Itô Calculus and Malliavin Calculus

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Dedicated to Professor Kiyosi Itô on his 90th birthday

1 Introduction.

Since the *Wiener space* was established by N. Wiener as a mathematical model of Brownian motion in 1923, a rigorous theory of integrations on a function space started. In these almost eighty years, it has been providing us with important methods in stochastic analysis and its applications.

Around 1942, R. Feynman ([F 1],[F 2]) had an epoch making idea of representing the propagators for Schrödinger equations by a path integral over trajectories of quantum mechanical particles. M. Kac noticed that its counterpart could be discussed rigorously on Wiener space and thus found the *Feynman-Kac formula*. He also applied probabilistic representations of heat kernels by Wiener functional expectations to study asymptotics of spectra of Schrödinger operators ([Ka 1],[Ka 2],[Ka 3]). This study was further developed in a fundamental paper by McKean and Singer [MS], which may be regarded as an origin of the *heat equation methods in the analysis of manifold*.

The approach by McKean and Singer is based on PDE theory, the method of parametrix for heat kernels, in particular. If we would give a similar probabilitistic approach as Kac in the problems of McKean and Singer, we have several difficulties to overcome. In the case of Kac, the second order term of the Schrödinger operator is Laplacian in Euclidean space so that a use of Wiener process and pinned Wiener process is sufficient, which could be easily set up on a Wiener space. In the case of McKean and Singer, however, we need a Brownian motion on a curved Riemannian manifold; also the analysis of pinned Brownian motion requires some fine properties of heat kernels.

As we review in this expository article, a Brownian motion on a Riemannian manifold can be well set up on a Wiener space by appealing to the *Itô calculus*, and the pinned Brownian motion can be well handled by appealing to the *Malliavin calulus* on the Wiener space. We would study the conditional expectations of a class of Wiener functionals as integrations on a 'submanifold' embedded in the Wiener space so that we can develop a 'smooth' desintegration theory on Wiener space. In this study, an important role is played by the notion of *generalized Wiener functionals*, a notion similar to that of Schwartz distributions on Wiener space.

2 Wiener space, Wiener functionals and Wiener maps.

Let $(W_0(\mathbf{R}^d), P^W)$ be the d-dimensional classical Wiener space: $W_0(\mathbf{R}^d)$ is a path space $W_0(\mathbf{R^d}) := \{\ w;\ [0,T] \ni t \mapsto w(t) \in \mathbf{R}^d,\ \text{continuous},\ w(0) = 0\ \}$, which is a Banach space with the usual maximum norm, and P^W is the d-dimensional Wiener measure on it. Here T is a positive constant; sometimes, the time interval is taken to be $[0, \infty)$ and then $W_0(\mathbf{R}^d)$ is a Fréchet space with a family of maximum (semi)norms on subintervals. Let $H \subset W_0(\mathbf{R}^d)$ be the *Cameron-Martin subspace*, which is a real Hilbert space given by

$$H = \left\{\ h \in \mathbf{W}_0(\mathbf{R^d}) \mid h(t) = \int_0^t \dot{h}(s)ds,\ \dot{h} \in L^2\left([0,T] \to \mathbf{R}^d\right)\ \right\},$$
$$||h||_H = ||\dot{h}||_{L^2}.$$

As we know well, the triple $(W_0(\mathbf{R}^d), H, P^W)$ is a typical example of more general notion of *abstract Wiener space*; that is, we may think of the Wiener space as a realization of *standard Gaussian measure on* H.

A P^W-measurable function $F : w \in W_0(\mathbf{R}^d) \mapsto F(w) \in S$, where S is a topological space endowed with the Borel σ-field $\mathcal{B}(S)$, is called an S-valued *Wiener functional* (or a *Wiener map* if we would regard F as a mapping). As usual, we identify two S-valued Wiener functionals F and F' if $P^W\{w; F(w) \neq F'(w)\} = 0$. Let B be any separable Banach space. Then $L_p(B) := L_p(W_0(\mathbf{R}^d) \to B)$, $1 \leq p \leq \infty$, is the usual L_p-space formed of B-valued Wiener functionals $F : W_0(\mathbf{R}^d) \to B$ such that $||F||_p := \left\{\int_{W_0(\mathbf{R}^d)} ||F(w)||_B^p P^W(dw)\right\}^{1/p} < \infty$. $L_p(\mathbf{R})$ is denoted simply by L_p. As usual, we denote the integral $\int_{W_0(\mathbf{R}^d)} F(w) P^W(dw)$ for $F \in L_1$ by $E(F)$ and call it the expectation of F.

If $F \in L_1$, then the conditional expectation $E_{0,0}^{T,x}(F) = E(F|w(T) = x)$ is defined, as usual, by a Radon-Nikodym density, so that, as a function of x, it is determined almost everywhere with an ambiguity of a set of Lebesgue measure 0. However, the Brownian bridge measure $P_{0,0}^{T,x}$ on $W_{0,0}^{T,x}(\mathbf{R}^d) := \{\ w;\ [0,T] \ni t \mapsto w(t) \in \mathbf{R}^d,\ \text{continuous},\ w(0) = 0,\ w(T) = x\ \}$, is well-defined for each x as the image measure of P^W under the map $w \in W_0(\mathbf{R}^d) \mapsto \hat{w} \in W_{0,0}^{T,x}(\mathbf{R}^d)$ defined by $\hat{w}(t) = w(t) + \frac{t}{T}(x - w(T))$, $0 \leq t \leq T$, and the conditional expectation $E_{0,0}^{T,x}(F)$ is defined without ambiguity if F is a Borel function on $W_{0,0}^{T,x}(\mathbf{R}^d)$ which is $P_{0,0}^{T,x}$-integrable. The expectation $E_{0,0}^{T,x}(F)$ may be symbolically written as $E[\delta_x(w(T))F(w)]/p(T,x)$, where $\delta_x(\cdot) = \delta_0(\cdot - x)$ is the Dirac delta function and

$$p(t,x) = (2\pi t)^{-d/2} \exp\left\{-\frac{|x|^2}{2t}\right\}, \quad t > 0, \quad x \in \mathbf{R}^d. \tag{1}$$

This kind of formal expressions will be rigorously and more generally defined in Section 3 below.

In the following, as a typical and important application of Wiener functional expectations, we would review probabilistic expressions of solution $u = (u(t,x))$ for initial value problem (IVP) of heat equations

$$\frac{\partial u}{\partial t} = Lu, \quad u|_{t=0} = f. \tag{2}$$

where L is a second-order semi-elliptic differential operator. In this section, we deal with the case of heat equations on $\mathbf{R}^d$ in which the principal second-order term of L is the half Laplacian: $\frac{1}{2}\Delta$. We introduce a usual notation $u = e^{tL}f$ or $u(t,x) = (e^{tL}f)(x)$ for the solution u of (2).

