Homogenization of diffusions on the lattice $\mathbf{Z}^d$ with periodic drift coefficients, applying a logarithmic Sobolev inequality or a weak Poincaré inequality

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1 Introduction

In this paper we treat limit theorems for diffusions on the lattice $\mathbf{Z}^d$ of the form of those constituting the solution of the homogenization problem of diffusions. For finite dimensional diffusion processes, various models of homogenization (generalized in several directions) have been studied in detail (cf. eg. [F2], [FNT], [FunU], [O], [PapV], [Par] and references therein). On the other hand, for corresponding problems of infinite dimensional diffusions only few results are known (cf. [FunU], [ABRY1,2,3]). In this paper we consider a homogenization problem of infinite dimensional diffusion processes indexed by $\mathbf{Z}^d$ having periodic drift coefficients with the period 2π (cf. (10)), by applying an L^2 type ergodic theorem for the corresponding quotient processes taking values in $[0, 2\pi)^{\mathbf{Z}^d}$ (cf. Prop. 1). The ergodic theorem which is based on a (weak) Poincaré inequality.

In [ABRY3] the same problem has been discussed by applying the uniform ergodic theorem for the corresponding quotient process, that is available by assuming that the Markov semi-group of the quotient process of the original process satisfies a logarithmic Sobolev inequality. In the same paper it has also been shown that a homogenization property of the processes starting from an almost every arbitrary point in the state space with respect to an invariant measure of the quotient process holds (cf. also [ABRY1], [ABRY2]). In this occasion, the main purpose of the present paper is the comparison between the results derived under the assumption of logarithmic Sobolev inequality and the corresponding results proven by assuming L^2 ergodic theorem based on (weak) Poincaré inequality, which is strictly weaker than the one for logarithmic Sobolev inequality (cf. [G], [AKR]). This paper is a series of works on the considerations of several types of homogenization models for infinite dimensional diffusion processes.

For an adequate understanding of crucial differences between homogenization problems in finite and infinite dimensional situations, we first briefly review a simple case of the homogenization problem for finite dimensional diffusions.

On some complete probability space, suppose that we are given a one dimensional standard Brownian motion process $\{B_t\}_{t\in\mathbf{R}_+}$ and consider the stochastic differential equation for each initial state $x \in \mathbf{R}$ and each scaling parameter $\epsilon > 0$ given by

$$X^\epsilon(t,x) = x + \frac{1}{\epsilon}\int_0^t b(\frac{X^\epsilon(s,x)}{\epsilon})ds + \sqrt{2}\int_0^t a(\frac{X^\epsilon(s,x)}{\epsilon})dB_s, \qquad t \in \mathbf{R}_+, \tag{1}$$

where $a \in C^\infty(\mathbf{R} \to \mathbf{R})$ is a periodic function with period 2π which satisfies

$$\lambda \le a(x) \le \lambda^{-1}, \qquad \forall x \in \mathbf{R},$$

for some constant $\lambda > 0$ and $b(x) \equiv \frac{d}{dx}a^2(x)$.

Let $p_t^\epsilon(x,y)$ be the transition density function corresponding to the diffusion process defined through (1). Then by Nash's inequality (cf. eg. [S]) we have that there exist constants $c_1, c_2 > 0$ such that

$$p_t^\epsilon(x,y) \le c_1 t^{-\frac{1}{2}} \exp\{-c_2|x-y|^2 t^{-1}\}, \qquad \forall t > 0, \quad \forall \epsilon \in (0,1]. \tag{2}$$

Also, there exists a periodic function $\chi \in C^2(\mathbf{R})$ with period 2π such that

$$a^2(x)\chi''(x) + b(x)\chi'(x) = b(x), \quad x \in \mathbf{R}, \qquad \chi(0) = 0. \tag{3}$$

Then by Itô's formula and using (3) we see that

$$\begin{aligned} X^\epsilon(t,x) &= x - \epsilon\chi(\frac{x}{\epsilon}) + \epsilon\chi(\frac{X^\epsilon(t,x)}{\epsilon}) \\ &\quad + \sqrt{2}\int_0^t (1-\chi'(\frac{X^\epsilon(s,x)}{\epsilon}))a(\frac{X^\epsilon(s,x)}{\epsilon})dB_s, \quad t \in \mathbf{R}_+. \end{aligned} \tag{4}$$

The (probabilistic) homogenization problem consists of proving weak convergence of the process $\{\{X^\epsilon(t,x)\}_{t\in\mathbf{R}_+}\}_{\epsilon>0}$. In this simple situation and also in various generalized models of finite dimensional diffusions, it is shown that

$$\lim_{\epsilon \downarrow 0} E^{P_x^\epsilon}[\varphi(\cdot)] = E^{P_x}[\varphi(\cdot)], \tag{5}$$

$$\forall x \in \mathbf{R}, \qquad \forall \varphi \in C_b(C(\mathbf{R}_+ \to \mathbf{R}) \to \mathbf{R}),$$

where P_x^ϵ is the probability law of the process $\{X^\epsilon(t,x)\}_{t\in\mathbf{R}_+}$, as a $C(\mathbf{R}_+ \to \mathbf{R})$ valued random variable, P_x is the probability law of the continuous Gaussian process starting at $x \in \mathbf{R}$ with constant diffusion coefficient given by

$$\sigma \equiv \left\{2 \int_0^{2\pi} \{(1-\chi'(y))a(y)\}^2 dy\right\}^{\frac{1}{2}}, \tag{6}$$

$E^{P_x^\epsilon}[\cdot]$, $E^{P_x}[\cdot]$ are respectively the expectations with respect to the corresponding probability measures, and $C_b(\cdot)$ is the space of bounded continuous functions.

Here, our problem is a homogenization problem of a diffusion

$$\{\mathbf{X}_\mathbf{k}^\epsilon(t,\mathbf{x})\}_{\mathbf{k}\in\mathbf{Z}^d}$$

with index set $\mathbf{Z}^d$ involving the periodic drift coefficients of perion 2π defined by (10) in the next section. Since the constants in Nash's inequality depend on the dimensions, for our infinite dimensional diffusions we can not use the uniform bound (2) for the Markovian transition density functions which are based on Nash's inequality. By Lemma 3 below for our infinite dimensional diffusions we can define a family of functions $\chi_\mathbf{k}$, ($\mathbf{k} \in \mathbf{Z}^d$) (cf. (19)) that is an infinite dimensional version of χ defined by (3). But except for some trivial cases, we can not expect the regularity $\chi_\mathbf{k} \in C^2$. Thus the same strategy developed for the consideration of finite dimensional problems can not be applied directly in infinite dimension. These are the main difficulties which appear in the alter situations.

But in Theorem 9 and Theorem 12, by using the L^2 ergodic theorem (17), which is a consequence of a weak Poincaré inequality for the corresponding quotient process of $\{\mathbf{X}_\mathbf{k}^1(t,\mathbf{x})\}_{\mathbf{k}\in\mathbf{Z}^d}$, instead we can show that in the infinite dimensional situation a homogenization holds, roughly speaking, in the following weaker sense than (5): Let a Polish space W be a subspace of $C(\mathbf{R}_+ \to \mathbf{R}^{\mathbf{Z}^d})$ equipped with a topology which is sufficiently stronger than the product topology on $C(\mathbf{R}_+ \to \mathbf{R}^{\mathbf{Z}^d})$ (cf. (13)), $C_b(W)$ be the space of bounded continuous functions on W, P_0 be the probability law of an infinite dimensional diffusion $\{\mathbb{Y}_t\}_{t\in\mathbf{R}_+}$ with a constant covariance matrix defined through $\chi_\mathbf{k}$, ($\mathbf{k} \in \mathbf{Z}^d$) starting from the initial state 0 (cf. (22) and Definition 5), also let $\tilde{P}_\mathbf{x}^\epsilon$ be the probability law of the process

$$\{\mathbf{X}_\mathbf{k}^\epsilon(t,\mathbf{x}) - \epsilon\chi_\mathbf{k}(\frac{\mathbf{X}_\mathbf{k}^\epsilon(t,\mathbf{x})}{\epsilon}) + \epsilon\chi_\mathbf{k}(\frac{\mathbf{x}}{\epsilon})\}_{\mathbf{k}\in\mathbf{Z}^d},$$

then it holds that

$$\lim_{\epsilon \downarrow 0} E^{\tilde{P}_{\nu_\epsilon}^\epsilon}[\varphi(\cdot)] = E^{P_0}[\varphi(\cdot)], \quad \forall \varphi \in C_b(W \to \mathbf{R}), \tag{7}$$

where for each $\epsilon \in [0,1)$, the probability measure $\tilde{P}^{\epsilon}_{\nu_\epsilon}$ on $(W, \mathcal{B}(W))$ is defined by

$$\tilde{P}^{\epsilon}_{\nu_\epsilon}(B) \equiv \int_{[0,2\pi)^{\mathbf{Z}^d}} \tilde{P}^{\epsilon}_{\epsilon\mathbf{y}}(B)\nu(d\mathbf{y}), \quad \forall B \in \mathcal{B}(W), \tag{8}$$

for a probability measure ν on $([0,2\pi)^{\mathbf{Z}^d}, \mathcal{B}([0,2\pi)^{\mathbf{Z}^d}))$ such that

$$\|\frac{d\nu}{d\mu}\|_{L^\infty([0,2\pi)^{\mathbf{Z}^d})} < \infty, \tag{9}$$

and μ is the unique invariant measure of the quotient diffusion process on the infinite dimensional torus (identified with $[0,2\pi)^{\mathbf{Z}^d}$) of the original diffusion $\{\mathbf{X}^1_{\mathbf{k}}(t,\mathbf{x})\}_{\mathbf{k}\in\mathbf{Z}^d}$ (cf. Proposition 1).

