G-Expectation, G-Brownian Motion and Related Stochastic Calculus of Itô Type

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Dedicated to Professor Kiyosi Itô for His 90th Birthday

Summary. We introduce a notion of nonlinear expectation — G-expectation — generated by a nonlinear heat equation with a given infinitesimal generator G. We first discuss the notion of G-standard normal distribution. With this nonlinear distribution we can introduce our G-expectation under which the canonical process is a G-Brownian motion. We then establish the related stochastic calculus, especially stochastic integrals of Itô's type with respect to our G-Brownian motion and derive the related Itô's formula. We have also given the existence and uniqueness of stochastic differential equation under our G-expectation. As compared with our previous framework of g-expectations, the theory of G-expectation is intrinsic in the sense that it is not based on a given (linear) probability space.

Keywords: g-expectation, G-expectation, G-normal distribution, BSDE, SDE, nonlinear probability theory, nonlinear expectation, Brownian motion, Itô's stochastic calculus, Itô's integral, Itô's formula, Gaussian process, quadratic variation process.
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1 Introduction

In 1933 Andrei Kolmogorov published his Foundation of Probability Theory (Grundbegriffe der Wahrscheinlichkeitsrechnung) which set out the axiomatic

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basis for modern probability theory. The whole theory is built on the Measure Theory created by Émile Borel and Henry Lebesgue and profoundly developed by Radon and Fréchet. The triple $(\Omega, \mathcal{F}, \mathbf{P})$, i.e., a measurable space $(\Omega, \mathcal{F})$ equipped with a probability measure $\mathbf{P}$ becomes a standard notion which appears in most papers of probability and mathematical finance. The second important notion, which is in fact at an equivalent place as the probability measure itself, is the notion of expectation. The expectation $\mathbf{E}[X]$ of a $\mathcal{F}$-measurable random variable X is defined as the integral $\int_\Omega XdP$. A very original idea of Kolmogorov's Grundbegriffe is to use Radon–Nikodym theorem to introduce the conditional probability and the related conditional expectation under a given σ-algebra $\mathcal{G} \subset \mathcal{F}$. It is hard to imagine the present state of arts of probability theory, especially of stochastic processes, e.g., martingale theory, without such notion of conditional expectations. A given time information $(\mathcal{F}_t)_{t\geq 0}$ is so ingeniously and consistently combined with the related conditional expectations $\mathbf{E}[X|\mathcal{F}_t]_{t\geq 0}$. Itô's calculus—Itô's integration, Itô's formula and Itô's equation since 1942 [21], is, I think, the most beautiful discovery on this ground.

A very interesting problem is to develop a nonlinear expectation $\mathbb{E}[\cdot]$ under which we still have such notion of conditional expectation. A notion of g-expectation was introduced by Peng, 1997 (see [32] and [33]) in which the conditional expectation $\mathbb{E}^g[X|\mathcal{F}_t]_{t\geq 0}$ is the solution of the backward stochastic differential equation (BSDE), within the classical framework of Itô's calculus, with X as its given terminal condition and with a given real function g as the generator of the BSDE. driven by a Brownian motion defined on a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It is completely and perfectly characterized by the function g. The above conditional expectation is characterized by the following well-known condition

$$\mathbb{E}^g[\mathbb{E}^g[X|\mathcal{F}_t]\mathbf{I}_A] = \mathbb{E}^g[X\mathbf{I}_A], \quad \forall A \in \mathcal{F}_t.$$

Since then many results have been obtained in this subject (see, among others, [3], [4], [5], [6], [10], [11], [7], [8], [22], [23], [34], [38], [39], [41], [43], [24]).

In [37] (see also [36]), we have constructed a kind of filtration-consistent nonlinear expectations through the so-called nonlinear Markov chain. As compared with the framework of g-expectation, the theory of G-expectation is intrinsic, a meaning similar to the "intrinsic geometry". in the sense that it is not based on a classical probability space given a priori.

In this paper, we concentrate ourselves to a concrete case of the above situation and introduce a notion of G-expectation which is generated by a very simple one dimensional fully nonlinear heat equation, called G-heat equation, whose coefficient has only one parameter more than the classical heat equation considered since Bachelier 1900, Einstein 1905 to describe the Brownian motion. But this slight generalization changes the whole things. Firstly, a random variable X with "G-normal distribution" is defined via the heat equation. With this single nonlinear distribution we manage to introduce our G-expectation under which the canonical process is a G-Brownian motion.

We then establish the related stochastic calculus, especially stochastic integrals of Itô's type with respect to our G-Brownian motion. A new type of Itô's formula is obtained. We have also established the existence and uniqueness of stochastic differential equation under our G-stochastic calculus.

In this paper we concentrate ourselves to 1-dimensional G-Brownian motion. But our method of [37] can be applied to multi-dimensional G-normal distribution, G-Brownian motion and the related stochastic calculus. This will be given in [40].

Recently a new type of second order BSDE was proposed to give a probabilistic approach for fully nonlinear 2nd order PDE, see [9]. In finance a type of uncertain volatility model in which the PDE of Black-Scholes type was modified to a fully nonlinear model, see [26].

As indicated in Remark 3, the nonlinear expectations discussed in this paper are equivalent to the notion of coherent risk measures. This with the related conditional expectations $\mathbb{E}[\cdot|\mathcal{F}_t]_{t\geq 0}$ makes a dynamic risk measure: G-risk measure.

This paper is organized as follows: in Section 2, we recall the framework established in [37] and adapt it to our objective. In section 3 we introduce 1-dimensional standard G-normal distribution and discuss its main properties. In Section 4 we introduce 1-dimensional G-Brownian motion, the corresponding G-expectation and their main properties. We then can establish stochastic integral with respect to our G-Brownian motion of Itô type and the corresponding Itô's formula in Section 5 and the existence and uniqueness theorem of SDE driven by G-Brownian motion in Section 6.

2 Nonlinear expectation: a general framework

We briefly recall the notion of nonlinear expectations introduced in [37]. Following Daniell (see Daniell 1918 [13]) in his famous Daniell's integration, we begin with a vector lattice. Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real functions defined on Ω containing 1, namely, $\mathcal{H}$ is a linear space such that $1 \in \mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. $\mathcal{H}$ is a space of random variables. We assume the functions on $\mathcal{H}$ are all bounded. Notice that

$$a \wedge b = \min\{a, b\} = \frac{1}{2}(a + b - |a - b|), \quad a \vee b = -[(-a) \wedge (-b)].$$

Thus X, $Y \in \mathcal{H}$ implies that $X \wedge Y$, $X \vee Y$, $X^+ = X \vee 0$ and $X^- = (-X)^+$ are all in $\mathcal{H}$.