Solutions of IVP (2) by Wiener functional expectations. I

(1) The case $L = \frac{1}{2}\Delta$. Then $u = e^{tL}f$ is given by

$$u(t,x) = E[f(x + w(t))]. \tag{3}$$

(2) (Feynman-Kac formula) The case of a Schrödinger operator $L = \frac{1}{2}\Delta - V$ where the potential $V(x)$ is a Borel function bounded from below. Then $u = e^{tL}f$ is given by

$$u(t,x) = E\left[\exp\left\{-\int_0^t V(x + w(s))ds\right\} f(x + w(t))\right]. \tag{4}$$

(3) The case of operator $L = \frac{1}{2}\Delta + \sum_{i=1}^d b^i \frac{\partial}{\partial x^i} - V$ where the drift coefficients $b^i(x)$, $i = 1, \ldots, d$, are bounded Borel functions and the potential $V(x)$ is a Borel function bounded from below. Then, setting $w(t) = (w^1(t), \cdots, w^d(t))$, $u = e^{tL}f$ is given by

$$\begin{aligned} u(t,x) = E\Bigg[\exp\Bigg\{&\sum_{i=1}^d \int_0^t b^i(x + w(s))dw^i(s) - \frac{1}{2}\sum_{i=1}^d \int_0^t b^i(x + w(s))^2 ds \\ &- \int_0^t V(x + w(s))ds\Bigg\} f(x + w(t))\Bigg]. \end{aligned} \tag{5}$$

Here, the Wiener functionals $\int_0^t b^i(x + w(s))dw^i(s)$ are defined by Itô's stochastic integrals. Thus, we have encountered now a case of Wiener functional expectations in which a use of the Itô calculus is indispensable.

Now we can see that the solution of (2) has, in each case, the Lebesgue integral representation by the heat kernel:

$$u(t,x) = \int_{\mathbf{R^d}} \langle x|e^{tL}|y\rangle f(y)dy.$$

and the heat kernel $\langle x|e^{tL}|y\rangle$ is given as follows:

In the case (1), $\langle x|e^{tL}|y\rangle = p(t, y-x)$, where $p(t,x)$ is the Gauss kernel given by (1).

In the case (2),

$$\begin{aligned}\langle x|e^{tL}|y\rangle &= E_{0,0}^{t,y-x}\left[\exp\left\{-\int_0^t V(x+w(s))ds\right\}\right]p(t,y-x)\\ &= E\left[\exp\left\{-\int_0^t V(x+w(s))ds\right\}\delta_y(x+w(t))\right].\end{aligned}$$

In the case (3),

$$\begin{aligned}\langle x|e^{tL}|y\rangle = E_{0,0}^{t,y-x}&\left[\exp\left\{\sum_{i=1}^d\int_0^t b^i(x+w(s))dw^i(s)\right.\right.\\ &-\frac{1}{2}\sum_{i=1}^d\int_0^t b^i(x+w(s))^2 ds\\ &\left.\left.-\int_0^t V(x+w(s))ds\right\}\right]p(t,y-x)\\ &= E\left[\exp\left\{\sum_{i=1}^d\int_0^t b^i(x+w(s))dw^i(s)-\frac{1}{2}\sum_{i=1}^d\int_0^t b^i(x+w(s))^2 ds\right.\right.\\ &\left.\left.-\int_0^t V(x+w(s))ds\right\}\delta_y(x+w(t))\right].\end{aligned}$$

Strictly speaking, it is by no means obvious in this case that the Wiener functional under the expectation $E_{0,0}^{t,y-x}$ is $P_{0,0}^{t,y-x}$-measurable. Such a difficulty will be completely resolved by a general theory of desintegrations and quasi sure analysis in the Malliavin calculus, as we review in Section 3.

3 Itô calculus on Wiener space, Itô functionals and Itô maps.

We would continue the same problem of probabilistic solutions of IVP (2) in which the second order differential operator L is of variable coefficients. If it is elliptic, then it is essentially the case of differential operators on a Riemannian manifold M in which the principal second order term is the half Laplacian

$\frac{1}{2}\Delta_M$. Hence we need a Brownian motion, i.e., the diffusion on M with the infinitesimal generator $\frac{1}{2}\Delta_M$. As we would review now, this can be realized on a Wiener space by an application of the Itô calculus.

Let $(W_0(\mathbf{R}^d), P^W)$ be the d-dimensional Wiener space. Then, the coordinate $w(t) = (w^1(t), \cdots, w^d(t))$ of $w \in W_0(\mathbf{R}^d)$ is a realization of the d-dimensional Wiener process.

Let M be a smooth manifold of dimension n and let $A_0, A_1, \ldots, A_d$ be a smooth and complete vector fields on M. Consider the following stochastic differential equation (SDE) on M in which $\circ$ denotes the stochastic differential in the Stratonovich sense:

$$dX(t) = \sum_{i=1}^{d} A_i(X(t)) \circ dw^i(t) + A_0(X(t))dt, \quad X(0) = x \tag{6}$$

and we obtain the pathwise unique solution $X(t) = (X(t, x; w))$. In this note, we assume for simplicity that solutions exist globally; otherwise, we must consider solutions which may tend to the point at infinity of M in a finite time and many of definitions given below need some modifications. For the global existence, it is sufficient to assume that M is compact, or assume that M is embedded into a higher dimensional Euclidean space and, in the global Euclidean coordinates, the coefficients of vector fields have all the derivatives of order ≥ 1 bounded, cf. e.g. [IW] for details.

Let $\mathcal{C}([0, T] \to M)$ be the space of continuous paths $\xi : [0, T] \ni t \mapsto \xi(t) \in M$ endowed with the topology of uniform convergence and, for $x \in M$, $\mathcal{C}_x([0, T] \to M)$ be its subspace consisting of paths ξ such that $\xi(0) = x$. Then the solution defines the following Wiener map (called also an *Itô map*); for each $x \in M$,

$$X^x : w \in W_0(\mathbf{R}^d) \to X^x(w) := [t \mapsto X(t, x; w)] \in \mathcal{C}_x([0, T] \to M).$$

If P_x is the image measure on $\mathcal{C}_x([0, T] \to M)$ of P^W under the Itô map X^x, then the system $\{P_x; x \in M\}$ defines a diffusion process on M with the infinitesimal generator $A = \frac{1}{2}\sum_{i=1}^{r}(A_i)^2 + A_0$. Thus, the Itô map provides us with the A-diffusion process. It actually provides us with something more; if we regard the solution as

$$w \in W_0(\mathbf{R}^d) \to X(w) := [t \mapsto [x \mapsto X(t, x; w)]] \,,$$

then $[x \mapsto X(t, x; w)] \in \mathrm{Diff}(M \to M)$, i.e. a diffeomorphism of M, so that we have a *stochastic flow* of diffeomorphisms (cf. [Ku]).

Wiener functionals which are defined by using the Itô calculus, particularly such functionals as are associated with an Itô map, are often called *Itô functionals*.

In order to discuss the Brownian motion on a Riemannian manifold, the following Itô map, called a *stochastic moving frame*, is very important and useful.

Stochastic moving frames

Let M be a Riemannian manifold of dimension d and $x \in M$. By a *frame* at x, we mean an orthonormal base (ONB) $\mathbf{e} = [\mathbf{e}_1, \ldots, \mathbf{e}_d]$ of the tangent space $T_x M$ at x. A frame at x is denoted by $r = (x, \mathbf{e})$. We denote the totality of frames at all points of M by $O(M)$. It is given a natural structure of manifold and the projection $\pi : O(M) \to M$ is defined by $\pi(r) = x$ if $r = (x, \mathbf{e})$. The d-dimensional orthogonal group $O(d)$ acts on $O(M)$ from the right: $rg = (x, \mathbf{e}g = [(\mathbf{e}g)_1, \ldots, (\mathbf{e}g)_d])$,
$g = (g^i_j) \in O(d)$, where $(\mathbf{e}g)_k = g^i_k \mathbf{e}_i$ (by the usual convention for summation), so that $O(M)$ forms a principal fibre bundle over M with the structure group $O(d)$, which we call the *bundle of orthonormal frames*.