2 Fundamental notations

Let $\mathbf{N}$ and $\mathbf{Z}$ be the set of natural numbers and integers respectively. For $d \in \mathbf{N}$ let $\mathbf{Z}^d$ be the d-dimensional lattice. We consider the problem for the diffusions taking values in $\mathbf{R}^{\mathbf{Z}^d}$. We use the following notions and notations: By $\mathbf{k}$ we denote $\mathbf{k} = (k^1, \ldots, k^d) \in \mathbf{Z}^d$. For a subset $\Lambda \subseteq \mathbf{Z}^d$, we define $|\Lambda| \equiv \mathrm{card}\,\Lambda$. For $\mathbf{k} \in \mathbf{Z}^d$ and $\Lambda \subseteq \mathbf{Z}^d$ let

$$\Lambda + \mathbf{k} \equiv \{\mathbf{l} + \mathbf{k} \,|\, \mathbf{l} \in \Lambda\}.$$

For any non-empty $\Lambda \subseteq \mathbf{Z}^d$, we assume that $\mathbf{R}^\Lambda$ is the topological space equipped with the direct product topology. For each non-empty $\Lambda \subseteq \mathbf{Z}^d$, by $\mathbf{x}_\Lambda$ we denote the image of the projection onto $\mathbf{R}^\Lambda$:

$$\mathbf{R}^{\mathbf{Z}^d} \ni \mathbf{x} \longmapsto \mathbf{x}_\Lambda \in \mathbf{R}^\Lambda.$$

For each $p \in \mathbf{N} \cup \{0\} \cup \{\infty\}$ we define the set of p-times continuously differentiable functions with support Λ: $C^p_\Lambda(\mathbf{R}^{\mathbf{Z}^d}) \equiv \{\varphi(\mathbf{x}_\Lambda) \,|\, \varphi \in C^p(\mathbf{R}^\Lambda)\}$, where $C^p(\mathbf{R}^\Lambda)$ is the set of real valued p-times continuously differentiable functions on $\mathbf{R}^\Lambda$. For $p = 0$, we simply denote $C^0_\Lambda(\mathbf{R}^{\mathbf{Z}^d})$ by $C_\Lambda(\mathbf{R}^{\mathbf{Z}^d})$. Also we set

$$C^p_0(\mathbf{R}^{\mathbf{Z}^d}) \equiv \{\varphi \in C^p_\Lambda(\mathbf{R}^{\mathbf{Z}^d}) \,|\, |\Lambda| < \infty\}.$$

$\mathcal{B}(\mathbf{R}^{\mathbf{Z}^d})$ is the Borel σ-field of $\mathbf{R}^{\mathbf{Z}^d}$ and $\mathcal{B}_\Lambda(\mathbf{R}^{\mathbf{Z}^d})$ is the sub σ-field of $\mathcal{B}(\mathbf{R}^{\mathbf{Z}^d})$ that is generated by the family $C_\Lambda(\mathbf{R}^{\mathbf{Z}^d})$. For each $\mathbf{k} \in \mathbf{Z}^d$, let $\vartheta^{\mathbf{k}}$ be the shift operator on $\mathbf{R}^{\mathbf{Z}^d}$ such that

$$(\vartheta^{\mathbf{k}}\mathbf{x})_{\{\mathbf{j}\}} \equiv \mathbf{x}_{\{\mathbf{k}+\mathbf{j}\}}, \, \mathbf{x} \in \mathbf{R}^{\mathbf{Z}^d}, \, \mathbf{j} \in \mathbf{Z}^d,$$

where $\mathbf{x}_{\{\mathbf{k}+\mathbf{j}\}}$ is the $\mathbf{k}+\mathbf{j}$-th component of the vector $\mathbf{x}$.

We shall define the infinite dimensional diffusions we are interested in through a stochastic differential equation (SDE). On a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ with an increasing family of sub σ-field $\{\mathcal{F}_t\}_{t\in\mathbf{R}_+}$ we are given a family of independent 1-dimensional $\mathcal{F}_t$-standard Brownian motion processes $\{B_{\mathbf{k}}(t)\}_{t\in\mathbf{R}_+}$, $\mathbf{k}\in\mathbf{Z}^d$. For each $\epsilon\in(0,1]$ and each $\mathbf{x}=\{x_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{Z}^d}\in\mathbf{R}^{\mathbf{Z}^d}$, consider the following system of SDE's :

$$X^{\epsilon}_{\mathbf{k}}(t,\mathbf{x}) = x_{\mathbf{k}} + \sqrt{2}B_{\mathbf{k}}(t) + \frac{1}{\epsilon}\int_0^t b_{\mathbf{k}}(\frac{\mathbb{X}^{\epsilon}(s,\mathbf{x})}{\epsilon})ds, \quad t\in\mathbf{R}_+,\ \mathbf{k}\in\mathbf{Z}^d, \tag{10}$$

where we set

$$\mathbb{X}^{\epsilon}(s,\mathbf{x}) \equiv \{X^{\epsilon}_{\mathbf{k}}(s,\mathbf{x})\}_{\mathbf{k}\in\mathbf{Z}^d}, \quad \text{and define} \quad b_{\mathbf{k}}(\mathbf{x}) \equiv \sum_{\Lambda\ni\mathbf{k}}(-\frac{\partial}{\partial x_{\mathbf{k}}}J_{\Lambda}(\mathbf{x})),$$

for a given family of potentials $\mathcal{J}\equiv\{J_{\Lambda}\,|\,\Lambda\subset\mathbf{Z}^d, |\Lambda|<\infty\}$ such that
J-1) (Periodicity) for each $\Lambda\subset\mathbf{Z}^d$ such that $|\Lambda|<\infty$,

$$J_{\Lambda}\in C^{\infty}_{\Lambda}(\mathbf{R}^{\mathbf{Z}^d}),$$

and it is a periodic function with respect to each variable with the period 2π;
J-2) (Shift invariance)

$$J_{\Lambda+\mathbf{k}} = J_{\Lambda}\circ\vartheta^{\mathbf{k}}, \qquad \forall\mathbf{k}\in\mathbf{Z}^d;$$

J-3) (Finite range) there exists an $L<\infty$ and $J_{\Lambda}=0$ holds for any Λ such that $\Lambda\ni 0$ and $\Lambda\not\subseteq[-L,+L]^d$.

By Lemma 1.2 of [HS] we have the following: Under the assumption J-1), J-2) and J-3), for each $\epsilon>0$, SDE (10) has a strong unique solution. Also, for any $T<\infty$ and $\epsilon>0$ there exists a constant $A^{\epsilon}_T<\infty$ and for any $\mathbf{x}=\{x_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{Z}^d}$, $\mathbf{x}'=\{x'_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{Z}^d}$ one has that

$$E[\sum_{\mathbf{k}\in\mathbf{Z}^d}\frac{1}{2^{|\mathbf{k}|}}\sup_{0\le t\le T}|X^{\epsilon}_{\mathbf{k}}(t,\mathbf{x})-x_{\mathbf{k}}|^2]\le A^{\epsilon}_T, \tag{11}$$

$$E[\sum_{\mathbf{k}\in\mathbf{Z}^d}\frac{1}{2^{|\mathbf{k}|}}\sup_{0\le t\le T}|X^{\epsilon}_{\mathbf{k}}(t,\mathbf{x})-X^{\epsilon}_{\mathbf{k}}(t,\mathbf{x}')|^2]\le A^{\epsilon}_T(\sum_{\mathbf{k}\in\mathbf{Z}^d}\frac{1}{2^{|\mathbf{k}|}}|x_{\mathbf{k}}-x'_{\mathbf{k}}|^2). \tag{12}$$

By (11) and (12) we can define a metric ρ on a linear subspace W of $C(\mathbf{R}_+\to\mathbf{R}^{\mathbf{Z}^d})$, on which the trajectories of the diffusions $\mathbb{X}^{\epsilon}(\cdot,\mathbf{x})$ exist if their initial states satisfy $\mathbf{x}\in\mathcal{H}$. Namely, let

$$\mathcal{H}\equiv\Big\{\mathbf{x}=\{x_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{Z}^d}\in\mathbf{R}^{\mathbf{Z}^d}\mid\sum_{\mathbf{k}\in\mathbf{Z}^d}\frac{1}{2^{|\mathbf{k}|}}x^2_{\mathbf{k}}<\infty\Big\},$$

denoting $\mathbf{x}(\cdot)\equiv\{x_{\mathbf{k}}(\cdot)\}_{\mathbf{k}\in\mathbf{Z}^d}$ and define

$$W\equiv\Big\{\mathbf{x}(\cdot)\in C(\mathbf{R}_+\to\mathbf{R}^{\mathbf{Z}^d})\Big|\sum_{\mathbf{k}\in\mathbf{Z}^d}\frac{1}{2^{|\mathbf{k}|}}\sup_{0\le t\le T}|x_{\mathbf{k}}(t)|^2<\infty,\,\forall T<\infty\Big\}.$$

We define a metric ρ on W, and denote the Polish space equipped with this metric by the same symbol W:

$$\rho(\mathbf{x}(\cdot),\mathbf{x}'(\cdot)) \equiv \sum_{n\in\mathbf{N}} \frac{1}{2^n}\Big\{\{\sum_{\mathbf{k}\in\mathbf{Z}^d} \frac{1}{2^{|\mathbf{k}|}} \sup_{0\le t\le n} |x_{\mathbf{k}}(t)-x'_{\mathbf{k}}(t)|^2\}^{\frac{1}{2}} \wedge 1\Big\}, \tag{13}$$

$$\text{for} \qquad \mathbf{x}(\cdot)\equiv\{x_{\mathbf{k}}(\cdot)\}_{\mathbf{k}\in\mathbf{Z}^d},\, \mathbf{x}'(\cdot)\equiv\{x'_{\mathbf{k}}(\cdot)\}_{\mathbf{k}\in\mathbf{Z}^d}\in W.$$

The metric ρ gives a stronger topology than the product topology and keeps the Borel structure unchanged. We note that the metric ρ is also stronger than the following metric (cf. [AKR])

$$\sum_{n\in\mathbf{N}} \frac{1}{2^n}\Big\{\sup_{0\le t\le n}\{\sum_{\mathbf{k}\in\mathbf{Z}^d} \frac{1}{2^{|\mathbf{k}|}}|x_{\mathbf{k}}(t)-x'_{\mathbf{k}}(t)|^2\}^{\frac{1}{2}} \bigwedge 1\Big\}.$$

Let $\mathcal{B}(W)$ be the Borel σ-field of W and $\mathcal{B}_t(W)$, $t\in\mathbf{R}_+$, be the sub σ-field of $\mathcal{B}(W)$ generated by the cylinder sets of $\big(C([0,t]\to\mathbf{R})\big)^{\mathbf{Z}^d}$.