Definition 1. ***A nonlinear expectation*** $\mathbb{E}$ *is a functional* $\mathcal{H} \mapsto \mathbb{R}$ *satisfying the following properties*

(a) Monotonicity: *If* $X, Y \in \mathcal{H}$ *and* $X \geq Y$ *then* $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
(b) Preserving of constants: $\mathbb{E}[c] = c$.

In this paper we are interested in the expectations which satisfy

(c) Sub-additivity (or self-dominated property):

$$\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y], \quad \forall X, Y \in \mathcal{H}.$$

(d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X],\ \forall \lambda \geq 0,\ X \in \mathcal{H}.$

(e) Constant translatability: $\mathbb{E}[X + c] = \mathbb{E}[X] + c.$

Remark 2. The above condition (d) has an equivalent form: $\mathbb{E}[\lambda X] = \lambda^+ \mathbb{E}[X] + \lambda^- \mathbb{E}[-X]$. This form will be very convenient for the conditional expectations studied in this paper (see (vi) of Proposition 16).

Remark 3. We recall the notion of the above expectations satisfying (c)–(e) was systematically introduced by Artzner, Delbaen, Eber and Heath [1], [2], in the case where Ω is a finite set, and by Delbaen [14] in general situation with the notation of risk measure: $\rho(X) = \mathbb{E}[-X]$. See also in Huber [20] for even early study of this notion $\mathbb{E}$ (called upper expectation $\mathbf{E}^*$ in Ch.10 of [20]) in a finite set Ω. See Rosazza Gianin [43] or Peng [35], El Karoui & Barrieu [15], [16] for dynamic risk measures using g-expectations. Super-hedging and super pricing (see [17] and [18]) are also closely related to this formulation.

Remark 4. We observe that $\mathcal{H}_0 = \{X \in \mathcal{H},\ \mathbb{E}[|X|] = 0\}$ is a linear subspace of $\mathcal{H}$. To take $\mathcal{H}_0$ as our null space, we introduce the quotient space $\mathcal{H}/\mathcal{H}_0$. Observe that, for every $\{X\} \in \mathcal{H}/\mathcal{H}_0$ with a representation $X \in \mathcal{H}$, we can define an expectation $\mathbb{E}[\{X\}] := \mathbb{E}[X]$ which still satisfies (a)–(e) of Definition 1. Following [37], we set $\|X\| := \mathbb{E}[|X|],\ X \in \mathcal{H}/\mathcal{H}_0$. It is easy to check that $\mathcal{H}/\mathcal{H}_0$ is a normed space under $\|\cdot\|$. We then extend $\mathcal{H}/\mathcal{H}_0$ to its completion $[\mathcal{H}]$ under this norm. $([\mathcal{H}], \|\cdot\|)$ is a Banach space. The nonlinear expectation $\mathbb{E}[\cdot]$ can also be continuously extended from $\mathcal{H}/\mathcal{H}_0$ to $[\mathcal{H}]$, which satisfies (a)–(e).

For any $X \in \mathcal{H}$, the mappings

$$X^+(\omega) : \mathcal{H} \longmapsto \mathcal{H} \quad \text{and} \quad X^-(\omega) : \mathcal{H} \longmapsto \mathcal{H}$$

satisfy

$$|X^+ - Y^+| \leq |X - Y| \quad \text{and} \quad |X^- - Y^-| = |(-X)^+ - (-Y)^+| \leq |X - Y|.$$

Thus they are both contraction mappings under $\|\cdot\|$ and can be continuously extended to the Banach space $([\mathcal{H}], \|\cdot\|)$.

We define the partial order "$\geq$" in this Banach space.

Definition 5. *An element X in $([\mathcal{H}], \|\cdot\|)$ is said to be nonnegative, or $X \geq 0$, $0 \leq X$, if $X = X^+$. We also denote by $X \geq Y$, or $Y \leq X$, if $X - Y \geq 0$.*

It is easy to check that $X \geq Y$ and $Y \geq X$ implies $X = Y$ in $([\mathcal{H}], \|\cdot\|)$.

The nonlinear expectation $\mathbb{E}[\cdot]$ can be continuously extended to $([\mathcal{H}], \|\cdot\|)$ on which **(a)**–**(e)** still hold.

3 *G*-normal distributions

For a given positive integer n, we denote by $lip(\mathbb{R}^n)$ the space of all bounded and Lipschitz real functions on $\mathbb{R}^n$. In this section $\mathbb{R}$ is considered as Ω and $lip(\mathbb{R})$ as $\mathcal{H}$.

In the classical linear situation, a random variable $X(x) = x$ with standard normal distribution, i.e., $X \sim N(0,1)$, can be characterized by

$$E[\phi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \phi(x)dx, \quad \forall \phi \in lip(\mathbb{R}).$$

It is known since Bachelier 1900 and Einstein 1950 that $E[\phi(X)] = u(1,0)$ where $u = u(t,x)$ is the solution of the heat equation

$$\partial_t u = \frac{1}{2}\partial^2_{xx} u \tag{1}$$

with Cauchy condition $u(0,x) = \phi(x)$.

In this paper we set $G(a) = \frac{1}{2}(a^+ - \sigma_0^2 a^-)$, $a \in \mathbb{R}$, where $\sigma_0 \in [0,1]$ is fixed.

Definition 6. *A real valued random variable X with the standard **G-normal distribution** is characterized by its G-expectation defined by*

$$\mathbb{E}[\phi(X)] = P_1^G(\phi) := u(1,0), \quad \phi \in lip(\mathbb{R}) \mapsto \mathbb{R}$$

where $u = u(t,x)$ is a bounded continuous function on $[0,\infty) \times \mathbb{R}$ which is the (unique) viscosity solution of the following nonlinear parabolic partial differential equation (PDE)

$$\partial_t u - G(\partial^2_{xx} u) = 0, \; u(0,x) = \phi(x). \tag{2}$$

In case no confusion is caused, we often call the functional $P_1^G(\cdot)$ the standard G-normal distribution. When $\sigma_0 = 1$, the above PDE becomes the standard heat equation (1) and thus this G-distribution is just the classical normal distribution $N(0,1)$:

$$P_1^G(\phi) = P_1(\phi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \phi(x)dx.$$

Remark 7. The function G can be written as $G(a) = \frac{1}{2}\sup_{\sigma_0 \le \sigma \le 1} \sigma^2 a$, thus the nonlinear heat equation (2) is a special kind of Hamilton–Jacobi–Bellman equation. The existence and uniqueness of (2) in the sense of viscosity solution can be found in, for example, [12], [19], [31], [44], and [25] for $C^{1,2}$-solution if $\sigma_0 > 0$ (see also in [29] for elliptic cases). Readers who are unfamililar with the notion of viscosity solution of PDE can just consider, in the whole paper, the case $\sigma_0 > 0$, under which the solution u becomes a classical smooth function.