We can identify $r \in O(M)$ with an isometric isomorphism $\tilde{r} : \mathbf{R}^d \to T_x M$, $x = \pi(r)$, defined by sending each of the canonical base δ_i, $\delta_i := (0, \cdots, 0, \overset{i-th}{1}, 0 \cdots 0)$, to $\mathbf{e}_i$, $i = 1, \ldots, d$, where $r = (x, \mathbf{e})$, $\mathbf{e} = [\mathbf{e}_1, \ldots, \mathbf{e}_d]$. This isomorphism $\tilde{r}$ is called the *canonical isomorphism* associated with the frame r. It holds that $\widetilde{rg} = \tilde{r} \circ g$, $g = (g^i_j) \in O(d)$; here g is identified with the orthogonal transformation $g : x = (x^i) \in \mathbf{R}^d \mapsto gx = (g^i_j x^j) \in \mathbf{R}^d$.

Before giving a formal definition of the stochastic moving frame in general, we explain the idea in a simple case of M being a two dimensional sphere $\mathbf{S}^2$. We take a plane $\mathbf{R}^2$ and consider a Brownian motion $w(t)$ on it canonically realized on the two dimensional Wiener space. We assign at each point $w(t) \in \mathbf{R}^2$ the canonical bases $\delta_1 = (1, 0)$ and $\delta_2 = (0, 1)$ so that $\delta = [\delta_1, \delta_2]$ forms an ONB in the tangent space $T_{w(t)} \mathbf{R}^2 \cong \mathbf{R}^2$. Then these bases at different points of the curve are parallel to each other. Given a sphere $\mathbf{S}^2$, choose a point x on it and an ONB $\mathbf{e} = [\mathbf{e}_1, \mathbf{e}_2]$ in the tangent space $T_x \mathbf{S}^2$. We put the sphere on the plane so that x touches at the origin of the plane and the ONB $\mathbf{e}$ coincides with the ONB δ. Now we roll the sphere on the plane along the Brownian curve $w(t)$ without slipping. Suppose that the Brownian curve is traced in ink. Then the trace of $w(t)$ together with the ONB δ at $w(t)$ is transferred into a curve $X(t)$ on $\mathbf{S}^2$ with an ONB $\mathbf{e}(t) = [\mathbf{e}_1(t), \mathbf{e}_1(t)]$ in $T_{X(t)} \mathbf{S}^2$. Thus, a random curve $r(t) = (X(t), \mathbf{e}(t))$ on the orthonormal frame bundle $O(\mathbf{S}^2)$ is obtained and this is precisely the stochastic moving frame we want. We can see that the random curve $X(t)$ thus obtained is a Brownian motion on the sphere.

Now we give a formal definition. There is a notion of the *system of canonical horizontal vector fields* $A_1, \ldots, A_d$ on $O(M)$: For each $i = 1, \cdots, d$, $A_i(r)$ is a smooth vector field on $O(M)$ uniquely determined by the property that the integral curve, i.e. the solution, of the following ordinary differential equation (ODE)

$$\frac{dr(t)}{dt} = A_i(r(t)), \quad r(0) = r, \quad r = (x, \mathbf{e}), \quad \mathbf{e} = [\mathbf{e}_1, \ldots, \mathbf{e}_d]$$

coincides with the curve $r(t) = (x(t), \mathbf{e}(t))$, $\mathbf{e}(t) = [\mathbf{e}_1(t), \ldots, \mathbf{e}_d(t)]$, where $x(t)$ is the geodesic with $x(0) = x$ and $\frac{dx}{dt}|_{t=0} = \mathbf{e}_i$, and $\mathbf{e}(t) = [\mathbf{e}_1(t), \ldots, \mathbf{e}_d(t)]$ is the parallel translate, in the sense of Lévi-Civita, of $\mathbf{e} = [\mathbf{e}_1, \ldots, \mathbf{e}_d]$ along the curve $x(t)$.

Let $(W_0(\mathbf{R}^d), P^W)$ be the d-dimensional Wiener space. The stochastic moving frames on M starting at a frame r is, by definition, the solution

$$r(t) = (r(t, r; w)), \quad r(t, r; w) = (X(t, r; w), \mathbf{e}(t, r; w)) \tag{7}$$

of the following SDE on $O(M)$:

$$dr(t) = A_k(r(t)) \circ dw^k(t), \quad r(0) = r. \tag{8}$$

The assumption that solutions exist globally is equivalent to that the manifold is *stochastically complete*. We have the following important property of the stochastic moving frame under the right action of the structure group $O(d)$; for each $g \in O(d)$,

$$r(t, r; w)g = r(t, rg; g^{-1}w), \quad t \geq 0, \quad r \in O(M),$$

where $g^{-1}w \in W_0(\mathbf{R}^d)$ is defined by $\left(g^{-1}w\right)(t) = g^{-1}[w(t)]$. This implies, in particular, that

$$X(t, r; w) = X(t, rg; g^{-1}w), \quad t \geq 0, \quad r \in O(M), \quad g \in O(d).$$

By the rotation invariance of Wiener process, we have $g^{-1}w \stackrel{d}{=} w$, and hence

$$\{X(t, rg; w);\ t \geq 0\} \stackrel{d}{=} \{X(t, r; w);\ t \geq 0\}, \quad r \in O(M), \quad g \in O(d).$$

In other words, the law P_r on $\mathcal{C}_x([0,T] \to M), x = \pi(r)$, of $[t \mapsto X(t, r; w)]$ satisfies $P_{rg} = P_r$ for all $g \in O(d)$. This implies that P_r depends only on $x = \pi(r)$ and we may write $P_r = P_x$. Then the family $\{P_x\}$ defines a diffusion process on M. If we note the identity: $\sum_{k=1}^d A_k^2 \tilde{f} = \widetilde{\Delta_M f}$, which holds for any smooth function f on M and $\tilde{f} := f \circ \pi$, we can see that its generator coincides with $\frac{1}{2}\Delta_M$ so that it is a Brownian motion on M. In this way, the Brownian motion on a Riemannian manifold can be obtained as the projection on the base manifold of the stochastic moving frame, cf. [IW] for details.

Solutions of IVP (2) **by Wiener functional expectations. II**

We consider the case of heat equations on a Riemannian manifold M of dimension d and we set up on d-dimensional Wiener space the stochastic moving frame $\{r(t, r; w) = (X(t, r; w), \mathbf{e}(t, r; w))\}$ as above.

(4) We consider the case $L = \frac{1}{2}\Delta_M$. Then $u = e^{tL}f$ is given by

$$u(t, x) = E[f(X(t, r; w))], \quad x = \pi(r). \tag{9}$$

As we explained above, the right-hand side (RHS) depends only on $x = \pi(r)$.