For each $t\ge 0$, let $\boldsymbol{\xi}_t$ be the measurable map given by

$$\boldsymbol{\xi}_t : W\ni\mathbf{x}(\cdot)\longmapsto\mathbf{x}(t)\in\mathbf{R}^{\mathbf{Z}^d},$$

then $\mathcal{B}_t(W)$ is the σ-field generated by $\boldsymbol{\xi}_s$, $s\in[0,t]$.
For each $\mathbf{x}\in\mathcal{H}$ and $\epsilon>0$, let $P^{\epsilon}_{\mathbf{x}}$ be the probability measure on $(W,\mathcal{B}(W))$ which is the probability law of the process $\{\mathbb{X}^{\epsilon}(t,\mathbf{x})\}_{t\in\mathbf{R}_+}$:

$$P(\{\omega\,|\,\mathbb{X}^{\epsilon}(\cdot,\mathbf{x})\in B\}) = P^{\epsilon}_{\mathbf{x}}(\boldsymbol{\xi}_{\cdot}\in B), \qquad \forall B\in\mathcal{B}(W).$$

Let $T=\{y\in\mathbf{R}^2 \,:\, |y|=1\}$ be the unit circle equipped with the natural Riemannian metric. Let $T^{\mathbf{Z}^d}$ be the product space of T endowed with the direct product topology, so that $T^{\mathbf{Z}^d}$ is a Polish space. Let $(W_T,\mathcal{B}(W_T);\mathcal{B}_t(W_T))$ be the measurable space of the Polish space $W_T\equiv C(\mathbf{R}_+\to T^{\mathbf{Z}^d})$, such that $\mathcal{B}(W_T)$ is the Borel σ-field of W_T and $\mathcal{B}_t(W_T)$, $(t\in\mathbf{R}_+)$, is the sub σ-field of $\mathcal{B}(W_T)$ generated by the cylinder sets of $C([0,t]\to T^{\mathbf{Z}^d})$.

Corresponding to the previously defined notations $\mathbf{x}_{\Lambda}$, resp. $C^p_{\Lambda}(\mathbf{R}^{\mathbf{Z}^d})$ and $C^p_0(\mathbf{R}^{\mathbf{Z}^d})$, we define the following: For each non-empty $\Lambda\subseteq\mathbf{Z}^d$, by $\mathbf{y}_{\Lambda}$ we denote the image of the projection onto T^{Λ}:

$$T^{\mathbf{Z}^d}\ni\mathbf{y}\longmapsto\mathbf{y}_{\Lambda}\in T^{\Lambda}.$$

Also, $C^p_{\Lambda}(T^{\mathbf{Z}^d})$, $C^p_0(T^{\mathbf{Z}^d})$, and $C_{\Lambda}(T^{\mathbf{Z}^d})$ are defined correspondigly.
We use the notation $\mathbf{y}=\{y_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{Z}^d}$ to denote a point in $T^{\mathbf{Z}^d}$.

In order to give a correspondence between the points in $\mathbf{R}^{\mathbf{Z}^d}$ and the points in $T^{\mathbf{Z}^d}$, we introduce the function

$$\Theta : T^{\mathbf{Z}^d}\ni\{y_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{Z}^d}\longmapsto\{\theta_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{Z}^d}\in[0,2\pi)^{\mathbf{Z}^d}$$

where $\theta_{\mathbf{k}} = \theta(y_{\mathbf{k}})$ and the function $\theta : T \longrightarrow [0, 2\pi)$ is defined by

$$y = \begin{pmatrix} \cos\theta(y) \\ \sin\theta(y) \end{pmatrix} \in T \subset \mathbf{R}^2.$$

Let $\widehat{C}([0, 2\pi)^{\mathbf{Z}^d})$ be the linear subspace of $C([0, 2\pi)^{\mathbf{Z}^d})$ such that

$$\widehat{C}([0, 2\pi)^{\mathbf{Z}^d}) \equiv \Big\{ \phi \in C([0, 2\pi)^{\mathbf{Z}^d}) \;\Big|\; \lim_{\theta_{\mathbf{k}} \uparrow 2\pi} \phi(\boldsymbol{\theta}) = \phi(\boldsymbol{\theta}|_{\theta_{\mathbf{k}}=0}), \\ \forall \boldsymbol{\theta} \equiv \{\theta_{\mathbf{l}}\}_{\mathbf{l} \in \mathbf{Z}^d} \in [0, 2\pi)^{\mathbf{Z}^d},\ \forall \mathbf{k} \in \mathbf{Z}^d \Big\},$$

where $\boldsymbol{\theta}|_{\theta_{\mathbf{k}}=0}$ is the vector defined by changing the $\mathbf{k}$-th component $\theta_{\mathbf{k}}$ of $\boldsymbol{\theta}$ to 0. Then, each $\varphi \in C(T^{\mathbf{Z}^d})$ has a corresponding element $\phi \in \widehat{C}([0, 2\pi)^{\mathbf{Z}^d})$ such that

$$\varphi(\mathbf{y}) = \phi \circ \Theta(\mathbf{y}), \qquad \forall \mathbf{y} \in T^{\mathbf{Z}^d}$$

By this, we will identify the elements of $C(T^{\mathbf{Z}^d})$ with the corresponding elements in $\widehat{C}([0, 2\pi)^{\mathbf{Z}^d})$.

In addition we define $\Phi(x_{\mathbf{k}}) = \theta_{\mathbf{k}} \in [0, 2\pi)$ if $x_{\mathbf{k}} = \theta_{\mathbf{k}} \, mod \, 2\pi$. Then we can define a surjection from $\mathbf{R}^{\mathbf{Z}^d}$ to $T^{\mathbf{Z}^d}$ such that

$$\Theta^{-1} \circ \boldsymbol{\Phi} : \mathbf{R}^{\mathbf{Z}^d} \ni \mathbf{x} = \{x_{\mathbf{k}}\}_{\mathbf{k} \in \mathbf{Z}^d} \longmapsto \{\theta^{-1} \circ \Phi(x_{\mathbf{k}})\}_{\mathbf{k} \in \mathbf{Z}^d} \in T^{\mathbf{Z}^d}. \tag{14}$$

In the sequel, if there is no ambiguity, to denote such interpretation $\varphi(\boldsymbol{\Theta}^{-1} \circ \boldsymbol{\Phi}(\cdot)) \in C(\mathbf{R}^{\mathbf{Z}^d})$ of $\varphi(\cdot) \in C(T^{\mathbf{Z}^d})$ we will use the same notation φ, i.e., we will not always write the corresponding **periodic function** by $\varphi(\boldsymbol{\Theta}^{-1} \circ \boldsymbol{\Phi}(\mathbf{x}))$ but simply $\varphi(\mathbf{x})$.

The following Proposition 1-i), ii) resp. and iii) are results of Theorem 2.23 of [HS] resp. and Proposition 1.2 of [S] (cf. also [BoRW]):

Proposition 1. ***(Quotient process of*** $\{\mathbb{X}^1(t, \mathbf{x})\}_{t \geq 0}$***)***
Let $\mathcal{J}$ be a potential that satisfies the conditions J-1), J-2) and J-3).
i) For each $t \geq 0$, let $\boldsymbol{\eta}_t$ be the measurable function defined by

$$\boldsymbol{\eta}_t : W_T \ni \mathbf{y}(\cdot) \longmapsto \mathbf{y}(t) \in T^{\mathbf{Z}^d}.$$

Let $\mathbf{y} \in T^{\mathbf{Z}^d}$ and take $\mathbf{x} \in \mathcal{H}$ such that $\mathbf{y} = \boldsymbol{\Theta}^{-1} \circ \boldsymbol{\Phi}(x)$.
On $(W_T, \mathcal{B}(W_T))$ define the probability measure

$$Q_{\mathbf{y}} \equiv P^1_{\mathbf{x}} \circ \Theta^{-1} \circ \boldsymbol{\Phi},$$

i.e.

$$Q_{\mathbf{y}}(B) \equiv P^1_{\mathbf{x}}\Big(\{\mathbf{x}(\cdot) \in W \,|\, \Theta^{-1} \circ \boldsymbol{\Phi}(\mathbf{x}(\cdot)) \in B\}\Big),\ \forall B \in \mathcal{B}(W_T),$$

where the probability measure $P^1_{\mathbf{x}}$ *on* $(W, \mathcal{B}(W))$ *is the probability law of the process* $\{\mathbb{X}^1(t,\mathbf{x})\}_{t\in\mathbf{R}_+}$. *Then,* $Q_{\mathbf{y}}$ *satisfies the following:*

$$Q_{\mathbf{y}}(\boldsymbol{\eta}_0 = \mathbf{y}) = 1 \quad \textit{and} \quad \big(f(\boldsymbol{\eta}_t) - \int_0^t (Lf)(\boldsymbol{\eta}_s)ds,\ \mathcal{B}_t(W_T),\ Q_{\mathbf{y}}\big)$$

is a martingale for each $f \in C_0^\infty(T^{\mathbf{Z}^d})$, *where*

$$(L\,f)(\mathbf{y}) = \sum_{k\in\mathbf{Z}^d} \left\{ \frac{\partial^2 f}{\partial y_{\mathbf{k}}^2}(\mathbf{y}) + b_{\mathbf{k}}(\Theta(\mathbf{y}))\frac{\partial}{\partial y_{\mathbf{k}}} f(\mathbf{y}) \right\}.$$