Remark 8. It is known that $u(t,\cdot)\in lip(\mathbb{R})$ (see e.g. [44] Ch.4, Prop.3.1. or [31] Lemma 3.1 for the Lipschitz continuity of $u(t,\cdot)$, or Lemma 5.5 and Proposition 5.6 in [36] for a more general conclusion). The boundedness is simply from the comparison theorem (or maximum principle) of this PDE. It is also easy to check that, for a given $\psi\in lip(\mathbb{R}^2)$, $P_1^G(\psi(x,\cdot))$ is still a bounded and Lipschitz function in x.

In general situations we have, from the comparison theorem of PDE,

$$P_1^G(\phi)\geq P_1(\phi),\ \forall\phi\in lip(\mathbb{R}). \tag{3}$$

The corresponding normal distribution with mean at $x\in\mathbb{R}$ and variance $t>0$ is $P_1^G(\phi(x+\sqrt{t}\times\cdot))$. Just like the classical situation, we have

Lemma 9. *For each $\phi\in lip(\mathbb{R})$, the function*

$$u(t,x)=P_1^G(\phi(x+\sqrt{t}\times\cdot)),\quad (t,x)\in[0,\infty)\times\mathbb{R} \tag{4}$$

is the solution of the nonlinear heat equation (2) with the initial condition $u(0,\cdot)=\phi(\cdot)$.

Proof. Let $u\in C([0,\infty)\times\mathbb{R})$ be the viscosity solution of (2) with $u(0,\cdot)=\phi(\cdot)\in lip(\mathbb{R})$. For a fixed $(\bar{t},\bar{x})\in(0,\infty)\times\mathbb{R}$, we denote $\bar{u}(t,x)=u(t\times\bar{t},x\sqrt{\bar{t}}+\bar{x})$. Then $\bar{u}$ is the viscosity solution of (2) with the initial condition $\bar{u}(0,x)=\phi(x\sqrt{\bar{t}}+\bar{x})$. Indeed, let ψ be a $C^{1,2}$ function on $(0,\infty)\times\mathbb{R}$ such that $\psi\geq\bar{u}$ (resp. $\psi\leq\bar{u}$) and $\psi(\tau,\xi)=\bar{u}(\tau,\xi)$ for a fixed $(\tau,\xi)\in(0,\infty)\times\mathbb{R}$. We have $\psi(\frac{t}{\bar{t}},\frac{x-\bar{x}}{\sqrt{\bar{t}}})\geq u(t,x)$, for all (t,x) and

$$\psi(\frac{t}{\bar{t}},\frac{x-\bar{x}}{\sqrt{\bar{t}}})=u(t,x),\ \text{at}\ (t,x)=(\tau\bar{t},\xi\sqrt{\bar{t}}+\bar{x}).$$

Since u is the viscosity solution of (2), at the point $(t,x)=(\tau\bar{t},\xi\sqrt{\bar{t}}+\bar{x})$, we have

$$\frac{\partial\psi(\frac{t}{\bar{t}},\frac{x-\bar{x}}{\sqrt{\bar{t}}})}{\partial t}-G(\frac{\partial^2\psi(\frac{t}{\bar{t}},\frac{x-\bar{x}}{\sqrt{\bar{t}}})}{\partial x^2})\leq 0\ \ (\text{resp.}\ \geq 0).$$

But since G is positive homogenous, i.e., $G(\lambda a)=\lambda G(a)$, we thus derive

$$(\frac{\partial\psi(t,x)}{\partial t}-G(\frac{\partial^2\psi(t,x)}{\partial x^2}))|_{(t,x)=(\tau,\xi)}\leq 0\ \ (\text{resp.}\ \geq 0).$$

This implies that $\bar{u}$ is the viscosity subsolution (resp. supersolution) of (2). According to the definition of $P^G(\cdot)$ we obtain (4).

Definition 10. *We denote*

$$P_t^G(\phi)(x)=P_1^G(\phi(x+\sqrt{t}\times\cdot))=u(t,x),\quad (t,x)\in[0,\infty)\times\mathbb{R}. \tag{5}$$

From the above lemma, for each $\phi \in lip(\mathbb{R})$, we have the following Kolmogorov–Chapman chain rule:

$$P_t^G(P_s^G(\phi))(x) = P_{t+s}^G(\phi)(x), \quad s,t \in [0,\infty),\ x \in \mathbb{R}. \tag{6}$$

Such type of nonlinear semigroup was studied in Nisio 1976 [27], [28].

Proposition 11. *For each $t > 0$, the G-normal distribution P_t^G is a nonlinear expectation on $\mathcal{H} = lip(\mathbb{R})$, with $\Omega = \mathbb{R}$, satisfying (a)–(e) of Definition 1. The corresponding completion space $[\mathcal{H}] = [lip(\mathbb{R})]_t$ under the norm $\|\phi\|_t := P_t^G(|\phi|)(0)$ contains $\phi(x) = x^n$, $n = 1, 2, \cdots$, as well as $x^n\psi$, $\psi \in lip(\mathbb{R})$ as its special elements. Relation (5) still holds. We also have the following properties:*
(1) Central symmetric: $P_t^G(\phi(\cdot)) = P_t^G(\phi(-\cdot))$;
(2) For each convex $\phi \in [lip(\mathbb{R})]$ we have

$$P_t^G(\phi)(0) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \phi(x) \exp(-\frac{x^2}{2t}) dx;$$

For each concave ϕ, we have, for $\sigma_0 > 0$,

$$P_t^G(\phi)(0) = \frac{1}{\sqrt{2\pi t}\sigma_0} \int_{-\infty}^{\infty} \phi(x) \exp(-\frac{x^2}{2t\sigma_0^2}) dx,$$

and $P_t^G(\phi)(0) = \phi(0)$ for $\sigma_0 = 0$. In particular, we have

$$P_t^G((x)_{x\in\mathbb{R}}) = 0, \quad P_t^G((x^{2n+1})_{x\in\mathbb{R}}) = P_t^G((-x^{2n+1})_{x\in\mathbb{R}}),\ n = 1, 2, \cdots,$$
$$P_t^G((x^2)_{x\in\mathbb{R}}) = t, \quad P_t^G((-x^2)_{x\in\mathbb{R}}) = -\sigma_0^2 t.$$

Remark 12. Corresponding to the above four expressions, a random variable X with the G-normal distribution P_t^G satisfies

$$\mathbb{E}[X] = 0, \quad \mathbb{E}[X^{2n+1}] = \mathbb{E}[-X^{2n+1}],$$
$$\mathbb{E}[X^2] = t, \quad \mathbb{E}[-X^2] = -\sigma_0^2 t.$$

See the next section for a detail study.