(5) (Feynman-Kac formula) We consider the case of a Schrödinger operator $L = \frac{1}{2}\Delta_M - V$ where the potential $V(x)$ is a real Borel function bounded from below. Then $u = e^{tL}f$ is given by

$$u(t,x) = E\left[\exp\left\{-\int_0^t V(X(s,r;w))ds\right\} f(X(t,r;w))\right], \quad x = \pi(r). \tag{10}$$

(6) (Schrödinger operators with magnetic fields) We consider the case of operator

$$Lu = \frac{1}{2}\left[\Delta_M u + 2\sqrt{-1}(du,\theta) - (\sqrt{-1}d^*\theta + ||\theta||^2 + 2V)u\right] := H(\theta, V),$$

where θ is a real one-form (called a vector potential) and V is a real function (called a scalar potential). $(*, **)$ and $||*||$ are the Riemannian inner product and norm on the cotangent space $T^*_\cdot(M)$, respectively. d^* is the adjoint of exterior differentiation d so that $d^*\theta$ is a real function. Then $u = e^{tL}f$ is given by

$$\begin{aligned} u(t,x) = E\bigg[\exp\bigg\{\sqrt{-1}\int_0^t \bar{\theta}_i(X(s,r;w)) \circ dw^i(s) \\ - \int_0^t V(X(s,r;w))ds\bigg\} f(X(t,r;w))\bigg], \quad x = \pi(r). \end{aligned} \tag{11}$$

Here, $\bar{\theta}_i(r) = e_i^k\theta_k(x)$ if $r = (x, \mathbf{e})$, $\mathbf{e} = [\mathbf{e}_1, \ldots, \mathbf{e}_d]$ and $\theta(x) = \theta_i(x)dx^i$, $\mathbf{e}_i = e_i^k\frac{\partial}{\partial x^k}$ in a local coordinate $x = (x^1, \cdots, x^d)$. Obviously, $\bar{\theta}$ is defined independently of a particular choice of local coordinates.

(7) (Heat equations on vector bundles) We consider the case of the exterior product $\bigwedge T^*M$ of cotangent bundle T^*M, so that its section is a differential form on M, and the case of $L = \frac{1}{2}\Box$, where $\Box := -(d^*d + dd^*)$ is the de Rham-Hodge-Kodaira Laplacian acting on differential forms. We assume that M is compact and orientable.
The canonical isomorphism $\tilde{r} : \mathbf{R}^d \to T_xM$ associated with a frame $r = (x, \mathbf{e}) \in O(M)$ naturally induces an isomorphism $\tilde{r} : \mathbf{R}^d \to T_x^*M$ and an isomorphim $\tilde{r} : \bigwedge \mathbf{R}^d \to \bigwedge T_x^*M$, by sending bases δ_i and $\delta_{i_1} \wedge \cdots \wedge \delta_{i_p}$ to $\mathbf{f}^i$ and $\mathbf{f}^{i_1} \wedge \cdots \wedge \mathbf{f}^{i_p}$, for $i = 1, \cdots, d$ and $1 \le i_1 < \cdots < i_p \le d$, respectively, where $\mathbf{f} = [\mathbf{f}^1, \cdots, \mathbf{f}^d]$ is the ONB in T_x^*M dual to the ONB $\mathbf{e} = [\mathbf{e}_1, \ldots, \mathbf{e}_d]$ in T_xM. Here, we recall that the exterior product $\bigwedge \mathbf{R}^d = \sum_{p=0}^d \oplus \bigwedge^p \mathbf{R}^d$ is a 2^d-dimensional Euclidean space with the canonical base $\delta_{i_1} \wedge \cdots \wedge \delta_{i_p}$, forming an algebra under the exterior product $\wedge$. Let $\mathrm{End}(\bigwedge \mathbf{R}^d)$ be the algebra of linear tranformations on $\bigwedge \mathbf{R}^d$ and let $a_i^* \in \mathrm{End}(\bigwedge \mathbf{R}^d)$ be defined by

$$a_i^*(\lambda) = \delta_i \wedge \lambda, \quad \lambda \in \bigwedge \mathbf{R}^d, \quad i = 1, \ldots, d.$$

Let a_i be the dual of a_i^*. Then the system $a_{i_1}a_{i_2}\cdots a_{i_p}a_{j_1}^*a_{j_2}^*\cdots a_{j_q}^*$, where $1 \le i_1 < \cdots < i_p \le d$, $1 \le j_1 < \cdots < j_q \le d$, $p, q = 0, 1, \ldots, d$, forms a basis in $\mathrm{End}(\bigwedge \mathbf{R}^d)$. Let $J^{ijkl}(r)$ be the scalarization (equivariant representation) of the Riemann curvature tensor; in a local coordinate, $J^{ijkl}(r) = R_{\alpha\beta\gamma\delta}(x)e_i^\alpha e_j^\beta e_k^\gamma e_l^\delta$, $r = (x, \mathbf{e})$. Let $D_2[J](r) \in \mathrm{End}(\bigwedge \mathbf{R}^d)$ be defined by $D_2[J](r) = J^{ijkl}(r)a_i^* a_j a_k^* a_l$. We define an $\mathrm{End}(\bigwedge \mathbf{R}^d)$-valued process $t \to M(t, r; w)$ by the solution to the following ODE on $\mathrm{End}(\bigwedge \mathbf{R}^d)$:

$$\frac{dM(t)}{dt} = \frac{1}{2} D_2[J](r(t, r; w)) \cdot M(t), \quad M(0) = I.$$

Then, $u = e^{tL}f$, $f \in \bigwedge(M)$, is given by

$$u(t, x) = E\left[\tilde{r} M(t, r; w) \widetilde{r(t, r; w)}^{-1} f(X(t, r; w))\right], \quad r = (x, \mathbf{e}). \tag{12}$$

Note that $\tilde{r} : \bigwedge \mathbf{R}^d \to \bigwedge T_x^* M$, $\widetilde{r(t, r; w)}^{-1} : \bigwedge T_{X(t,r;w)}^* M \to \bigwedge \mathbf{R}^d$, and $f(X(t, r; w)) \in \bigwedge T_{X(t,r;w)}^* M$, so that the Itô functional under the expectation takes values in $\bigwedge T_x^* M$. Hence the expectation is well-defined and takes its value in $\bigwedge T_x^* M$. Also, it does not depend on a particular choice of $r \in O(M)$ over the point x and so, we may write it as $u(t, x)$. Cf. [IW], for details.

The solutions $u = e^{tL}f$ obtained by Wiener functional expectations as above can also possess *heat kernel representations* in the form

$$u(t, x) = \int_M \langle x | e^{tL} | y \rangle f(y) dy$$

where dy is the Riemannian volume of M. The heat kernel $\langle x | e^{tL} | y \rangle$ is usually constructed by the method of parametrix in PDE theory. Here, we would apply our probabilistic approach by Wiener functional expectations also to this problem; this is indeed possible by appealing to the Malliavin calculus on Wiener space.

4 Malliavin calculus on Wiener space

The Malliavin calculus is a differential and integral calculus on an infinite dimensional vector space endowed with a Gaussian measure. Here, we restrict ourselves to the case of the r-dimensional Wiener space $(W_0(\mathbf{R}^r), P^W)$; the Malliavin calculus in this case is well suited to the analysis of Itô fuctionals as we shall see. We would develop the Malliavin calculus as a Sobolev differential calculus on Wiener space by introducing a family of *Sobolev spaces* of Wiener functionals.