Furthermore, $Q_{\mathbf{y}}$ *is the unique solution of the above martingale problem.*
ii) *Let* $p(t,\mathbf{y},\cdot)$ *be the transition function associated with the diffusion process* $(\boldsymbol{\eta}_t, Q_{\mathbf{y}} : \mathbf{y} \in T^{\mathbf{Z}^d})$. *For* $N \in \mathbf{N}$, $\mathbf{y} \in T^{\mathbf{Z}^d}$, *let* $p^{(N)}(t,\mathbf{y},\cdot)$ *be such that*

$$p^{(N)}(t,\mathbf{y},\Gamma) = p(t,\mathbf{y},\tilde{\Gamma}) \qquad \textit{for} \quad \Gamma \in \mathcal{B}(T^{[-N,+N]^d}),$$

where $\tilde{\Gamma} = \{\mathbf{y} \in T^{\mathbf{Z}^d} \,|\, \mathbf{y}_{(N)} \equiv \mathbf{y}_{[-N,N]^d} \in \Gamma\}$. *Then* $p^{(N)}(t,\mathbf{y},d\mathbf{y}_{(N)})$ *has a density* $p^{(N)}(t,\mathbf{y},\mathbf{y}_{(N)})$ *with respect to Lebesgue measure on* $T^{[-N,+N]^d}$ *whose partial derivatives in the variable* $\mathbf{y}_{(N)}$ *of all order exist and are continuous functions of* $(t,\mathbf{y},\mathbf{y}_{(N)})$ *in* $(0,\infty)\times T^{\mathbf{Z}^d} \times T^{[-N,+N]^d}$.
iii) *There exists at least one Gibbs probability measure* μ *on* $(T^{\mathbf{Z}^d}, \mathcal{B}(T^{\mathbf{Z}^d}))$ *such that*

$$\langle \mathbb{E}^\Lambda \varphi, \mu\rangle = \langle \varphi, \mu\rangle, \quad \forall \Lambda \subset \mathbf{Z}^d \textit{ s.t. } |\Lambda| < \infty, \quad \forall \varphi \in C_0(T^{\mathbf{Z}^d}), \tag{15}$$

where

$$[\mathbb{E}^\Lambda \varphi](\mathbf{y}) = \frac{1}{Z_\Lambda(\mathbf{y}_{\Lambda^c})} \int_{T^{\mathbf{Z}^d}} \varphi(\mathbf{y'}_\Lambda \cdot \mathbf{y}_{\Lambda^c}) e^{-U^\Lambda(\Theta(\mathbf{y'}_\Lambda \cdot \mathbf{y}_{\Lambda^c}))} d\mathbf{y'},$$

with

$$U^\Lambda(\mathbf{x}) \equiv \sum_{\Lambda'\cap\Lambda\neq\emptyset} J_{\Lambda'}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^{\mathbf{Z}^d},$$

$$Z_\Lambda(\mathbf{y}_{\Lambda^c}) = \int_{T^{\mathbf{Z}^d}} e^{-U^\Lambda(\Theta(\mathbf{y}'_\Lambda \cdot \mathbf{y}_{\Lambda^c}))} d\mathbf{y'}.$$

Here we use the notation $\mathbf{y}'_\Lambda \cdot \mathbf{y}_{\Lambda^c} \equiv \mathbf{y}'' \in T^{\mathbf{Z}^d}$,
so that $\mathbf{y}''_\Lambda = \mathbf{y}'_\Lambda$ *and* $\mathbf{y}''_{\Lambda^c} = \mathbf{y}_{\Lambda^c}$.

■

Remark 2. **(Q_y and Dirichlet forms)**
Let μ be some Gibbs measure on $(T^{\mathbf{Z}^d}, \mathcal{B}(T^{\mathbf{Z}^d}))$, and consider the Dirichlet space $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ on $L^2(\mu)$ that is a quasi-regular Markovian extension of the form

$$\sum_{\mathbf{k}\in\mathbf{Z}^d}\int_{T^{\mathbf{Z}^d}}\frac{\partial\varphi}{\partial y_{\mathbf{k}}}\cdot\frac{\partial\psi}{\partial y_{\mathbf{k}}}\mu(d\mathbf{y}),\qquad \varphi,\psi\in C_0^\infty(T^{\mathbf{Z}^d}),\quad \text{on } L^2(\mu).$$

(closability holds according to [AKR].) Let $\mathbf{M}$ be the (strong) Markov process properly associated with the Dirichlet space $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$, which exists by [MR]. Denote the corresponding Markovian transition function and the probability law of the process $\mathbf{M}$ starting at $\mathbf{y}\in T^{\mathbf{Z}^d}$ by $p^{\mathbf{M}}(t,\mathbf{y},\cdot)$ ($t\in\mathbf{R}_+$, $\mathbf{y}\in T^{\mathbf{Z}^d}$) and $Q^{\mathbf{M}}_{\mathbf{y}}(\cdot)$ on $(W_T,\mathcal{B}(W_T))$ respectively.

By the uniqueness statement given in Prop. 1 we see that the Markov process $(\{\boldsymbol{\eta}_t\}_{t\ge0},Q_{\mathbf{y}}:\mathbf{y}\in T^{\mathbf{Z}^d})$ defined by Prop. 1 is equivalent to $\mathbf{M}$ above, hence properly associated to the Dirichlet space $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ (for precise arguments cf. Remark 1.1 of [ABRY3], also cf. [F1], [MR]).

■

3 Theorems

In [ABRY3] we have considered the homogenization problem of the sequence of the diffusions $\{\{\mathbb{X}^\epsilon(t,\mathbf{x})\}_{t\in\mathbf{R}_+}\}_{\epsilon>0}$ in the case where the the following uniform ergodicity (16) holds for the quotient process $(\{\boldsymbol{\eta}_t\}_{t\ge0},Q_{\mathbf{y}}:\mathbf{y}\in T^{\mathbf{Z}^d})$. Here we consider the same problem for $\{\{\mathbb{X}^\epsilon(t,\mathbf{x})\}_{t\in\mathbf{R}_+}\}_{\epsilon>0}$ in the case where the L^2-type ergodicity holds for $(\boldsymbol{\eta}_t,Q_{\mathbf{y}}:\mathbf{y}\in T^{\mathbf{Z}^d})$, and compare the results available under these two different assumptions of (16) and (17). Each comparison will be given as a Remark following each Theorem resp. Lemma.

In the sequel we denote the uniform ergodicity (16) as (LS) and the L^2-type ergodicity (17) as (WP) respectively. We have to remark that if the potential $\mathcal{J}$, that satisfies J-1), J-2) and J-3), satisfies in addition *Dobrushin-Shlosman mixing condition*, then (16) holds, more precisely in this case the *logarithmic Sobolev inequality (LS)* holds for the Dirichlet form $\mathcal{E}(u(\cdot),v(\cdot))$ defined in Remark 2, then the stronger inequality such that the term $(c+t)^{-\alpha}$ in (16) is replaced by $e^{-\alpha t}$ for some $\alpha>0$ holds (cf. [S]).

Correspondingly, if $\mathcal{E}(u(\cdot),v(\cdot))$ satisfies the *weak Poincaré (WP) inequality*, then (17) holds. We remark that the *logarithmic Sobolev inequality* is strictly stronger than the *the weak Poincaré inequality* (cf. [RWang]).

Precisely, we define the ergodicities (LS) and (WP) as follows:
(LS) For some Gibbs state μ, there exists a $c=c(\mathcal{J})>0$ and an $\alpha=\alpha(\mathcal{J})>1$ which depend only on $\mathcal{J}$, such that for each $\Lambda\in\mathbf{Z}^d$ with $|\Lambda|<\infty$ there exists $K(\Lambda)\in(0,\infty)$ and for $\forall t>0$, $\forall\varphi\in C^\infty_\Lambda(T^{\mathbf{Z}^d})$ the following holds

$$\|\int_{T^{\mathbf{Z}}}\varphi(\mathbf{y}_\Lambda)p(t,\cdot,d\mathbf{y})-\langle\varphi,\mu\rangle\|_{L^\infty}\le K(\Lambda)(c+t)^{-\alpha}(\|\nabla\varphi\|_{L^\infty}+\|\varphi\|_{L^\infty}),\quad (16)$$

(WP) There exist $c=c(\mathcal{J})>0$, $\alpha=\alpha(\mathcal{J})>1$ and $K>0$, that depends only on $\mathcal{J}$, and the following holds

$$\|\mathcal{P}_t\varphi - <\varphi,\mu>\|_{L^2(\mu)} \le K(c+t)^{-\alpha}\|\varphi\|_{L^2(\mu)},\ \forall t>0,\ \forall\varphi\in C(T^{\mathbf{Z}^d}). \quad (17)$$

We also remark that (16) or (17) gives the uniqueness of the Gibbs state, since by (16) or (17) we see that a Gibbs state μ that satisfies (16) or (17) is the only invariant measure for $p(t,\cdot,d\mathbf{y})$, but every Gibbs state is an invariant measure. From now on we denote the unique Gibbs measure by μ (cf. [ABRY3] and [AKR]).