4 1-dimensional *G*-Brownian motion under *G*-expectation

In the rest of this paper, we denote by $\Omega = C_0(\mathbb{R}^+)$ the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

We set, for each $t \in [0,\infty)$,

$$\begin{aligned} \mathbf{W}_t &:= \{\omega_{\cdot\wedge t} : \omega \in \mathbf{\Omega}\}, \\ \mathcal{F}_t &:= \mathcal{B}_t(\mathbf{W}) = \mathcal{B}(\mathbf{W}_t), \\ \mathcal{F}_{t+} &:= \mathcal{B}_{t+}(\mathbf{W}) = \bigcap_{s>t} \mathcal{B}_s(\mathbf{W}), \\ \mathcal{F} &:= \bigvee_{s>t} \mathcal{F}_s. \end{aligned}$$

$(\mathbf{\Omega},\mathcal{F})$ is the canonical space equipped with the natural filtration and $\omega = (\omega_t)_{t\geq 0}$ is the corresponding canonical process.

For each fixed $T \geq 0$, we consider the following space of random variables:

$$L^0_{ip}(\mathcal{F}_T) := \{X(\omega) = \phi(\omega_{t_1}, \cdots, \omega_{t_m}), \forall m \geq 1, \\ t_1, \cdots, t_m \in [0,T], \forall \phi \in lip(\mathbb{R}^m)\}.$$

It is clear that $L^0_{ip}(\mathcal{F}_t) \subseteq L^0_{ip}(\mathcal{F}_T)$, for $t \leq T$. We also denote

$$L^0_{ip}(\mathcal{F}) := \bigcup_{n=1}^{\infty} L^0_{ip}(\mathcal{F}_n).$$

Remark 13. It is clear that $lip(\mathbb{R}^m)$ and then $L^0_{ip}(\mathcal{F}_T)$ and $L^0_{ip}(\mathcal{F})$ are vector lattices. Moreover, since $\phi, \psi \in lip(\mathbb{R}^m)$ implies $\phi \cdot \psi \in lip(\mathbb{R}^m)$ thus X, $Y \in L^0_{ip}(\mathcal{F}_T)$ implies $X \cdot Y \in L^0_{ip}(\mathcal{F}_T)$.

We will consider the canonical space and set $B_t(\omega) = \omega_t$, $t \in [0,\infty)$, for $\omega \in \Omega$.

Definition 14. *The canonical process B is called a G-**Brownian** motion under a nonlinear expectation $\mathbb{E}$ defined on $L^0_{ip}(\mathcal{F})$ if for each $T > 0$, $m = 1, 2, \cdots$, and for each $\phi \in lip(\mathbb{R}^m)$, $0 \leq t_1 < \cdots < t_m \leq T$, we have*

$$\mathbb{E}[\phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})] = \phi_m,$$

where $\phi_m \in \mathbb{R}$ is obtained via the following procedure:

$$\begin{aligned} \phi_1(x_1, \cdots, x_{m-1}) &= P^G_{t_m - t_{m-1}}(\phi(x_1, \cdots, x_{m-1}, \cdot)); \\ \phi_2(x_1, \cdots, x_{m-2}) &= P^G_{t_{m-1} - t_{m-2}}(\phi_1(x_1, \cdots, x_{m-2}, \cdot)); \\ &\vdots \\ \phi_{m-1}(x_1) &= P^G_{t_2 - t_1}(\phi_{m-2}(x_1, \cdot)); \\ \phi_m &= P^G_{t_1}(\phi_{m-1}(\cdot)). \end{aligned}$$

The related conditional expectation of $X = \phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$ under $\mathcal{F}_{t_j}$ is defined by

$$\begin{aligned} \mathbb{E}[X|\mathcal{F}_{t_j}] &= \mathbb{E}[\phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})|\mathcal{F}_{t_j}] \\ &= \phi_{m-j}(B_{t_1}, \cdots, B_{t_j} - B_{t_{j-1}}). \end{aligned} \tag{7}$$

It is proved in [37] that $\mathbb{E}[\cdot]$ consistently defines a nonlinear expectation on the vector lattice $L^0_{ip}(\mathcal{F}_T)$ as well as on $L^0_{ip}(\mathcal{F})$ satisfying (a)–(e) in Definition 1. It follows that $\mathbb{E}[|X|]$, $X \in L^0_{ip}(\mathcal{F}_T)$ (resp. $L^0_{ip}(\mathcal{F})$) forms a norm and that $L^0_{ip}(\mathcal{F}_T)$ (resp. $L^0_{ip}(\mathcal{F})$) can be continuously extended to a Banach space, denoted by $L^1_G(\mathcal{F}_T)$ (resp. $L^1_G(\mathcal{F})$). For each $0 \le t \le T < \infty$, we have $L^1_G(\mathcal{F}_t) \subseteq L^1_G(\mathcal{F}_T) \subset L^1_G(\mathcal{F})$. It is easy to check that, in $L^1_G(\mathcal{F}_T)$ (resp. $L^1_G(\mathcal{F})$), $\mathbb{E}[\cdot]$ still satisfies (a)–(e) in Definition 1.

Definition 15. *The expectation $\mathbb{E}[\cdot] : L^1_G(\mathcal{F}) \mapsto \mathbb{R}$ introduced through above procedure is called **G-expectation**. The corresponding canonical process B is called a G-Brownian motion under $\mathbb{E}[\cdot]$.*

For a given $p > 1$, we also denote $L^p_G(\mathcal{F}) = \{X \in L^1_G(\mathcal{F}) | X|^p \in L^1_G(\mathcal{F})\}$. $L^p_G(\mathcal{F})$ is also a Banach space under the norm $\|X\|_p := (\mathbb{E}[|X|^p])^{1/p}$. We have (see Appendix)

$$\|X + Y\|_p \le \|X\|_p + \|Y\|_p$$

and, for each $X \in L^p_G$, $Y \in L^q_G(Q)$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|XY\| = \mathbb{E}[|XY|] \le \|X\|_p \|X\|_q .$$

With this we have $\|X\|_p \le \|X\|_{p'}$ if $p \le p'$.

We now consider the conditional expectation introduced in (7). For each fixed $t = t_j \le T$, the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t] : L^0_{ip}(\mathcal{F}_T) \mapsto L^0_{ip}(\mathcal{F}_t)$ is a continuous mapping under $\|\cdot\|$ since $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]] = \mathbb{E}[X]$, $X \in L^0_{ip}(\mathcal{F}_T)$ and

$$\mathbb{E}[\mathbb{E}[X|\mathcal{F}_t] - \mathbb{E}[Y|\mathcal{F}_t]] \le \mathbb{E}[X - Y],$$
$$\|\mathbb{E}[X|\mathcal{F}_t] - \mathbb{E}[Y|\mathcal{F}_t]\| \le \|X - Y\| .$$

It follows that $\mathbb{E}[\cdot|\mathcal{F}_t]$ can be also extended as a continuous mapping $L^1_G(\mathcal{F}_T) \mapsto L^1_G(\mathcal{F}_t)$. If the above T is not fixed, then we can obtain $\mathbb{E}[\cdot|\mathcal{F}_t] : L^1_G(\mathcal{F}) \mapsto L^1_G(\mathcal{F}_t)$.