For a real separable Hilbert space E, we denote by $L_p(E)$, $1 \leq p < \infty$, the usual L^p-space of E-valued Wiener functionals. It is convenient to introduce the Fréchet space $L_{\infty-}(E) := \cap_{1<p<\infty} L_p(E)$ and its dual $L_{1+}(E) :=$

$\cup_{1<p<\infty} L_p(E)$, (the dual E^* being always identified with E by the Riesz theorem). When $E = \mathbf{R}$, $L_p(E)$, $(L_{\infty-}(E),\ L_{1+}(E))$ is denoted simply by L_p, (resp. $L_{\infty-}$, L_{1+}).

Typical differential operators are, the Gross-Malliavin-Shigekawa gradient operator D which sends an E-valued Wiener functional to an $H \otimes E$-valued Wiener functional, its dual operator or Skorohod operator D^* and the Ornstein-Uhrenbeck operator $L = -D^*D$. D is defined formally, for a Wiener functional $F = (F(w))$, by

$$\langle DF(w), h \otimes e\rangle_{H\otimes E} = \left\langle \lim_{\epsilon \to 0} (F(w + \epsilon h) - F(w))/\epsilon, e \right\rangle_E \quad h \in H,\ e \in E.$$

These operators are defined, first, for polynomial functionals and also, the fractional power $(I - L)^\alpha$, $\alpha \in \mathbf{R}$, is defined for polynomial functionals by using the Wiener chaos expansion; a polynomial functional F is a finite sum $F = \oplus \sum_n F_n$ where F_n is a polynomial functional in the chaos subspace of order n, and then, $(I - L)^\alpha F$ is defined to be $\oplus \sum_n (1 + n)^\alpha F_n$, which is also a polynomial functional. Let $\mathcal{P}(E)$ be the real vector space of all E-valued polynomial functionals. Noting that $\mathcal{P}(E) \subset L_{\infty-}(E)$, we define the norm $|| * ||_{p,\alpha}$ on $\mathcal{P}(E)$, $1 < p < \infty$, $\alpha \in \mathbf{R}$, by $||F||_{p,\alpha} = ||(I - L)^{\alpha/2} F||_p$. Let $\mathcal{P}(E)^*$ is the algebraic dual of $\mathcal{P}(E)$, which is a real vector space formed of all $\mathbf{R}$-linear mappings $T : F \in \mathcal{P}(E) \mapsto T(F) \in \mathbf{R}$. For $G \in L_2(E)$, we define $T_G \in \mathcal{P}(E)^*$ by $T_G(F) = E(\langle G, F\rangle_E)$, and identify G with T_G. Then, $L_2(E) \subset \mathcal{P}(E)^*$ and hence, $\mathcal{P}(E) \subset L_2(E) \subset \mathcal{P}(E)^*$. Define the norm on $\mathcal{P}(E)^*$ by setting

$$||T||_{p,\alpha} = \sup \{T(F); F \in \mathcal{P}(E),\ ||F||_{q,-\alpha} \le 1\},$$
$$1 < p < \infty,\ \alpha \in \mathbf{R},\ \frac{1}{p} + \frac{1}{q} = 1.$$

This definition is compatible with the norm defined already on $\mathcal{P}(E)$, which is a subspace of $\mathcal{P}(E)^*$ as we saw above. We now define the family of Sobolev spaces:

$$\mathbf{D}_p^\alpha(E) = \{\ T \in \mathcal{P}(E)^*;\ \ ||T||_{p,\alpha} < \infty\ \}, \quad 1 < p < \infty, \quad \alpha \in \mathbf{R}.$$

Then $\mathbf{D}_p^\alpha(E)$, endowed with the norm $|| * ||_{p,\alpha}$, is a real separable Banach space in which the space $\mathcal{P}(E)$ of E-valued polynomial functionals is densely included. Our definition given here is of course equivalent to the usual one given by the completion of $\mathcal{P}(E)$ with respect to the norm $|| * ||_{p,\alpha}$ (cf. e.g. [IW]); this elegant idea of avoiding the use of an abstract notion like completion is due to Itô [It].

We have $\mathbf{D}_p^0(E) = L_p(E)$, $\mathbf{D}_p^\alpha(E) \subset \mathbf{D}_{p'}^{\alpha'}(E)$ if $p' \le p,\ \alpha' \le \alpha$, and $\mathbf{D}_p^\alpha(E)^* = \mathbf{D}_q^{-\alpha}(E)$ if $p^{-1} + q^{-1} = 1$. Again, $\mathbf{D}_p^\alpha(E)$ in the case of $E = \mathbf{R}$ is denoted simply by $\mathbf{D}_p^\alpha$.

We set $\mathbf{D}_p^\infty(E) = \bigcap_{\alpha>0} \mathbf{D}_p^\alpha(E)$, $\mathbf{D}_p^{-\infty}(E) = \bigcup_{\alpha>0} \mathbf{D}_p^{-\alpha}(E)$. We also denote $\mathbf{D}_{\infty-}^\infty(E) = \bigcap_{1<p<\infty} \mathbf{D}_p^\infty(E)$ and $\mathbf{D}_{1+}^{-\infty}(E) = \bigcup_{1<p<\infty} \mathbf{D}_p^{-\infty}(E)$. Again, we omit E in these notations when $E = \mathbf{R}$.

Now, the differential operators D and L can be extended uniquely to act on the space $\mathbf{D}_{1+}^{-\infty}(E)$ and D^* on $\mathbf{D}_{1+}^{-\infty}(H \otimes E)$, so that $D : \mathbf{D}_p^{\alpha+1}(E) \to \mathbf{D}_p^{\alpha}(H \otimes E)$, $D^* : \mathbf{D}_p^{\alpha+1}(H \otimes E) \to \mathbf{D}_p^{\alpha}(E)$ and $L : \mathbf{D}_p^{\alpha+2}(E) \to \mathbf{D}_p^{\alpha}(E)$ are continuous for every $1 < p < \infty$ and $\alpha \in \mathbf{R}$.

An element F in the space $\mathbf{D}_p^{\infty}(E)$ may be called *smooth* because it is infinitely differentiable. Typical examples of smooth functionals are give by Itô functionals; if $F(w) = f(X(t,x;w))$, $t > 0, x \in M$, where $X = (X(t,x;w))$ is the solution of SDE (2.1) and f is a smooth function on M with a suitable growth condition at the point at infinity of M, then $F \in \mathbf{D}_{\infty-}^{\infty}$. It should be remarked, however, that smooth functionals are not continuous, in general. A typical example is Lévy's stochastic area $S(t,w) = \frac{1}{2}\int_0^t w^1(s)dw^2(s) - w^2(s)dw^1(s)$ on the two-dimensional Wiener space, which is an element in $\mathbf{D}_{\infty-}^{\infty}$ for each $t > 0$. However, there is no continuous function on the Wiener space which coincides with $S(t,w)$, P^W-a.e.; in fact, Sugita ([S 2]) showed more strongly that, on any separable Banach space continuously included in $W_0(\mathbf{R}^2)$ which has nonetheless P^W-measure 1, there exists no continuous function which coincides with $S(t,w)$, P^W-a.e. Thus, we see that Sobolev's embedding theorem no longer holds in our Sobolev differential calculus on Wiener space.