Lemma 3. *Assume that J-1), J-2), J-3) and the L^2 ergodicity (WP) is satisfied. Then, for any $\mathbf{k}\in\mathbf{Z}^d$,*

$$\chi_{\mathbf{k}}(\mathbf{y}) \equiv E^{Q_{\mathbf{y}}}[\int_0^\infty \{b_{\mathbf{k}}(\Theta(\eta_s(\cdot))\}ds],$$

is well defined as a measurable function of $\mathbf{y}\in T^{\mathbf{Z}^d}$.
Let for u, $v\in\mathcal{D}(\mathcal{E})$

$$\mathcal{E}(u(\cdot),v(\cdot)) \equiv \sum_{\mathbf{j}\in\mathbf{Z}^d}\int_{T^{\mathbf{Z}^d}}(\frac{\partial}{\partial y_{\mathbf{j}}}u(\mathbf{y}))(\frac{\partial}{\partial y_{\mathbf{j}}}v(\mathbf{y}))\mu(d\mathbf{y}),$$

then for any $\mathbf{k}\in\mathbf{Z}^d$

$$\chi_{\mathbf{k}}(\cdot)\in\mathcal{D}(\mathcal{E}), \qquad \chi_{\mathbf{k}} \text{ is quasi-continuous} \quad (18)$$

$$\mathcal{E}(\chi_{\mathbf{k}},\chi_{\mathbf{k}}) \le \frac{5}{4},$$
$$\mathcal{E}(\chi_{\mathbf{k}}(\cdot),v(\cdot)) = -\int_{T^{\mathbf{Z}^d}} b_{\mathbf{k}}(\Theta(\mathbf{y}))v(\mathbf{y})\mu(d\mathbf{y}) \quad \forall v\in C_0^\infty(T^{\mathbf{Z}^d}).$$

■

Remark 4. Under the assumption (LS), we have a stronger result than (18) (cf. Lemma 2.1 of [ABRY3]):

$$\chi_{\mathbf{k}}(\cdot)\in\mathcal{D}(\mathcal{E}) \quad \textit{and} \quad \chi_{\mathbf{k}}(\cdot)\in C(T^{\mathbf{Z}^d}).$$

■

By Lemma 3 we define

$$\chi'_{\mathbf{k},\mathbf{j}}(\mathbf{y}) = \sqrt{2}\frac{\partial}{\partial y_{\mathbf{j}}}\chi_{\mathbf{k}}(\mathbf{y}) \quad \text{if } \mathbf{j}\ne\mathbf{k}$$

and

$$\chi'_{\mathbf{k},\mathbf{j}}(\mathbf{y}) = \sqrt{2}(1-\frac{\partial}{\partial y_{\mathbf{k}}}\chi_{\mathbf{k}}(\mathbf{y})) \quad \text{if } \mathbf{j}=\mathbf{k}.$$

Let $\boldsymbol{\chi}'$ and $\mathbb{A}$ be the matrices whose components are functions such that respectively

$$\boldsymbol{\chi}'(\mathbf{y}) \equiv \Big(\chi'_{\mathbf{k},\mathbf{j}}(\mathbf{y})\Big)_{\mathbf{k},\mathbf{j}\in\mathbf{Z}^d}, \tag{19}$$

$$\mathbb{A}(\mathbf{y}) \equiv \Big(a_{\mathbf{k},\mathbf{j}}(\mathbf{y})\Big)_{\mathbf{k},\mathbf{j}\in\mathbf{Z}^d}, \quad \text{with} \quad a_{\mathbf{k},\mathbf{l}}(\mathbf{y}) \equiv \sum_{\mathbf{j}\in\mathbf{Z}^d} \chi'_{\mathbf{k},\mathbf{j}}(\mathbf{y}) \cdot \chi'_{\mathbf{l},\mathbf{j}}(\mathbf{y}). \tag{20}$$

By (19), we can define a matrix $\bar{\mathbb{A}}$ whose components are constants as follows:

$$\bar{\mathbb{A}} \equiv \Big(\bar{a}_{\mathbf{k},\mathbf{j}}\Big)_{\mathbf{k},\mathbf{j}\in\mathbf{Z}^d}, \quad \text{with} \quad \bar{a}_{\mathbf{k},\mathbf{l}} \equiv \sum_{\mathbf{j}\in\mathbf{Z}^d} \int_{T^{\mathbf{Z}^d}} \chi'_{\mathbf{k},\mathbf{j}}(\mathbf{y}) \cdot \chi'_{\mathbf{l},\mathbf{j}}(\mathbf{y})\mu(d\mathbf{y}). \tag{21}$$

For each $M \in \mathbf{N}$ let $\bar{\mathbb{A}}|_M$ be the submatrix of $\bar{\mathbb{A}}$ such that

$$\bar{\mathbb{A}}|_M = \big\{\bar{a}_{\mathbf{k},\mathbf{l}}\big\}_{|\mathbf{k}|,|\mathbf{l}|\le M},$$

then by (19) and Fubini's Lemma, for $\mathbf{z} = \{z_{\mathbf{k}}\}_{|\mathbf{k}|\le M}$

$$\begin{aligned} 0 \le\ & \mathbf{z} \cdot \bar{\mathbb{A}}|_M \cdot {}^t\mathbf{z} \\ =& \int_{T^{\mathbf{Z}^d}} \sum_{\mathbf{j}\in\mathbf{Z}^d} \Big(\sum_{|\mathbf{k}|\le M} \big(\chi'_{\mathbf{k},\mathbf{j}}(\mathbf{y})\big) z_{\mathbf{k}} \Big)^2 \mu(d\mathbf{y}) < +\infty. \end{aligned}$$

Hence, by the martingale representation theorem by means of the Brownian motion processes (cf. for e.g. Section II-6 of [IW]) the finite dimensional quadratic variation matrix $\bar{\mathbb{A}}|_M$ determines uniquely an $M' \equiv \sharp\{\mathbf{k}\big|\,|\mathbf{k}| \le M\}$-dimensional continuous Gaussian process on some adequate probability space. Since the sequence of the probability laws of such M'-dimensional processes, that is a sequence of Borel probability measures on $C(\mathbf{R}_+ \to \mathbf{R}^{M'})$, is consistent, by the Kolmogorov's extention theorem there exists a unique probability measure on $\big(C(\mathbf{R}_+ \to \mathbf{R}^{\mathbf{Z}^d}), \mathcal{B}(C(\mathbf{R}_+ \to \mathbf{R}^{\mathbf{Z}^d}))\big)$, such that any of its M'-dimensional marginals is identical to the probability law of the continuous Gaussian process characterized by $\bar{\mathbb{A}}|_M$.

By this construction, we denote by $\{\mathbb{Y}_t\}_{t\in\mathbf{R}_+}$ with $\mathbb{Y}_0 = 0$ as the unique continuous Gaussian process taking values in $\mathbf{R}^{\mathbf{Z}^d}$ (namely $\mathbb{Y}.$ is a $C(\mathbf{R}_+ \to \mathbf{R}^{\mathbf{Z}^d})$ valued random variable) with covariance matrix $t \cdot \bar{\mathbb{A}}$ ($t \in \mathbf{R}_+$) defined on a complete probability space.

If $\mathbf{x} \equiv \{x_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{Z}^d} \in \mathcal{H}$, then by (19), (22) and above mentioned construction of $\{\mathbb{Y}_t\}_{t\in\mathbf{R}_+}$ with $\mathbb{Y}_0 = 0$ by means of $\bar{\mathbb{A}}|_M$, by using the martingale inequality we see that the trajectories of $\{\mathbb{Y}_t + \mathbf{x}\}_{t\in\mathbf{R}_+}$ stay in W with probability 1, and $\mathcal{B}(C(\mathbf{R}_+ \to \mathbf{R}^{\mathbf{Z}^d})) \cap W = \mathcal{B}(W)$ is identical with $(W, \mathcal{B}(W))$. We can then set the following definition:

Definition 5. *Let $\{\mathbb{Y}_t\}_{t\in\mathbf{R}_+}$ with $\mathbb{Y}_0 = 0$ be the unique continuous Gaussian process defined above, with a law which is a Borel probability measure on $(W, \mathcal{B}(W))$. For each $\mathbf{x} \in \mathcal{H}$, let $P_{\mathbf{x}}$ be the probability measure on $(W, \mathcal{B}(W))$ that is the probability law of the process $\{\mathbf{x} + \mathbb{Y}_t\}_{t\in\mathbf{R}_+}$.*

■

Heuristically, $\{\mathbb{Y}_t\}_{t\in\mathbf{R}_+}$ can be expressed by

$$\mathbb{Y}_t = \int_0^t \bar{\mathbb{A}}^{\frac{1}{2}} d\mathbf{B}_t, \quad t \in \mathbf{R}_+,$$

where $\{\mathbf{B}_t\}_{t\in\mathbf{R}_+} \equiv \big\{\{B_{\mathbf{k},t}\}_{t\in\mathbf{R}_+}\big\}_{\mathbf{k}\in\mathbf{Z}^d}$ and $\{B_{\mathbf{k},t}\}_{t\in\mathbf{R}_+}$ ($\mathbf{k}\in\mathbf{Z}^d$) are some independent sequences of one-dimensional standard Brownian motion processes.

Lemma 6. *Assume that J-1), J-2), J-3) and that the L^2 ergodicity (WP) is satisfied. Let $\chi_{\mathbf{k}}(\cdot) \in \mathcal{D}(\mathcal{E})$ ($\mathbf{k} \in \mathbf{Z}^d$) be the functions defined by Lemma 3, denote $\chi_{\mathbf{k}}(\boldsymbol{\Theta}^{-1} \circ \boldsymbol{\Phi}(\mathbf{x}))$ simply by $\chi_{\mathbf{k}}(\mathbf{x})$. For each $\epsilon > 0$ let*

$$M_t^{\epsilon,\mathbf{k}}(\cdot) = \big(\xi_t^{\mathbf{k}}(\cdot) - \xi_0^{k}(\cdot)\big) - \big(\epsilon\chi_{\mathbf{k}}(\frac{\boldsymbol{\xi}_t(\cdot)}{\epsilon}) - \epsilon\chi_{\mathbf{k}}(\frac{\boldsymbol{\xi}_0(\cdot)}{\epsilon})\big). \tag{22}$$