Proposition 16. *We list the properties of $\mathbb{E}[\cdot|\mathcal{F}_t]$ that hold in $L^0_{ip}(\mathcal{F}_T)$ and still hold for X, $Y \in L^1_G(\mathcal{F})$:*

(i) $\mathbb{E}[X|\mathcal{F}_t] = X$, *for* $X \in L^1_G(\mathcal{F}_t)$, $t \le T$.
(ii) *If* $X \ge Y$, *then* $\mathbb{E}[X|\mathcal{F}_t] \ge \mathbb{E}[Y|\mathcal{F}_t]$.
(iii) $\mathbb{E}[X|\mathcal{F}_t] - \mathbb{E}[Y|\mathcal{F}_t] \le \mathbb{E}[X - Y|\mathcal{F}_t]$.
(iv) $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[X|\mathcal{F}_{t\wedge s}]$, $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]] = \mathbb{E}[X]$.
(v) $\mathbb{E}[X + \eta|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{F}_t] + \eta$, $\eta \in L^1_G(\mathcal{F}_t)$.
(vi) $\mathbb{E}[\eta X|\mathcal{F}_t] = \eta^+\mathbb{E}[X|\mathcal{F}_t] + \eta^-\mathbb{E}[-X|\mathcal{F}_t]$, *for each bounded* $\eta \in L^1_G(\mathcal{F}_t)$.
(vii) *For each* $X \in L^1_G(\mathcal{F}^t_T)$, $\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}[X]$,

where $L^1_G(\mathcal{F}^t_T)$ is the extension, under $\|\cdot\|$, of $L^0_{ip}(\mathcal{F}^t_T)$ which consists of random variables of the form $\phi(B_{t_1} - B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$, $m = 1, 2, \cdots$, $\phi \in lip(\mathbb{R}^m)$, $t_1, \cdots, t_m \in [t, T]$. Condition (vi) is the positive homogeneity, see Remark 2.

Definition 17. *An $X \in L_G^1(\mathcal{F})$ is said to be independent of $\mathcal{F}_t$ under the G-expectation $\mathbb{E}$ for some given $t \in [0,\infty)$, if for each real function Φ suitably defined on $\mathbb{R}$ such that $\Phi(X) \in L_G^1(\mathcal{F})$ we have*

$$\mathbb{E}[\Phi(X)|\mathcal{F}_t] = \mathbb{E}[\Phi(X)].$$

Remark 18. It is clear that all elements in $L_G^1(\mathcal{F})$ are independent of $\mathcal{F}_0$. Just like the classical situation, the increments of G-Brownian motion $(B_{t+s} - B_s)_{t\geq 0}$ is independent of $\mathcal{F}_s$. In fact it is a new G-Brownian motion since, just like the classical situation, the increments of B are identically distributed.

Example 19. For each $n = 0, 1, 2, \cdots$, $0 \leq s - t$, we have $\mathbb{E}[B_t - B_s|\mathcal{F}_s] = 0$ and, for $n = 1, 2, \cdots$,

$$\mathbb{E}[|B_t - B_s|^n|\mathcal{F}_s] = \mathbb{E}[|B_{t-s}|^{2n}] = \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}|x|^n \exp(-\frac{x^2}{2(t-s)})dx.$$

But we have

$$\mathbb{E}[-|B_t - B_s|^n|\mathcal{F}_s] = \mathbb{E}[-|B_{t-s}|^n] = -\sigma_0^n\mathbb{E}[|B_{t-s}|^n].$$

Exactly as in classical cases, we have

$$\begin{aligned}
\mathbb{E}[(B_t - B_s)^2|\mathcal{F}_s] &= t - s, \quad \mathbb{E}[(B_t - B_s)^4|\mathcal{F}_s] = 3(t-s)^2,\\
\mathbb{E}[(B_t - B_s)^6|\mathcal{F}_s] &= 15(t-s)^3, \quad \mathbb{E}[(B_t - B_s)^8|\mathcal{F}_s] = 105(t-s)^4,\\
\mathbb{E}[|B_t - B_s||\mathcal{F}_s] &= \frac{\sqrt{2(t-s)}}{\sqrt{\pi}}, \quad \mathbb{E}[|B_t - B_s|^3|\mathcal{F}_s] = \frac{2\sqrt{2}(t-s)^{3/2}}{\sqrt{\pi}},\\
\mathbb{E}[|B_t - B_s|^5|\mathcal{F}_s] &= 8\frac{\sqrt{2}(t-s)^{5/2}}{\sqrt{\pi}}.
\end{aligned}$$

Example 20. For each $n = 1, 2, \cdots$, $0 \leq s \leq t < T$ and $X \in L_G^1(\mathcal{F}_s)$, since $\mathbb{E}[B_{T-t}^{2n-1}] = \mathbb{E}[-B_{T-t}^{2n-1}]$, we have, by (vi) of Proposition 16,

$$\begin{aligned}
\mathbb{E}[X(B_T - B_t)^{2n-1}] &= \mathbb{E}[X^+\mathbb{E}[(B_T - B_t)^{2n-1}|\mathcal{F}_t]\\
&\quad + X^-\mathbb{E}[-(B_T - B_t)^{2n-1}|\mathcal{F}_t]]\\
&= \mathbb{E}[|X|] \cdot \mathbb{E}[B_{T-t}^{2n-1}],\\
\mathbb{E}[X(B_T - B_t)|\mathcal{F}_s] &= \mathbb{E}[-X(B_T - B_t)|\mathcal{F}_s] = 0.
\end{aligned}$$

We also have

$$\mathbb{E}[X(B_T - B_t)^2|\mathcal{F}_t] = X^+(T-t) - \sigma_0^2X^-(T-t).$$

Remark 21. It is clear that we can define an expectation $E[\cdot]$ on $L_{ip}^0(\mathcal{F})$ in the same way as in Definition 14 with the standard normal distribution $P_1(\cdot)$ in the place of $P_1^G(\cdot)$. Since $P_1(\cdot)$ is dominated by $P_1^G(\cdot)$ in the sense $P_1(\phi) - P_1(\psi) \leq$

$P_1^G(\phi-\psi)$, then $E[\cdot]$ can be continuously extended to $L_G^1(\mathcal{F})$. $E[\cdot]$ is a linear expectation under which $(B_t)_{t\geq 0}$ behaves as a Brownian motion. We have

$$E[X]\leq \mathbb{E}[X],\quad \forall X\in L_G^1(\mathcal{F}). \tag{8}$$

In particular, $\mathbb{E}[B_{T-t}^{2n-1}]=\mathbb{E}[-B_{T-t}^{2n-1}]\geq E[-B_{T-t}^{2n-1}]=0$. Such kind of extension under a domination relation was discussed in details in [37].