When $\alpha > 0$, some elements in $\mathbf{D}_p^{-\alpha}(E)$ are no more Wiener functionals in the sense of P^W-measurable functions; they are something like Schwartz distributions on Wiener space. We call them *generalized Wiener functionals*. The natural coupling between $F \in \mathbf{D}_p^{\alpha}$ and $G \in (\mathbf{D}_p^{\alpha})^* = \mathbf{D}_q^{-\alpha}$ is denoted by $E(FG)$; this notation is compatible with the usual one when $\alpha = 0$. In particular, the natural coupling of $F \in \mathbf{D}_{1+}^{-\infty}$ with $\mathbf{1} \in \mathbf{D}_{\infty-}^{\infty}$, $\mathbf{1}$ being the Wiener fuctional identically equal to 1, is denoted by $E(F)$ and is called the *generalized expectation* of F.

Typical examples of generalized Wiener functionals are obtained by the composite of Schwartz distributions on $\mathbf{R}^n$ (or on a manifold M) with a smooth $\mathbf{R}^n$-valued (resp. M-valued) Wiener functional which is *nondegenerate* in the sense given below. We mainly discuss the case of $\mathbf{R}^n$; the case of manifold can be discussed similarly and, indeed, can be reduced to the case of $\mathbf{R}^n$ by choosing a suitable local coordinate.

Let $F = (F^i(w)) \in \mathbf{D}_{\infty-}^{\infty}(\mathbf{R}^n)$ and define the *Malliavin covariance* $\sigma_F = (\sigma_F^{ij}(w))$ of F by

$$\sigma_F^{ij}(w) = \langle DF^i(w), DF^j(w)\rangle_H, \quad i,j = 1.\cdots,n.$$

It is nonnegative definite so that $\det \sigma_F \geq 0$, P^W-a.s.. We set $(\det \sigma_F)^{-1} = +\infty$ if $\det \sigma_F = 0$. For a domain U in $\mathbf{R}^n$, we say that F *is nondegenerate in* U if $1_U(F) \cdot (\det \sigma_F)^{-1} \in L_{\infty-} = \cap_{1<p<\infty} L_p$. Then, for every Schwartz distribution T on $\mathbf{R}^n$ with support contained in U, a generalized Wiener functional $T \circ F = T(F(w))$ can be defined uniquely as an element in $\mathbf{D}_{\infty-}^{-\infty} := \bigcap_{1<p<\infty} \mathbf{D}_p^{-\infty}$ so that this notion has the following two properties: (i) if T is given by a smooth function $f(x)$ on $\mathbf{R}^n$ with support contained in U, then

$T \circ F = f(F(w))$, (ii) if $T_\nu \to T$ in the sense of a Sobolev norm with negative differentiability index, then $T_\nu \circ F \to T \circ F$ in $\bigcap_{1<p<\infty} \mathbf{D}_p^{-\alpha}$ for some $\alpha > 0$. $T \circ F$ is called the *composite* of the Schwartz distribution T on $\mathbf{R}^n$ and the Wiener functional F, or the *pull-back* of the Schwartz distribution T on $\mathbf{R}^n$ by the Wiener map $F : W_0(\mathbf{R}^d) \to \mathbf{R}^n$.

In particular, for Dirac δ-functions δ_x, $x \in U$, $\delta_x(F)$ is defined as an element in $\mathbf{D}_{\infty-}^{-\infty}$. By the continuity property (ii), we can deduce that $x \in U \mapsto \delta_x(F) \in \mathbf{D}_{\infty-}^{-\alpha}$ is C^∞ and hence, $x \in U \mapsto E[\Phi \cdot \delta_x(F)]$ is a C^∞-function for $\Phi \in \mathbf{D}_{1+}^{\infty} =: \bigcup_{1<p<\infty} \mathbf{D}_p^{\infty}$. We can easily deduce that $p_F(x) := E(\delta_x(F))$, $x \in U$, is density in U, with respect to the Lebesgue measure, of the law of F and $E[\Phi \cdot \delta_x(F)] = p_F(x) E[\Phi | F = x]$, so that the conditional expectation of Φ given $F = x$ can be defined smoothly and pointwise on a set $\{ x \in U \mid p_F(x) > 0 \}$.

Let $r(t,r;w) = (X(t,r;w), \mathbf{e}(t,r,w))$ be the stochastic moving frame on a Riemannian manifold M as introduced in Section 2. Let δ_x, $x \in M$, be the Dirac delta function on M with pole at x defined with respect to the Riemannian volume dx. For each $t > 0$ and $r \in O(M)$, M-valued Wiener functional $X(t,r;w)$ is smooth and nondegenerate, so that the composite $\delta_y(X(t,r;w))$ is defined as an element in $\mathbf{D}_{\infty-}^{-\infty}$ for each $y \in M$. Using this notion, we can now give a probabilistic expression for heat kernels $\langle x | e^{tL} | y \rangle$ for heat equations studied in Section 2: here, heat kernels are defined with respect to the Riemannian volume dy on M, so that $u = e^{tL} f$ is given by $u(t,x) = \int_M \langle x | e^{tL} | y \rangle f(y) dy$.

In the case (4), i.e., $L = \frac{1}{2} \Delta_M$,

$$\langle x | e^{tL} | y \rangle = E[\delta_y(X(t,r;w))], \quad x = \pi(r).$$

By considering the right action of $O(d)$, we deduce as above that the expectation in the RHS does not depend on a particular choice of $r \in O(M)$ such that $x = \pi(r)$.

In the case of (5), i.e., $L = \frac{1}{2} \Delta_M - V$,

$$\langle x | e^{tL} | y \rangle = E\left[\exp\left\{ -\int_0^t V(X(s,r;w)) ds \right\} \cdot \delta_y(X(t,r;w)) \right], \quad x = \pi(r).$$

In the case of (6), i.e,, $L = H(\theta, V)$,

$$\begin{aligned} \langle x | e^{tL} | y \rangle = E\Bigg[\exp\Bigg\{ &\sqrt{-1} \int_0^t \bar{\theta}_i(X(s,r;w)) \circ dw^i(s) \\ &- \int_0^t V(X(s,r;w)) ds \Bigg\} \cdot \delta_y(X(t,r;w)) \Bigg], \quad x = \pi(r). \end{aligned}$$

In the case of (7), i.e., $L = \frac{1}{2} \Box$ acting on $\bigwedge(M)$, we need a more careful consideration; the heat kernel $\langle x | e^{tL} | y \rangle$ takes its value in the vector space $\mathrm{Hom}(\bigwedge T_y^* M, T_x^* M)$ formed of all linear mappings from $\bigwedge T_y^* M$ to $\bigwedge T_x^* M$, and it should be given formally by

$$\langle x|e^{tL}|y\rangle = E\left[\tilde{r}M(t,r;w)\widetilde{r(t,r;w)}^{-1}\cdot\delta_y(X(t,r;w))\right],\quad r=(x,\mathbf{e}).$$

However, the meaning of the generalized expectation in the RHS is not clear because the Wiener functional $\tilde{r}M(t,r;w)\widetilde{r(t,r;w)}^{-1}$ takes its value in the vector space $\mathrm{Hom}(\bigwedge T^*_{X(t,r;w)}M, \bigwedge T^*_x M)$, which is not a fixed vector space when w varies. We can overcome this difficulty by appealing to the *quasi-sure analysis* in the Malliavin calculus (cf. e.g., Malliavin [M], Lescot [Le], Itô [It]).