Set $\tilde{\mathbf{y}} = \boldsymbol{\Theta}(\mathbf{y})$, for $\boldsymbol{\Theta}$ the mapping from $T^{\mathbf{Z}^d}$ to $[0,2\pi)^{\mathbf{Z}^d}$ defined in the previous section. Then, for each $\epsilon > 0$ and $\mathcal{E}$-q.e. the processes $\{M_t^{\epsilon,\mathbf{k}}\}_{t\in\mathbf{R}_+}$, $\mathbf{k}\in\mathbf{Z}^d$, on $(W, \mathcal{B}(W), P^{\epsilon}_{\epsilon\tilde{\mathbf{y}}})$ are $L^2(P^{\epsilon}_{\epsilon\tilde{\mathbf{y}}})$, continuous $\mathcal{B}_t(W)$-martingales whose quadratic variations are given by

$$< M^{\epsilon,\mathbf{k}}(\cdot), M^{\epsilon,\mathbf{l}}(\cdot) >_t = \int_0^t a_{\mathbf{k},\mathbf{l}}(\frac{\boldsymbol{\xi}_s(\cdot)}{\epsilon}) ds, \quad \mathbf{k},\mathbf{l} \in \mathbf{Z}^d, \tag{23}$$

where

$$a_{\mathbf{k},\mathbf{l}}(\mathbf{y}) \equiv \sum_{\mathbf{j}\in\mathbf{Z}^d} {\chi'}_{\mathbf{k},\mathbf{j}}(\mathbf{y}){\chi'}_{\mathbf{l},\mathbf{j}}(\mathbf{y}),$$

with

$${\chi'}_{\mathbf{k},\mathbf{j}}(\mathbf{y}) = \begin{cases} \sqrt{2}\dfrac{\partial}{\partial y_{\mathbf{j}}}\chi_{\mathbf{k}}(\mathbf{y}) & \mathbf{j} \neq \mathbf{k} \\ \sqrt{2}(1 - \dfrac{\partial}{\partial y_{\mathbf{k}}}\chi_{\mathbf{k}}(\mathbf{y})) & \mathbf{j} = \mathbf{k}. \end{cases}$$

■

Remark 7. If we assume (LS), then for each $\epsilon > 0$ and each $\mathbf{x} \in \mathcal{H}$ the process $\{M_t^{\epsilon,\mathbf{k}}\}_{t\in\mathbf{R}_+}$, $\mathbf{k}\in\mathbf{Z}^d$, on $(W, \mathcal{B}(W), P^{\epsilon}_{\mathbf{x}})$ is an $L^2(P^{\epsilon}_{\mathbf{x}})$ continuous $\mathcal{B}_t(W)$-martingale, with quadratic variations given by (24).

■

Let ν be a probability measure on $(T^{\mathbf{Z}^d}, \mathcal{B}(T^{\mathbf{Z}^d}))$ such that

$$\|\frac{d\nu}{d\mu}\|_{L^\infty(T^{\mathbf{Z}^d})} < \infty. \tag{24}$$

For each $\epsilon \in [0,1)$, define a probability measure $P^{\epsilon}_{\nu_\epsilon}$ on $(W, \mathcal{B}(W))$ such that

$$P^{\epsilon}_{\nu_\epsilon}(B) \equiv \int_{T^{\mathbf{Z}^d}} P^{\epsilon}_{\epsilon\tilde{\mathbf{y}}}(B)\nu(d\mathbf{y}), \quad \forall B \in \mathcal{B}(W), \tag{25}$$

where as above (and in the sequel) $\tilde{\mathbf{y}} = \boldsymbol{\Theta}(\mathbf{y})$.

Remark 8. We remark that by Lemma 6, the processes $\{M_t^{\epsilon,\mathbf{k}}\}_{t\in\mathbf{R}_+}$, $\mathbf{k}\in\mathbf{Z}^d$, on
$(W,\mathcal{B}(W),P^\epsilon_{\nu_\epsilon})$ are $L^2(P^\epsilon_{\nu_\epsilon})$, continuous $\mathcal{B}_t(W)$-martingales with quadratic variations given by (24).

■

Theorem 9. *Assume that J-1), J-2), J-3) and (WP) are satisfied. Then, for each* $\epsilon>0$ *and each probability measure* ν *on* $(T^{\mathbf{Z}^d},\mathcal{B}(T^{\mathbf{Z}^d}))$ *satisfying (25), it is possible to construct a probability space* $(\bar W,\mathcal{B}(\bar W),\bar P^\epsilon_{\nu_\epsilon};\mathcal{B}_t(\bar W))$*, which is a standard extension of* $(W,\mathcal{B}(W),P^\epsilon_{\nu_\epsilon};\mathcal{B}_t(W))$*, and a* $\mathcal{B}_t(W)$*-adapted* $\mathbf{R}^{\mathbf{Z}^d}$*-valued continuous process* $\{\boldsymbol{\zeta}^\epsilon_t\}_{t\in\mathbf{R}_+}$ *(defined precisely in the next section) that satisfies the following:* $\boldsymbol{\zeta}^\epsilon_\cdot$ *is a* W *valued random variable whose probability law* $\bar P^\epsilon_{\nu_\epsilon}\circ\boldsymbol{\zeta}^\epsilon_\cdot$ *forms a relatively compact set* $\{\bar P^\epsilon_{\nu_\epsilon}\circ\boldsymbol{\zeta}^\epsilon_\cdot\}_{\epsilon>0}$ *in the space of probability measures on* $(W,\mathcal{B}(W))$ *equipped with the weak topology, and for any* $\varphi\in C_b(W\to\mathbf{R})$*, the following holds:*

$$\lim_{\epsilon\downarrow 0} E^{\bar P^\epsilon_{\nu_\epsilon}}[\varphi(\boldsymbol{\zeta}^\epsilon_\cdot(\cdot))] = E^{P_0}[\varphi(\hat{\boldsymbol{\xi}}_\cdot(\cdot))], \tag{26}$$

$$\lim_{\epsilon\downarrow 0} E^{\bar P^\epsilon_{\nu_\epsilon}}[\rho(\hat{\boldsymbol{\xi}}_\cdot(\cdot),\boldsymbol{\zeta}^\epsilon_\cdot(\cdot))] = 0, \tag{27}$$

where

$$\hat{\boldsymbol{\xi}}_t(\cdot)\equiv\boldsymbol{\xi}_t(\cdot)-\epsilon\boldsymbol{\chi}(\frac{\boldsymbol{\xi}_t(\cdot)}{\epsilon})+\epsilon\boldsymbol{\chi}(\frac{\boldsymbol{\xi}_0(\cdot)}{\epsilon}).$$

■

Remark 10. Under the assumption (LS), we can take the initial states as Dirac point measures (cf. Theorem 2.1 of [ABRY3]):

$$\lim_{\epsilon\downarrow 0} E^{\bar P^\epsilon_{\mathbf{x}}}[\varphi(\boldsymbol{\zeta}^\epsilon_\cdot(\cdot))] = E^{P_{\mathbf{x}}}[\varphi(\boldsymbol{\xi}_\cdot(\cdot))]. \qquad \forall\mathbf{x}\in\mathcal{H}, \tag{28}$$

One then also have :

$$\lim_{\epsilon\downarrow 0} E^{\bar P^\epsilon_{\epsilon\mathbf{x}}}[\varphi(\boldsymbol{\zeta}^\epsilon_\cdot(\cdot))] = E^{P_0}[\varphi(\boldsymbol{\xi}_\cdot(\cdot))], \qquad \forall\mathbf{x}\in[0,2\pi)^{\mathbf{Z}^d}, \tag{29}$$

where the approximation sequence $\big\{\{\boldsymbol{\zeta}^\epsilon_t\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$ satisfies

$$\lim_{\epsilon\downarrow 0}\int_{T^{\mathbf{Z}^d}} E^{\bar P^\epsilon_{\epsilon\tilde{\mathbf{y}}}}[\rho(\boldsymbol{\xi}_\cdot(\cdot),\boldsymbol{\zeta}^\epsilon_\cdot(\cdot))]\mu(d\mathbf{y}) = 0. \tag{30}$$

■

Remark 11. In order to show that $\chi_{\mathbf{k}}\in\mathcal{D}(\mathcal{E})$ satisfies $\chi_{\mathbf{k}}\in C(T^{\mathbf{Z}^d}\to\mathbf{R})$ we used crucially (LS) in [ABRY3]. Here, in Lemma 3 we assume (WP) and we can show $\chi_{\mathbf{k}}\in\mathcal{D}(\mathcal{E})$ only, and we can not see in general that $\chi_{\mathbf{k}}$ is bounded. By this we are not able to assert that the term

$$-\epsilon\boldsymbol{\chi}(\frac{\boldsymbol{\xi}_t(\cdot)}{\epsilon}) + \epsilon\boldsymbol{\chi}(\frac{\boldsymbol{\xi}_0(\cdot)}{\epsilon})$$

vanishes as $\epsilon \downarrow 0$, and we have to modify $\boldsymbol{\xi}_t(\cdot)$ by $\hat{\boldsymbol{\xi}}_t(\cdot)$ in Theorem 9 above and 12 below (cf. Remarks 4 and 10).