The following property is very useful

Proposition 22. *Let $X,Y\in L_G^1(\mathcal{F})$ be such that $\mathbb{E}[Y]=-\mathbb{E}[-Y]$ (thus $\mathbb{E}[Y]=E[Y]$), then we have*

$$\mathbb{E}[X+Y]=\mathbb{E}[X]+\mathbb{E}[Y].$$

In particular, if $\mathbb{E}[Y]=\mathbb{E}[-Y]=0$, then $\mathbb{E}[X+Y]=\mathbb{E}[X]$.

Proof. It is simply because we have $\mathbb{E}[X+Y]\leq\mathbb{E}[X]+\mathbb{E}[Y]$ and

$$\mathbb{E}[X+Y]\geq \mathbb{E}[X]-\mathbb{E}[-Y]=\mathbb{E}[X]+\mathbb{E}[Y].$$

Example 23. We have

$$\begin{aligned}\mathbb{E}[B_t^2-B_s^2|\mathcal{F}_s]&=\mathbb{E}[(B_t-B_s+B_s)^2-B_s^2|\mathcal{F}_s]\\&=E[(B_t-B_s)^2+2(B_t-B_s)B_s|\mathcal{F}_s]\\&=t-s,\end{aligned}$$

since $2(B_t-B_s)B_s$ satisfies the condition for Y in Proposition 22, and

$$\begin{aligned}\mathbb{E}[(B_t^2-B_s^2)^2|\mathcal{F}_s]&=\mathbb{E}[\{(B_t-B_s+B_s)^2-B_s^2\}^2|\mathcal{F}_s]\\&=\mathbb{E}[\{(B_t-B_s)^2+2(B_t-B_s)B_s\}^2|\mathcal{F}_s]\\&=\mathbb{E}[(B_t-B_s)^4+4(B_t-B_s)^3B_s+4(B_t-B_s)^2B_s^2|\mathcal{F}_s]\\&\leq\mathbb{E}[(B_t-B_s)^4]+4\mathbb{E}[|B_t-B_s|^3]|B_s|+4(t-s)B_s^2\\&=3(t-s)^2+8(t-s)^{3/2}|B_s|+4(t-s)B_s^2.\end{aligned}$$

5 Itô's integral of G-Brownian motion

5.1 Bochner's integral

Definition 24. *For $T\in\mathbb{R}_+$, a partition π_T of $[0,T]$ is a finite ordered subset $\pi=\{t_1,\cdots,t_N\}$ such that $0=t_0<t_1<\cdots<t_N=T$. We denote*

$$\mu(\pi_T)=\max\{|t_{i+1}-t_i|,i=0,1,\cdots,N-1\}.$$

We use $\pi_T^N=\{t_0^N<t_1^N<\cdots<t_N^N\}$ to denote a sequence of partitions of $[0,T]$ such that $\lim_{N\to\infty}\mu(\pi_T^N)=0$.

Let $p \geq 1$ be fixed. We consider the following type of simple processes: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0, T]$, we set

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_G^p(\mathcal{F}_{t_i})$, $i = 0, 1, 2, \cdots, N-1$, are given. The collection of these type of processes is denoted by $M_G^{p,0}(0, T)$.

Definition 25. *For an $\eta \in M_G^{1,0}(0, T)$ with $\eta_t = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j, t_{j+1})}(t)$, the related Bochner integral is*

$$\int_0^T \eta_t(\omega) dt = \sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j).$$

Remark 26. We set, for each $\eta \in M_G^{1,0}(0, T)$,

$$\tilde{\mathbb{E}}_T[\eta] := \frac{1}{T} \int_0^T \mathbb{E}[\eta_t] dt = \frac{1}{T} \sum_{j=0}^{N-1} \mathbb{E}[\xi_j(\omega)](t_{j+1} - t_j).$$

It is easy to check that $\tilde{\mathbb{E}}_T : M_G^{1,0}(0, T) \longmapsto \mathbb{R}$ forms a nonlinear expectation satisfying (a)–(e) of Definition 1. By Remark 4, we can introduce a natural norm $\|\eta\|_T^1 = \tilde{\mathbb{E}}_T[|\eta|] = \frac{1}{T} \int_0^T \mathbb{E}[|\eta_t|] dt$. Under this norm $M_G^{1,0}(0, T)$ can be continuously extended to $M_G^1(0, T)$ which is a Banach space.

Definition 27. *For each $p \geq 1$, we will denote by $M_G^p(0, T)$ the completion of $M_G^{p,0}(0, T)$ under the norm*

$$(\frac{1}{T} \int_0^T \|\eta_t^p\| \, dt)^{1/p} = \left(\frac{1}{T} \sum_{j=0}^{N-1} \mathbb{E}[|\xi_j(\omega)|^p](t_{j+1} - t_j) \right)^{1/p}.$$

We observe that,

$$\mathbb{E}[| \int_0^T \eta_t(\omega) dt |] \leq \sum_{j=0}^{N-1} \|\xi_j(\omega)\| \, (t_{j+1} - t_j) = \int_0^T \mathbb{E}[|\eta_t|] dt.$$

We then have

Proposition 28. *The linear mapping $\int_0^T \eta_t(\omega) dt : M_G^{1,0}(0, T) \mapsto L_G^1(\mathcal{F}_T)$ is continuous. and thus can be continuously extended to $M_G^1(0, T) \mapsto L_G^1(\mathcal{F}_T)$. We still denote this extended mapping by $\int_0^T \eta_t(\omega) dt$, $\eta \in M_G^1(0, T)$. We have*

$$\mathbb{E}[| \int_0^T \eta_t(\omega) dt |] \leq \int_0^T \mathbb{E}[|\eta_t|] dt, \quad \forall \eta \in M_G^1(0, T). \tag{9}$$

Since $M_G^1(0, T) \supset M_G^p(0, T)$, for $p \geq 1$, this definition holds for $\eta \in M_G^p(0, T)$.

5.2 Itô's integral of G-Brownian motion

Definition 29. *For each $\eta \in M_G^{2,0}(0,T)$ with the form*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j, t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

Lemma 30. *The mapping $I : M_G^{2,0}(0,T) \longmapsto L_G^2(\mathcal{F}_T)$ is a linear continuous mapping and thus can be continuously extended to $I : M_G^2(0,T) \longmapsto L_G^2(\mathcal{F}_T)$. In fact we have*

$$\mathbb{E}[\int_0^T \eta(s) dB_s] = 0, \tag{10}$$

$$\mathbb{E}[(\int_0^T \eta(s) dB_s)^2] \leq \int_0^T \mathbb{E}[(\eta(t))^2] dt. \tag{11}$$

Definition 31. *We define, for a fixed $\eta \in M_G^2(0,T)$, the stochastic integral*

$$\int_0^T \eta(s) dB_s := I(\eta).$$

It is clear that (10), (11) still hold for $\eta \in M_G^2(0,T)$.