As we remarked above, a smooth Wiener functional cannot have a continuous modification, in general. It can possess however a modification, called *quasi-continuous modification* or *redefinition* of it. If $F \in \mathbf{D}^\infty_{\infty-}(\mathbf{R}^n)$ is non-degenerate in $U \subset \mathbf{R}^n$, then, as was shown by Airault- Malliavin ([AM]) and Sugita ([S 1]), there exists a finite Borel measure μ_x on the Wiener space $W_0(\mathbf{R}^d)$ associated uniquely with $x \in U$ such that, for every $\Phi \in \mathbf{D}^\infty_{1+}$, its quasi-continuous modification $\tilde{\Phi}$ is μ_x-integrable and the following identity holds:

$$\int_{W_0(\mathbf{R}^d)} \tilde{\Phi}(w)\mu_x(dw) = E[\Phi\cdot\delta_x(F)].$$

The measure μ_x has its full measure on the set $\mathcal{S}_x := \{\ w \in W_0(\mathbf{R}^d) \mid \tilde{F}(w) = x\ \}$. We may think of the measure μ_x as having the formal density $\delta_x(F)$, or we may think of it as the 'surface measure' on a 'hypersurface' $\mathcal{S}_x$ embedded in the Wiener space.

A similar theory can be developed in the case of $O(M)$ and we have a measure $\mu^{t,r}_y$ associated with $\delta_y(X(t,r;w))$, $y \in M$. If $\tilde{X}(t,r;w)$ is a quasi-continous modification of $X(t,r;w)$, then $\mu^{t,r}_y(\{\ w \mid \tilde{X}(t,r;w) \neq y\ \}) = 0$. Then, a quasi-continuous modification $\left[\tilde{r}M(t,r;w)\widetilde{r(t,r;w)}^{-1}\right]^{\sim}$ of $\tilde{r}M(t,r;w)\widetilde{r(t,r;w)}^{-1}$ takes values in $\mathrm{Hom}(\bigwedge T^*_y M, T^*_x M)$ quasi-surely and hence $\mu^{t,r}_y$-almost surely, so that it can be integrated by the measure $\mu^{t,r}_y$ to get an element in $\mathrm{Hom}(\bigwedge T^*_y M, T^*_x M)$. Now we have

$$\langle x|e^{tL}|y\rangle = \int_{W_0(\mathbf{R}^d)} \left[\tilde{r}M(t,r;w)\widetilde{r(t,r;w)}^{-1}\right]^{\sim} \mu^{t,r}_y(dw),\quad r=(x,\mathbf{e}).$$

5 Concluding remarks.

Probabilistic representations of heat kernels given above can be applied to study various properties of heat kernels; regularities, estimates, short time asymptotic expansions and so on. There are a huge amount of literatures and it is beyond the scope of this work to review of them. We would only refer to two survey articles [Ik] and [W] in which we can see several effective applications of our heat kernel representation by generalized Wiener functional expectations to the problems of McKean and Singer. Here we would content ourselves with giving a remark on quasi-sure analysis discussed above.

Quasi-sure analysis and the theory of rough paths by T. Lyons

We saw in Section 3 an example of heat kernel representations in which some refinement of generalized expectations is necessary and we did it by appealing to the quasi-sure analysis on Wiener space. We would remark that another approach is possible to such a refinement by using the theory of rough paths due to T. Lyons. We first recall this theory: (cf. [Ly])

Let $W_0(\mathbf{R^d}) := \{ \ w; \ [0,T] \ni t \mapsto w(t) \in \mathbf{R}^d, \text{ continuous}, \ w(0) = 0 \ \}$ be the d-dimensional path space as above and H be its Cameron-Martin subspace. Let

$$\mathcal{H}_d \left(\cong \mathbf{R}^{d(d+1)/2} \cong \mathbf{R}^d \times so(d) \right) := \left\{ \ x = (x^i, x^{(i,j)}) \mid 1 \le i < j \le d \ \right\}$$

endowed with the group multiplication $x \cdot y, \ x, y \in \mathcal{H}_d$, defined by $x \cdot y := z = (z^i, z^{(i,j)})$ where $z^i = x^i + y^i, \ z^{(i,j)} = x^{(i,j)} + y^{(i,j)} + \frac{1}{2}(x^i y^j - x^j y^i)$. $\mathcal{H}_d$ is called the free nilpotent Lie group with step 2 and d generators.

Let $\triangle_T = \{ \ (s,t) \mid 0 \le s \le t \le T \ \} (\subset [0,T]^2)$ and set

$$\begin{aligned} \Omega(\mathbf{R}^d) = \{ \ \omega = (\omega(s,t)) : \triangle_T \ni (s,t) \mapsto \omega(s,t) \in \mathcal{H}_d, \quad &\text{continuous}, \\ \omega(s,u) = \omega(s,t) \cdot \omega(t,u) \ \text{ for every } \ 0 \le s \le t \le u \le T \ \}&. \end{aligned}$$

Let $2 < p < 3$ and define a metric d_p on $\Omega(\mathbf{R}^d)$ by

$$d_p(\omega, \theta) = \sup_{0 \le s < t \le T} \left\{ \frac{|\omega^{(1)}(s,t) - \theta^{(1)}(s,t)|}{(t-s)^{1/p}} + \frac{|\omega^{(2)}(s,t) - \theta^{(2)}(s,t)|}{(t-s)^{2/p}} \right\}$$

where we denote $x^{(1)} = (x^i) \in \mathbf{R}^d$ and $x^{(2)} = (x^{(i,j)}) \in \mathbf{R}^{d(d-1)/2}$ for $x = (x^i, x^{(i,j)}) \in \mathcal{H}_d$.

Define a subspace $\Omega^{smooth}(\mathbf{R}^d)$ of $\Omega(\mathbf{R}^d)$ by setting $\Omega^{smooth}(\mathbf{R}^d) = \{ \ \omega(h) \mid h \in H \ \}$, where $\omega(h) \in \Omega(\mathbf{R}^d)$ is defined by

$$\begin{aligned} \omega(h)(s,t)^i &= h^i(t) - h^i(s), \\ \omega(h)(s,t)^{(i,j)} &= \frac{1}{2} \int \int_{s \le t_1 \le t_2 \le t} (\dot{h}^i(t_1) \dot{h}^j(t_2) - \dot{h}^j(t_1) \dot{h}^i(t_2)) dt_1 dt_2. \end{aligned}$$

Finally, we set

$$G\Omega_p(\mathbf{R}^d) = \overline{\Omega^{smooth}(\mathbf{R}^d))}^{d_p}$$

and call it the *space of geometric rouph paths*. It is a separable and complete metric space (i.e. a Polish space) under the metric d_p.