■

Theorem 12. *Let P_0 be the probability law of the process $\{\mathbb{Y}_t\}_{t\in\mathbf{R}_+}$. Assume that the assumptions of Theorem 9 are satisfied, then the following hold:*

$$\lim_{\epsilon\downarrow 0} E^{P^{\epsilon}_{\tilde{\nu}_\epsilon}}[\varphi(\hat{\boldsymbol{\xi}}_{\cdot}(\cdot))] = E^{P_0}[\varphi(\boldsymbol{\xi}_{\cdot}(\cdot))], \quad \forall\varphi \in C_b(W \to \mathbf{R}). \tag{31}$$

■

Remark 13. Under the assumptions J-1), J-2), J-3) and (LS), in [ABRY3] we proved the following: For $\tilde{\mathbf{y}} = \Theta(\mathbf{y})$ with $\Theta : T^{\mathbf{Z}^d} \to [0, 2\pi)^{\mathbf{Z}^d}$,

$$\lim_{\epsilon\downarrow 0} \int_{T^{\mathbf{Z}^d}} \left| E^{P^{\epsilon}_{\epsilon\tilde{\mathbf{y}}}}[\varphi(\boldsymbol{\xi}_{\cdot}(\cdot))] - E^{P_0}[\varphi(\boldsymbol{\xi}_{\cdot}(\cdot))] \right| \mu(d\mathbf{y}) = 0, \quad \forall\varphi \in C_b(W \to \mathbf{R}). \tag{32}$$

Also there exists an $\mathcal{N} \in \mathcal{B}(T^{\mathbf{Z}^d})$ such that $\mu(\mathcal{N}) = 0$, and a subsequence

$$\big\{\epsilon_n\big\}_{n\in\mathbf{N}} \subset \big\{\epsilon \,|\, \epsilon \in (0, 1]\big\},$$

and the following holds (cf. [PapV] for the finite dimensional case):

$$\lim_{\epsilon_n\downarrow 0} E^{P^{\epsilon_n}_{\epsilon_n\mathbf{x}}}[\varphi(\boldsymbol{\xi}_{\cdot}(\cdot))] = E^{P_0}[\varphi(\boldsymbol{\xi}_{\cdot}(\cdot))], \quad \forall\varphi \in C_b(W \to \mathbf{R}) \tag{33}$$

$$\forall\mathbf{x} \equiv \{x_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{Z}^d} \in \mathbf{R}^{\mathbf{Z}^d} \quad \text{such that} \quad \Theta^{-1}\boldsymbol{\Phi}(\mathbf{x}) \in T^{\mathbf{Z}^d} \setminus \mathcal{N}, \quad \sup_{\mathbf{k}\in\mathbf{Z}^d} |x_{\mathbf{k}}| < \infty.$$

■

Remark 14. For the present problem we use crucially the ergodicity of the corresponding quotient process $(\{\boldsymbol{\eta}_t\}_{t\geq 0}, Q_{\mathbf{y}} : \mathbf{y} \in T^{\mathbf{Z}^d})$ whose existence depends essentially on the periodicity of the coefficients of original process. If we consider the homogenization problems based on the diffusions processes taking values in $\mathbf{R}^{\mathbf{Z}^d}$ with the index set $\mathbf{Z}^d$ which are defined through Dirichlet forms with convex potential terms (cf. [AKR]), then there are no corresponding quotient processes and the present formulation is impossible.

■

4 Construction of $\big\{\{\boldsymbol{\zeta}_t^\epsilon\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$ and an outline of the proofs

In order to get the results on the homogenization problem for the infinite dimensional diffusions $\{\mathbb{X}^\epsilon(t,\mathbf{x})\}_{t\in\mathbf{R}_+}$ ($\epsilon > 0$), we firstly pass through the discussion of a sequence of approximating processes $\{\boldsymbol{\zeta}_t^\epsilon\}_{t\in\mathbf{R}_+}$ ($\epsilon > 0$) of the original diffusions introduced in Theorem 9. $\big\{\{\boldsymbol{\zeta}_t^\epsilon\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$ is composed in order that the sequence of probability laws of $\big\{\{\boldsymbol{\zeta}_t^\epsilon\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$ forms a relatively compact set in the space of Borel probability measures on $(W,\mathcal{B}(W))$ equipped with the relative topology. For each $\epsilon > 0$ the dimension of $\{\boldsymbol{\zeta}_t^\epsilon\}_{t\in\mathbf{R}_+}$ is essentially finite, that is controlled by the parameter $\epsilon > 0$ with a tricky way composed by using the **uniform ergodic theorem** (LS) or **L^2 ergodic theorem** (WP). In [ABRY3] under the assumption (LS) this subsidiary sequence of processes $\big\{\{\boldsymbol{\zeta}_t^\epsilon\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$ has been constructed in order that it satisfies the pointwise homogenization property given by (29).

Here, we show how the approximating processes $\big\{\{\boldsymbol{\zeta}_t^\epsilon\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$ that satisfy the homogenization property given by (27) are constructed by using the assumption (WP). Onece $\big\{\{\boldsymbol{\zeta}_t^\epsilon\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$ is constructed, the proofs of Theorems 9 and 12 of the present paper are very similar to the ones of Theorems 2.1 and 2.2 in [ABRY3], we do not repeat them here. Also, since Lemmas 3 and 6 in this paper are included in Lemmas 2.1 and 3.1 of [ABRY3], therefore we also omit these proofs here.

For the (WP) case we construct $\big\{\{\boldsymbol{\zeta}_t^\epsilon\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$ as follows. Let $\mathbb{A}(\mathbf{y})$ be the matrix valued function defined by (21), and for each $N \in \mathbf{N}$ let

$$N' \equiv \sharp\{\mathbf{k} \,|\, |\mathbf{k}| \le N\},$$

and define an $N' \times N'$ matrix that is a submatrix of $\mathbb{A}(\mathbf{y})$ such that

$$\mathbb{A}(\mathbf{y})|_N \equiv \Big(a_{\mathbf{k},\mathbf{j}}(\mathbf{y})\Big)_{|\mathbf{k}|,|\mathbf{j}|\le N}, \qquad \mathbf{y} \in T^{\mathbf{Z}^d}.$$

Then by (19) and Fubini's Lemma, for any real vector $\mathbf{z} = \{z_{\mathbf{k}}\}_{|\mathbf{k}|\le N}$

$$\begin{aligned} 0 \le\ & \mathbf{z}\cdot \mathbb{A}(\mathbf{y})|_N \cdot {}^t\mathbf{z} \\ =& \sum_{\mathbf{j}\in\mathbf{Z}^d}\Big(\sum_{|\mathbf{k}|\le N}\big(\chi'_{\mathbf{k},\mathbf{j}}(\mathbf{y})\big)z_{\mathbf{k}}\Big)^2 < +\infty, \qquad \mu - a.s. \quad \mathbf{y} \in T^{\mathbf{Z}^d}. \end{aligned}$$

By this for each $N \in \mathbf{N}$, there exists a matrix $\big(\sigma^N_{\mathbf{k},\mathbf{l}}(\mathbf{y})\big)_{|\mathbf{k}|,|\mathbf{j}|\le N}$ such that

$$a_{\mathbf{k},\mathbf{l}}(\mathbf{y}) = \sum_{|\mathbf{j}|\le N}\sigma^N_{\mathbf{k},\mathbf{j}}(\mathbf{y})\cdot\sigma^N_{\mathbf{l},\mathbf{j}}(\mathbf{y}), \quad |\mathbf{k}|,\ |\mathbf{l}| \le N, \quad \mu - a.s.\ \mathbf{y} \in T^{\mathbf{Z}^d}. \tag{34}$$

By (19), (20) and (21) since for any $\mathbf{k} \in \mathbf{Z}^d$

$$\int_{T^{\mathbf{Z}^d}} a_{\mathbf{k},\mathbf{k}}(\mathbf{y})\mu(d\mathbf{y}) \le \frac{5}{2},$$

we see that

$$\sum_{|\mathbf{l}|\le N} \|\sigma^N_{\mathbf{k},\mathbf{l}}\|^2_{L^2(\mu)} \le \frac{5}{2}, \qquad \text{for any } \mathbf{k} \text{ such that} \quad |\mathbf{k}| \le N. \tag{35}$$

By this, for each $N \in \mathbf{N}$, there exists a sequence of $N' \times N'$ matrices

$$\big\{\sigma^{N,n}_{\mathbf{k},\mathbf{l}}(\mathbf{y})\big\}_{|\mathbf{k}|,|\mathbf{l}|\le N} \qquad n = 1, 2, \cdots,$$

such that

$$\sigma^{N,n}_{\mathbf{k},\mathbf{l}} \in C^\infty_{\Lambda_{N,n}}(T^{\mathbf{Z}^d} \to \mathbf{R}), \quad \text{for some bounded } \Lambda_{N,n} \subset \mathbf{Z}^d, \quad n \in \mathbf{N},$$

$$\lim_{n\to\infty} \|\sigma^{N,n}_{\mathbf{k},\mathbf{l}}(\cdot) - \sigma^N_{\mathbf{k},\mathbf{l}}(\cdot)\|_{L^2(\mu)} = 0, \qquad |\mathbf{k}|, |\mathbf{l}| \le N.$$

Next, define a natural number valued function $n(\cdot)$ as follows:

$$n(N) \equiv \min\Big\{n \in \mathbf{N} \,\Big|\, \sum_{|\mathbf{l}|\le N} \|\sigma^{N,n}_{\mathbf{k},\mathbf{l}} - \sigma^N_{\mathbf{k},\mathbf{l}}\|_{L^2(\mu)} < \frac{1}{N},\ \forall |\mathbf{k}| \le N\Big\}, \tag{36}$$

and then define

$$\tilde\sigma^N_{\mathbf{k},\mathbf{l}}(\mathbf{y}) \equiv \sigma^{N,n(N)}_{\mathbf{k},\mathbf{l}}(\mathbf{y}), \qquad \mathbf{y} \in T^{\mathbf{Z}^d}, \qquad |\mathbf{k}|,\ |\mathbf{l}| \le N. \tag{37}$$

By construction we see that

$$\tilde\sigma^N_{\mathbf{k},\mathbf{l}} \in C^\infty_{\Lambda_N}(T^{\mathbf{Z}^d} \to \mathbf{R}), \qquad \text{where} \quad \Lambda_N \equiv \textstyle\bigcup_{n\le n(N)} \Lambda_{N,n}.$$

Let

$$\tilde a^N_{\mathbf{k},\mathbf{k}}(\mathbf{y}) \equiv \textstyle\sum_{|\mathbf{j}|\le N} \tilde\sigma^N_{\mathbf{k},\mathbf{j}}(\mathbf{y}) \cdot \tilde\sigma^N_{\mathbf{k},\mathbf{j}}(\mathbf{y}) \ge 0, \qquad \mathbf{y} \in T^{\mathbf{Z}^d}. \tag{38}$$