Proof of Lemma 30. From Example 20, for each j,

$$\mathbb{E}[\xi_j (B_{t_{j+1}} - B_{t_j}) | \mathcal{F}_{t_j}] = 0.$$

We have

$$\begin{aligned}
\mathbb{E}[\int_0^T \eta(s) dB_s] &= \mathbb{E}[\int_0^{t_{N-1}} \eta(s) dB_s + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})] \\
&= \mathbb{E}[\int_0^{t_{N-1}} \eta(s) dB_s + \mathbb{E}[\xi_{N-1}(B_{t_N} - B_{t_{N-1}}) | \mathcal{F}_{t_{N-1}}]] \\
&= \mathbb{E}[\int_0^{t_{N-1}} \eta(s) dB_s].
\end{aligned}$$

We then can repeat this procedure to obtain (10). We now prove (11):

$$\begin{aligned}\mathbb{E}[(\int_0^T \eta(s)dB_s)^2] &= \mathbb{E}[\left(\int_0^{t_{N-1}} \eta(s)dB_s + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})\right)^2] \\ &= \mathbb{E}[\left(\int_0^{t_{N-1}} \eta(s)dB_s\right)^2 \\ &\quad + \mathbb{E}[2\left(\int_0^{t_{N-1}} \eta(s)dB_s\right)\xi_{N-1}(B_{t_N} - B_{t_{N-1}}) \\ &\quad + \xi_{N-1}^2(B_{t_N} - B_{t_{N-1}})^2|\mathcal{F}_{t_{N-1}}]] \\ &= \mathbb{E}[\left(\int_0^{t_{N-1}} \eta(s)dB_s\right)^2 + \xi_{N-1}^2(t_N - t_{N-1})].\end{aligned}$$

Thus $\mathbb{E}[(\int_0^{t_N} \eta(s)dB_s)^2] \leq \mathbb{E}[\left(\int_0^{t_{N-1}} \eta(s)dB_s\right)^2] + \mathbb{E}[\xi_{N-1}^2](t_N - t_{N-1})]$. We then repeat this procedure to deduce

$$\mathbb{E}[(\int_0^T \eta(s)dB_s)^2] \leq \sum_{j=0}^{N-1} \mathbb{E}[(\xi_j)^2](t_{j+1} - t_j) = \int_0^T \mathbb{E}[(\eta(t))^2]dt.$$

We list some main properties of the Itô's integral of G-Brownian motion. We denote for some $0 \leq s \leq t \leq T$,

$$\int_s^t \eta_u dB_u := \int_0^T \mathbf{I}_{[s,t]}(u)\eta_u dB_u.$$

We have

Proposition 32. *Let $\eta, \theta \in M_G^2(0,T)$ and let $0 \leq s \leq r \leq t \leq T$. Then in $L_G^1(\mathcal{F}_T)$ we have*
(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$,
(ii) $\int_s^t (\alpha\eta_u + \theta_u)dB_u = \alpha\int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$, *if α is bounded and in $L_G^1(\mathcal{F}_s)$,*
(iii) $\mathbb{E}[X + \int_r^T \eta_u dB_u|\mathcal{F}_s] = \mathbb{E}[X]$, $\forall X \in L_G^1(\mathcal{F})$.

5.3 Quadratic variation process of *G*-Brownian motion

We now study a very interesting process of the G-Brownian motion. Let π_t^N, $N = 1, 2, \cdots$, be a sequence of partitions of $[0,t]$. We consider

$$\begin{aligned}B_t^2 &= \sum_{j=0}^{N-1}[B_{t_{j+1}^N}^2 - B_{t_j^N}^2] \\ &= \sum_{j=0}^{N-1} 2B_{t_j^N}(B_{t_{j+1}^N} - B_{t_j^N}) + \sum_{j=0}^{N-1}(B_{t_{j+1}^N} - B_{t_j^N})^2.\end{aligned}$$

As $\mu(\pi_t^N) \to 0$, the first term of the right side tends to $\int_0^t B_s dB_s$. The second term must converge. We denote its limit by $\langle B \rangle_t$, i.e.,

$$\langle B \rangle_t = \lim_{\mu(\pi_t^N)\to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2 = B_t^2 - 2\int_0^t B_s dB_s. \tag{12}$$

By the above construction, $\langle B \rangle_t$, $t \geq 0$, is an increasing process with $\langle B \rangle_0 = 0$. We call it the **quadratic variation process** of the G-Brownian motion B. Clearly $\langle B \rangle$ is an increasing process. It perfectly characterizes the part of uncertainty, or ambiguity, of G-Brownian motion. It is important to keep in mind that $\langle B \rangle_t$ is not a deterministic process unless the case $\sigma = 1$, i.e., when B is a classical Brownian motion. In fact we have

Lemma 33. *We have, for each* $0 \leq s \leq t < \infty$

$$\mathbb{E}[\langle B \rangle_t - \langle B \rangle_s | \mathcal{F}_s] = t - s, \tag{13}$$

$$\mathbb{E}[-(\langle B \rangle_t - \langle B \rangle_s)|\mathcal{F}_s] = -\sigma_0^2(t - s). \tag{14}$$

Proof. By the definition of $\langle B \rangle$ and Proposition 32-(iii),

$$\begin{aligned}\mathbb{E}[\langle B \rangle_t - \langle B \rangle_s | \mathcal{F}_s] &= \mathbb{E}[B_t^2 - B_s^2 - 2\int_s^t B_u dB_u | \mathcal{F}_s] \\ &= \mathbb{E}[B_t^2 - B_s^2 | \mathcal{F}_s] = t - s.\end{aligned}$$

The last step can be check as in Example 23. We then have (13). (14) can be proved analogously with the consideration of $\mathbb{E}[-(B_t^2 - B_s^2)|\mathcal{F}_s] = -\sigma^2(t - s)$.