We can define a Wiener map $\rho : W_0(\mathbf{R}^d) \ni w \mapsto \rho[w] \in G\Omega_p(\mathbf{R}^d)$, by setting

$$\begin{aligned}
\rho[w](s,t)^i &= w^i(t) - w^i(s), \\
\rho[w](s,t)^{(i,j)} &= \frac{1}{2}\int\int_{s\le t_1\le t_2\le t} \left(dw^i(t_1)dw^j(t_2) - dw^j(t_1)dw^i(t_2)\right) \\
&= \frac{1}{2}\int_s^t \left([w^i(\tau) - w^i(s)]dw^j(\tau) - [w^j(\tau) - w^j(s)]dw^i(\tau)\right),
\end{aligned}$$

the integral being in the sense of Itô's stochastic integrals. The image measure $\rho_*(P^W)$ on $G\Omega_p(\mathbf{R}^d)$ is denoted by $\bar{P}_p$ or simply by $\bar{P}$ when p is well understood.

Define the projection $\pi : G\Omega_p(\mathbf{R}^d) \ni \omega \mapsto \pi[\omega] \in W_0(\mathbf{R}^d)$ by setting $\pi[\omega](t)^i = \omega(0,t)^i$, $t \in [0,T]$, $i = 1,\ldots,d$. Then, $\pi : G\Omega_p(\mathbf{R}^d) \to W_0(\mathbf{R}^d)$ is $\bar{P}$-measurable and the image measure $\pi_*(\bar{P})$ coincides with the Wiener measure P^W on $W_0(\mathbf{R}^d)$. Then every Wiener functional (i.e. P^W-measurable function) F on $W_0(\mathbf{R}^d)$ can be lifted to a $\bar{P}$-measurable function $\bar{F}$ on $G\Omega_p(\mathbf{R}^d)$ by setting $\bar{F} = F \circ \pi (:= \pi_*(F))$. We call $\bar{F}$ the *lift of* F *on* $G\Omega_p(\mathbf{R}^d)$. We have $\rho \circ \pi = \mathrm{id}|_{G\Omega_p(\mathbf{R}^d)}$, $\bar{P}$-a.s., and $\pi \circ \rho = \mathrm{id}|_{W_0(\mathbf{R}^d)}$, P^W-a.s., so that the lifting is obviously an isomorphism between P^W-measurable functions and $\bar{P}$-measurable functions, (strictly speaking, an isomorphism between the equivalence classes of functions coinciding each other almost surely.) F can be recovered from $\bar{F}$ as $F = \bar{F} \circ \rho (:= \rho_*(\bar{F}))$, P^W-a.s.

By this isomorphism, every notion concerning Wiener functionals can be lifted to that concerning $\bar{P}$-measurable functions on $G\Omega_p(\mathbf{R}^d)$; for example, differential operators D, D^*, L are lifted to $\bar{D} := \pi_* D \rho_*$, $\bar{D}^* := \pi_* D^* \rho_*$, $\bar{L} := \pi_* L \rho_*$, and so on. So the lifting gives isomorphisms between Sobolev spaces $\mathbf{D}_p^\alpha(E)$ on the the Wiener space and Sobolev spaces $\bar{\mathbf{D}}_p^\alpha(E)$ on the space of geometric rough paths.

We consider a SDE like (2.1) on $\mathbf{R}^n$, which is set up on the Wiener space $W_0(\mathbf{R}^d)$:

$$dX(t) = \sum_{i=1}^d A_i(X(t)) \circ dw^i(t) + A_0(X(t))dt, \quad X(0) = x \tag{13}$$

and denote the solution by $X = (X(t,x;w))$. We assume that all coefficients of SDE are smooth with bounded derivatives of all orders. By the *skeleton* of X, we mean the solution $\Xi = (\xi(t,x;h))$ of ODE for given $h \in H$:

$$\frac{d\xi}{dt}(t) = \sum_{i=1}^d A_i(\xi(t)) \cdot \dot{h}^i(t) + A_0(\xi(t)), \quad \xi(0) = x. \tag{14}$$

Note that $\xi(t,x;h)$ is, for fixed $t > 0$ and $x \in \mathbf{R}^n$, a smooth functional of $h \in H$ and also the map $\mathbf{R}^n \times H \ni (x,h) \mapsto [t \mapsto \xi(t,x;h)] \in \mathcal{C}([0,T] \to \mathbf{R}^n)$ is continuous. A skeleton is something like a restriction of X on H; since $P^W(H) = 0$, however, the restriction is usually meaningless.

Now, one of the fundamental theorems of T. Lyons can be stated as follows: There exists a *continuous* map

$$\phi : (x,\omega) \in \mathbf{R}^n \times G\Omega_p(\mathbf{R}^d) \mapsto \phi(x,\omega) := [t \mapsto \phi(x,\omega)(t)] \in \mathcal{C}([0,T] \mapsto \mathbf{R}^n),$$

and hence a continuous map $\phi_{(x,t)} : \omega \in G\Omega_p(\mathbf{R}^d) \mapsto \phi_{(x,t)}(\omega) := \phi(x,\omega)(t) \in \mathbf{R}^n$, for each fixed x and t, such that the following hold:

(i) $\xi(t,x;h) = \phi(x,\omega[h])(t)$, for all $h \in H$, $t \in [0,T]$ and $x \in \mathbf{R}^n$,
(ii) If $\bar{X} = \big(\bar{X}(t,x;\omega)\big)$ is the lift of $X = (X(t,x;w))$ on $G\Omega_p(\mathbf{R}^d)$, then it holds that $\phi(x,\omega)(t) = \bar{X}(t,x;\omega)$, $\bar{P}$-a.s.

Hence, $\bar{X} = \big(\bar{X}(t,x;\omega)\big)$ has a modification *which is continuous in* ω. This continuous modification can be used in place of a quasi-continuous modification in quasi-sure analysis. (We should note that the original result of T. Lyons is much stonger than what we stated above: He introduced the notion of a differential equation driven by rough paths and constructed its solution which is given as rough paths in $\mathbf{R}^n$. The function ϕ above is obtained from the first component of the solution. This theory of Lyons, indeed, is a pure real analysis.)

Consider the stochastic moving frame $r = (r(t,r;w))$ realized on the Wiener space as above. Then its lift $\bar{r} = (\bar{r}(t,r;\omega))$ has a continuous modification on $G\Omega_p(\mathbf{R}^d)$ which we denote by the same notation $\bar{r}$. The measure $\mu_y^{t,r}(dw)$ on $W_0(\mathbf{R}^d)$ is now lifted to a measure $\bar{\mu}_y^{t,r}(d\omega)$ on $G\Omega_p(\mathbf{R}^d)$ and it is supported on the *closed set* $\{\ \omega \mid \bar{r}(t,r;\omega) = y\ \}$. Now, returning to the heat kernel $\langle x|e^{tL}|y\rangle$ in the case of de Rham-Hodge-Kodaira Laplacian $\frac{1}{2}\Box$, it is easy to see that the lift $\tilde{r}\bar{M}(t,r;\omega)\widetilde{\bar{r}(t,r;\omega)}^{-1}$ is continuous in ω which takes values in $\mathrm{Hom}(\bigwedge T_y^*M, T_x^*M)$ on the set $\{\ \omega \mid \bar{r}(t,r;\omega) = y\ \}$, where $x = \pi(r)$. Then we have

$$\langle x|e^{tL}|y\rangle = \int_{\{\omega|\bar{r}(t,r;\omega)=y\}} \left[\tilde{r}\bar{M}(t,r;\omega)\widetilde{\bar{r}(t,r;\omega)}^{-1}\right] \bar{\mu}_y^{t,r}(d\omega), \quad r = (x,\mathbf{e}).$$

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