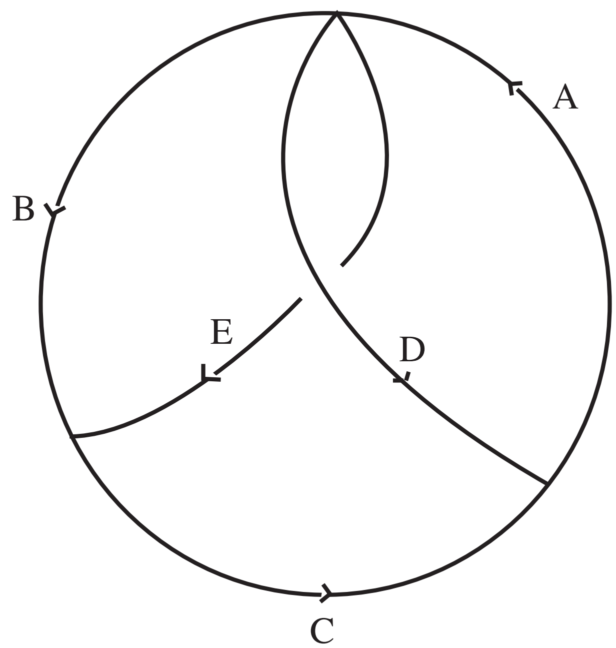
ORDER IS IMPORTANT!



Study the combinatorics:

Let E_Γ = set of edges of Γ, $\tilde{E}_\Gamma$ = set of oriented edges. So each edge of Γ appears twice in $\tilde{E}_\Gamma$: $e, \bar{e}$

Have a partition of $\tilde{E}_\Gamma$:



$$(A, B, C), \quad (\bar{A}, \bar{D}, E, \bar{B}, D, \bar{C}, \bar{E})$$

These are called *“boundary cycles”*.

Given a fat graph Γ of type (g, n), designate p boundary cycles as "incoming" and $q = n - p$ boundary cycles as "outgoing".

An important class of marked fat graph was defined by V. Godin.

Definition 1 An "admissible" marked fat graph is one with the property that for every oriented edge E that is part of an incoming boundary cycle, its conjugate $\bar{E}$ (i.e the same edge with the opposite orientation) is part of an *outgoing* boundary cycle. □

Theorem 6 *(V. Godin) The space of admissible, marked, metric fat graphs of topological type $(g, p+q)$, $\mathcal{G}_{g,p+q}$ is homotopy equivalent to the moduli space of bordered surfaces, $\mathcal{M}_{g,p+q}$.* □

Let $\mathcal{G}_{p,g+q}(M) = \{(\Gamma, f) : \Gamma \in \mathcal{G}_{g,p+q},\ f : \Gamma \to M\}$.

$$\mathcal{G}_{p,g+q}(M) \simeq \mathcal{M}^{top}_{g,p+q}(M) = \{(\Sigma, f) : \Sigma \in \mathcal{M}_{g,p+q},\ f : \Sigma \to M\}$$
$$\simeq Map(\Sigma_{g,p+q}, M)_{hDiff^+(\Sigma,\partial)}$$

$$\mathcal{G}_{g,p+q} \times (LM)^p \xleftarrow{\rho_{in}} \mathcal{G}_{g,p+q}(M) \xrightarrow{\rho_{out}} (LM)^q$$

Idea: With these types of graphs, there are sufficient transversality properties to define umkehr map,

$$(\rho_{in})_! : H_*(\mathcal{G}_{g,p+q} \times (LM)^p) \to H_{*+k(g)}(\mathcal{G}_{g,p+q}(M)$$

The string topology operation is defined to be the composition,

$$\rho_{out} \circ (\rho_{in})_! : H_*(\mathcal{G}_{g,p+q} \times (LM)^p) \to H_{*+k(g)}(LM^p).$$

Here $k(g) = 2 - 2g - (p+q)n$.

Question: How do these operations translate into operations in $HF_*(T^*M)$?

(Combination of work of Abbondandolo-Schwarz, and Cohen-Schwarz)

Given a graph $\Gamma \in \mathcal{G}_{p+q}$, define the surface Σ_Γ

$$\Sigma_\Gamma = \left(\coprod_p S^1 \times (-\infty, 0] \right) \sqcup \left(\coprod_q S^1 \times [0, +\infty) \right) \bigcup \Gamma / \sim \qquad (2)$$

where $(t, 0) \in S^1 \times (-\infty, 0] \sim \alpha^-(t) \in \Gamma$, and
$(t, 0) \in S^1 \times [0, +\infty) \sim \alpha^+(t) \in \Gamma$