Finally, by using the constants $c(\mathcal{J}) > 0$, $\alpha \equiv \alpha(\mathcal{J}) > 1$ and the constant $K > 0$ which appeared in (WP) we define

$$K_N = K\Big\{\max\big(\frac{2}{c^\alpha}, 1\big)\Big\} \cdot \big\{\max_{|\mathbf{k}|\le N} \|\tilde a^N_{\mathbf{k},\mathbf{k}}\|_{L^2(\mu)}\big\}, \tag{39}$$

and then, for each $\epsilon > 0$, we define

$$N(\epsilon) \equiv \max\Big\{N \in \mathbf{N} \,\big|\, \sqrt{\epsilon} K_N M_{N,\mathbf{k},\mathbf{l}} \le 1, \quad \forall |\mathbf{k}|, \forall |\mathbf{l}| \le N\Big\}, \tag{40}$$

where (cf. (39))

$$M_{N,\mathbf{k},\mathbf{l}} \equiv \sup_{\mathbf{y}\in T^{\Lambda_N}} \big(\tilde a^N_{\mathbf{k},\mathbf{k}}(\mathbf{y}) \cdot \tilde a^N_{\mathbf{l},\mathbf{l}}(\mathbf{y})\big)^{\frac{1}{2}}. \tag{41}$$

Now, we define the approximation sequence of the original process as follows. By Lemma 6, for each $\epsilon > 0$ (hence for $N(\epsilon)$ defined by (41)), since the quadratic variation of the $L^2(P^\epsilon_{\nu_\epsilon})$ continuous $\mathcal{B}_t(W)$-martingale $\{M^{\epsilon,\mathbf{k}}_t\}_{t\in\mathbf{R}_+}$, $\mathbf{k}\in\mathbf{Z}^d$, on $(W,\mathcal{B}(W),P^\epsilon_{\nu_\epsilon})$ is given by

$$< M^{\epsilon,\mathbf{k}}(\cdot), M^{\epsilon,\mathbf{l}}(\cdot) >_t = \int_0^t a_{\mathbf{k},\mathbf{l}}(\frac{\boldsymbol{\xi}_s(\cdot)}{\epsilon})ds, \quad \mathbf{k},\mathbf{l}\in\mathbf{Z}^d,$$

from the expression of $a_{\mathbf{k},\mathbf{l}}(\mathbf{y})$ given by (35), by applying the martingale representation theorem by means of the Brownian motion processes for the finite dimensional continuous L^2 martingales (cf., for e.g., section II-7 of [IW]), we see that on a probability space $(\bar{W},\mathcal{B}(\bar{W}),\bar{P}^\epsilon_{\nu_\epsilon};\mathcal{B}_t(\bar{W}))$ there exists an $N'(\epsilon)\equiv\sharp\{\mathbf{k}|\,|\mathbf{k}|\le N(\epsilon)\}$ dimensional standard Brownian motion process

$$\{B^\epsilon_{\mathbf{k}}(t)\}_{t\in\mathbf{R}_+}, \qquad |\mathbf{k}|\le N(\epsilon),$$

and the following holds:

$$\begin{aligned}\xi^{\mathbf{k}}_t(\cdot) &= \xi^{\mathbf{k}}_0(\cdot) - \epsilon\chi_{\mathbf{k}}(\frac{\boldsymbol{\xi}_0(\cdot)}{\epsilon}) + \epsilon\chi_{\mathbf{k}}(\frac{\boldsymbol{\xi}_t(\cdot)}{\epsilon}) \\ &+ \sum_{|\mathbf{l}|\le N(\epsilon)}\int_0^t \sigma^{N(\epsilon)}_{\mathbf{k},\mathbf{l}}(\frac{\boldsymbol{\xi}_s(\cdot)}{\epsilon})dB^\epsilon_l(s),\ \bar{P}^\epsilon_{\nu_\epsilon}-a.s., |\mathbf{k}|\le N(\epsilon). \qquad (42)\end{aligned}$$

Then, by using $\tilde{\sigma}^{N(\epsilon)}_{\mathbf{k},\mathbf{l}}$ defined by (38) and (41), we define the approximating process $\{\boldsymbol{\zeta}^\epsilon_t\}_{t\in\mathbf{R}_+} = \big\{\{\zeta^{\epsilon,\mathbf{k}}_t\}_{t\in\mathbf{R}_+}\big\}_{\mathbf{k}\in\mathbf{Z}^d}$ on $(\bar{W},\mathcal{B}(\bar{W}),\bar{P}^\epsilon_{\nu_\epsilon};\mathcal{B}_t(\bar{W}))$ as follows:

$$\zeta^{\epsilon,\mathbf{k}}_t(\cdot) = \xi^{\mathbf{k}}_0(\cdot) + \sum_{|\mathbf{l}|\le N(\epsilon)}\int_0^t \tilde{\sigma}^{N(\epsilon)}_{\mathbf{k},\mathbf{l}}(\frac{\boldsymbol{\xi}_s(\cdot)}{\epsilon})dB^\epsilon_l(s),\ |\mathbf{k}|\le N(\epsilon); \qquad (43)$$

$$\zeta^{\epsilon,\mathbf{k}}_t(\cdot) = \xi^{\mathbf{k}}_0(\cdot), \qquad |\mathbf{k}|> N(\epsilon), \quad \forall t\in\mathbf{R}_+. \qquad (44)$$

Let us explain the key point of the proof of the tightness of $\big\{\{\boldsymbol{\zeta}^\epsilon_t\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$. Let

$$\overline{\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}} = \|\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}\|_{L^1(\mu)}.$$

By (39) using

$$\begin{aligned}&p_{u_1}\big(\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}(\cdot)\cdot p_{u_2-u_1}(\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}(\cdot))\big)(\mathbf{y}) \\ &\le \big(\overline{\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}}\big)^2 + \overline{\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}}|p_{u_1}\big(\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}(\cdot)\big)(\mathbf{y}) - \overline{\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}}| \\ &+\|\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}\|_{L^\infty}p_{u_1}\big(|p_{u_2-u_1}\big(\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}(\cdot)\big)(\mathbf{y}) - \overline{\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}}|\big),\end{aligned}$$

by (43), (44), Fubini's Lemma and (WP) we see that

$$
\begin{aligned}
&E^{\bar{P}^{\epsilon}_{\nu_\epsilon}}[|\zeta^{\epsilon,\mathbf{k}}_t(\cdot)-\zeta^{\epsilon,\mathbf{k}}_0(\cdot)|^4] \\
&= E^{\bar{P}^{\epsilon}_{\nu_\epsilon}}\left[\Big|\sum_{|\mathbf{l}|\le N(\epsilon)}\int_0^t \tilde{\sigma}^{N(\epsilon)}_{\mathbf{k},\mathbf{l}}\Big(\frac{\boldsymbol{\xi}_s(\cdot)}{\epsilon}\Big)dB^{\epsilon}_{\mathbf{l}}(s)\Big|^4\right] \\
&\le \epsilon^4\int_{T^{\mathbf{Z}^d}}\Big\{\int_0^{\frac{t}{\epsilon^2}}\int_{u_1}^{\frac{t}{\epsilon^2}} p_{u_1}\big(\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}(\cdot)\cdot p_{u_2-u_1}(\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}(\cdot))\big)(\mathbf{y})du_1du_2\Big\}\nu(d\mathbf{y}) \\
&\le \frac{1}{2}(\overline{\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}})^2t^2+\epsilon^4\overline{\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}}\|\frac{d\nu}{d\mu}\|_{L^\infty}\|\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}\|_{L^2(\mu)}\int_0^{\frac{t}{\epsilon^2}}K(\frac{t}{\epsilon^2}-u_1)(c+u_1)^{-\alpha}du_1 \\
&+\epsilon^4\|\frac{d\nu}{d\mu}\|_{L^\infty}\|\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}\|_{L^2(\mu)}\|\tilde{a}^{N(\epsilon)}_{\mathbf{k},\mathbf{k}}\|_{L^\infty}\int_0^{\frac{t}{\epsilon^2}}\int_{u_1}^{\frac{t}{\epsilon^2}}K(c+(u_2-u_1))^{-\alpha}du_1du_2. \qquad (45)
\end{aligned}
$$

But, for $\alpha > 1$, using

$$
\epsilon^2\int_0^{\frac{t}{\epsilon^2}}\frac{1}{(c+s)^\alpha}ds \le \epsilon\max(\frac{2}{c^\alpha},1)t^{\frac{1}{2}}, \quad \forall\epsilon\in(0,1],
$$

togetherwith (40), (41), (42) and using the bounds (36) and (37) we see that the RHS of (46) is dominated by

$$
c't^2+c''t^{\frac{3}{2}},
$$

for some constants $c',\ c''>0$. Thus we have

$$
E^{\bar{P}^{\epsilon}_{\nu_\epsilon}}[|\zeta^{\epsilon,\mathbf{k}}_t(\cdot)-\zeta^{\epsilon,\mathbf{k}}_0(\cdot)|^4] \le c't^2+c''t^{\frac{3}{2}}, \qquad \forall t>0, \quad \forall\mathbf{k}\in\mathbf{Z}^d, \quad \forall\epsilon>0. \quad (46)
$$

From (47) through a similar discussion as for the proof of Theorem 2.1 of [ABRY3] we can complete the proof of Theorem 9 in the present paper.

■

Remark 15. In [ABRY3] by using the ergodicity (LS), we constructed the corresponding approximation sequence $\big\{\{\boldsymbol{\zeta}^\epsilon_t\}_{t\in\mathbf{R}_+}\big\}_{\epsilon>0}$ which satisfies

$$
\begin{aligned}
&E^{\bar{P}^{\epsilon}_{\mathbf{x}}}[|\zeta^{\epsilon,\mathbf{k}}_t(\cdot)-\zeta^{\epsilon,\mathbf{k}}_0(\cdot)|^4] \le c't^2+c''t^{\frac{3}{2}}, \\
&\qquad\qquad \forall t>0, \quad \forall\mathbf{k}\in\mathbf{Z}^d, \quad \forall\epsilon>0, \quad \forall\mathbf{x}\in\mathcal{H}. \quad (47)
\end{aligned}
$$

■

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