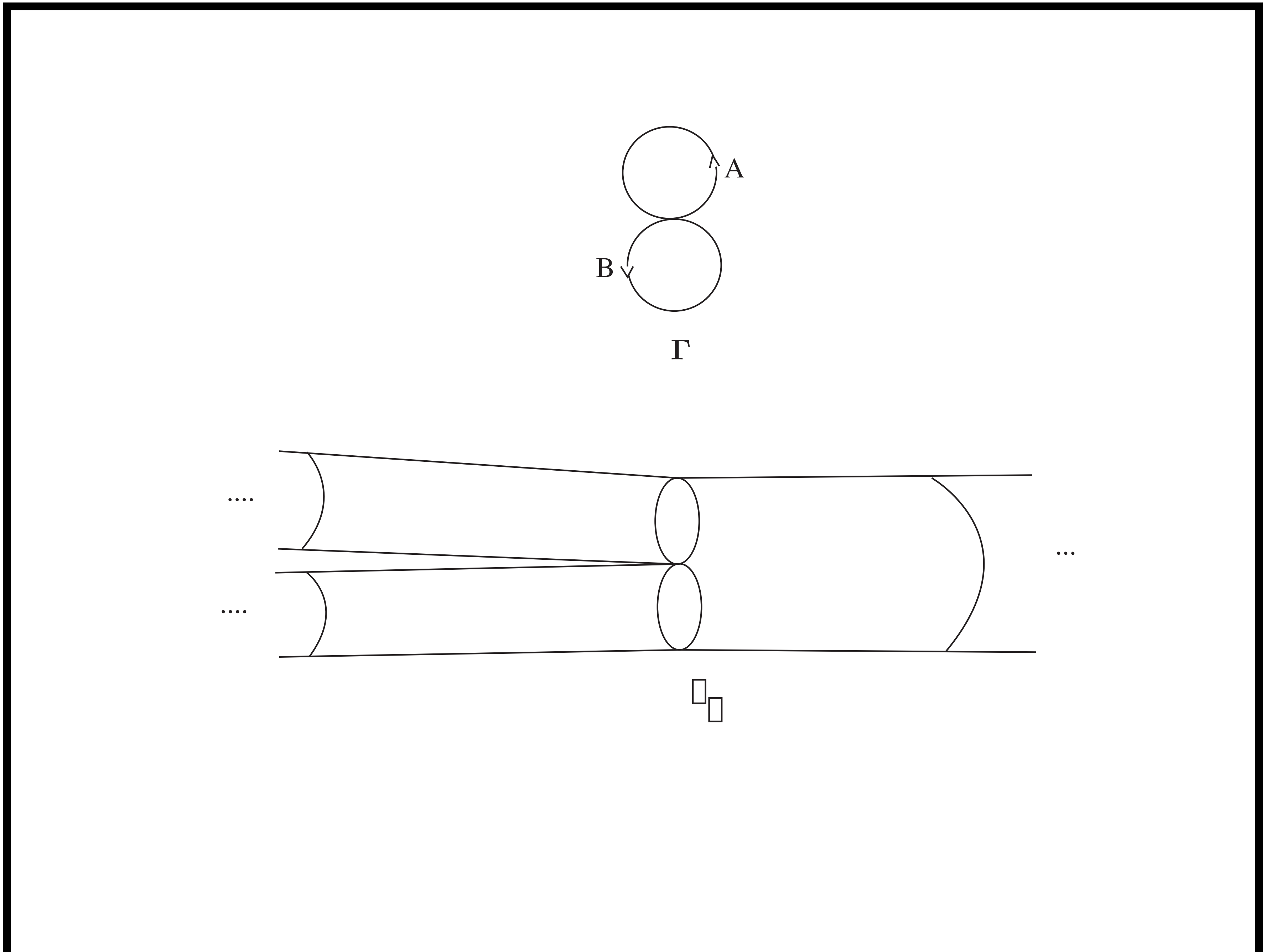


Figure 2: Σ_Γ

For a fixed Γ, define $Hol(\Sigma_\Gamma, T^*M) = \{\phi : \Sigma_\Gamma \to T^*M$ is J-holomorphic on the cylinders$\}$.

For $\vec{\alpha} = (\alpha_1, \cdots \alpha_p, \alpha_{p+1}, \cdots, \alpha_{p+q})$ where each $\alpha_i \in \mathcal{P}_H$, define

$$Hol_{\vec{\alpha}}(\Sigma_\Gamma, T^*M) = \{\phi \in Hol(\Sigma_\Gamma, T^*M) \text{ that converges to the periodic orbit } \alpha_i \text{ on the } i^{th}\text{-cylinder}\}$$

This is a manifold of dimension $\sum_{i=1}^{p} Ind(\alpha_i) - \sum_{j=1}^{q} Ind(\alpha_{p+j}) - \chi(\Sigma_\Gamma)n$, with a well understood compactification.

Define an operation

$$\mu_\Gamma : HF_*(T^*M)^{\otimes p} \to HF_*(T^*M)^{\otimes q}$$
$$[\alpha_1] \otimes \cdots \otimes [\alpha_p] \to \sum \# Hol_{\vec{\alpha}}(\Sigma_\Gamma, T^*M)\, [\alpha_{p+1}] \otimes \cdots \otimes [\alpha_{p+q}]$$

Theorem 7 *With respect to the isomorphism,* $HF_*(T^*M) \cong H_*(LM)$, *this is the string topology operation.* □