To define the integration of a process $\eta \in M_G^1(0,T)$ with respect to $d\langle B \rangle$, we first define a mapping:

$$Q_{0,T}(\eta) = \int_0^T \eta(s) d\langle B \rangle_s := \sum_{j=0}^{N-1} \xi_j(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) : M_G^{1,0}(0,T) \mapsto L^1(\mathcal{F}_T).$$

Lemma 34. *For each* $\eta \in M_G^{1,0}(0,T)$,

$$\mathbb{E}[|Q_{0,T}(\eta)|] \leq \int_0^T \mathbb{E}[|\eta_s|] ds. \tag{15}$$

Thus $Q_{0,T} : M_G^{1,0}(0,T) \mapsto L^1(\mathcal{F}_T)$ *is a continuous linear mapping. Consequently,* $Q_{0,T}$ *can be uniquely extended to* $L_{\mathcal{F}}^1(0,T)$*. We still denote this mapping by*

$$\int_0^T \eta(s) d\langle B \rangle_s = Q_{0,T}(\eta), \quad \eta \in M_G^1(0,T).$$

We still have

$$\mathbb{E}[|\int_0^T \eta(s) d\langle B \rangle_s|] \leq \int_0^T \mathbb{E}[|\eta_s|] ds, \quad \forall \eta \in M_G^1(0,T). \tag{16}$$

Proof. By applying Lemma 33, (15) can be checked as follows:

$$\begin{aligned}\mathbb{E}[|\sum_{j=0}^{N-1}\xi_j(\langle B\rangle_{t_{j+1}}-\langle B\rangle_{t_j})|] &\leq \sum_{j=0}^{N-1}\mathbb{E}[|\xi_j|\cdot\mathbb{E}[\langle B\rangle_{t_{j+1}}-\langle B\rangle_{t_j}|\mathcal{F}_{t_j}]] \\ &= \sum_{j=0}^{N-1}\mathbb{E}[|\xi_j|](t_{j+1}-t_j) \\ &= \int_0^T \mathbb{E}[|\eta_s|]ds.\end{aligned}$$

A very interesting point of the quadratic variation process $\langle B\rangle$ is, just like the G-Brownian motion B it's self, the increment $\langle B\rangle_{t+s}-\langle B\rangle_s$ is independent of $\mathcal{F}_s$ and identically distributed like $\langle B\rangle_t$. In fact we have

Lemma 35. *For each fixed $s\geq 0$, $(\langle B\rangle_{s+t}-\langle B\rangle_s)_{t\geq 0}$ is independent of $\mathcal{F}_s$. It is the quadratic variation process of the Brownian motion $B_t^s=B_{s+t}-B_s$, $t\geq 0$, i.e., $\langle B\rangle_{s+t}-\langle B\rangle_s=\langle B^s\rangle_t$. We have*

$$\mathbb{E}[\langle B^s\rangle_t^2|\mathcal{F}_s]=\mathbb{E}[\langle B\rangle_t^2]=t^2 \tag{17}$$

as well as

$$\mathbb{E}[\langle B^s\rangle_t^3|\mathcal{F}_s]=\mathbb{E}[\langle B\rangle_t^2]=t^3,\quad \mathbb{E}[\langle B^s\rangle_t^4|\mathcal{F}_s]=\mathbb{E}[\langle B\rangle_t^4]=t^4.$$

Proof. The independence is simply from

$$\begin{aligned}\langle B\rangle_{s+t}-\langle B\rangle_s &= B_{t+s}^2-2\int_0^{s+t}B_r dB_r-[B_s^2-2\int_0^s B_r dB_r] \\ &= (B_{t+s}-B_s)^2-2\int_s^{s+t}(B_r-B_s)d(B_r-B_s) \\ &= \langle B^s\rangle_t .\end{aligned}$$

We set $\phi(t):=\mathbb{E}[\langle B\rangle_t^2]$.

$$\begin{aligned}\phi(t) &= \mathbb{E}[\{(B_t)^2-2\int_0^t B_u dB_u\}^2] \\ &\leq 2\mathbb{E}[(B_t)^4]+8\mathbb{E}[(\int_0^t B_u dB_u)^2] \\ &\leq 6t^2+8\int_0^t \mathbb{E}[(B_u)^2]du \\ &= 10t^2.\end{aligned}$$

This also implies $\mathbb{E}[(\langle B\rangle_{t+s}-\langle B\rangle_s)^2]=\phi(t)\leq 14t$. Thus

$$
\begin{aligned}
\phi(t) &= \mathbb{E}[\{\langle B\rangle_s + \langle B\rangle_{s+t} - \langle B\rangle_s\}^2] \\
&\leq \mathbb{E}[(\langle B\rangle_s)^2] + \mathbb{E}[(\langle B^s\rangle_t)^2] + 2\mathbb{E}[\langle B\rangle_s \langle B^s\rangle_t] \\
&= \phi(s) + \phi(t) + 2\mathbb{E}[\langle B\rangle_s \mathbb{E}[\langle B^s\rangle_t]] \\
&= \phi(s) + \phi(t) + 2st.
\end{aligned}
$$

We set $\delta_N = t/N$, $t_k^N = kt/N = k\delta_N$ for a positive integer N. By the above inequalities

$$
\begin{aligned}
\phi(t_N^N) &\leq \phi(t_{N-1}^N) + \phi(\delta_N) + 2t_{N-1}^N\delta_N \\
&\leq \phi(t_{N-2}^N) + 2\phi(\delta_N) + 2(t_{N-1}^N + t_{N-2}^N)\delta_N \\
&\vdots
\end{aligned}
$$

We then have

$$
\phi(t) \leq N\phi(\delta_N) + 2\sum_{k=0}^{N-1} t_k^N\delta_N \leq 10\frac{t^2}{N} + 2\sum_{k=0}^{N-1} t_k^N\delta_N.
$$

Let $N \to \infty$ we have $\phi(t) \leq 2\int_0^t sds = t^2$. Thus $\mathbb{E}[\langle B_t\rangle^2] \leq t^2$. This with $\mathbb{E}[\langle B_t\rangle^2] \geq E[\langle B_t\rangle^2] = t^2$ implies (17).

Proposition 36. *Let* $0 \leq s \leq t$, $\xi \in L_G^1(\mathcal{F}_s)$. *Then*

$$
\begin{aligned}
\mathbb{E}[X + \xi(B_t^2 - B_s^2)] &= \mathbb{E}[X + \xi(B_t - B_s)^2] \\
&= \mathbb{E}[X + \xi(\langle B\rangle_t - \langle B\rangle_s)].
\end{aligned}
$$

Proof. By (12) and Proposition 22, we have

$$
\begin{aligned}
\mathbb{E}[X + \xi(B_t^2 - B_s^2)] &= \mathbb{E}[X + \xi(\langle B\rangle_t - \langle B\rangle_s + 2\int_s^t B_u dB_u)] \\
&= \mathbb{E}[X + \xi(\langle B\rangle_t - \langle B\rangle_s)].
\end{aligned}
$$

We also have

$$
\begin{aligned}
\mathbb{E}[X + \xi(B_t^2 - B_s^2)] &= \mathbb{E}[X + \xi\{(B_t - B_s)^2 + 2(B_t - B_s)B_s\}] \\
&= \mathbb{E}[X + \xi(B_t - B_s)^2].
\end{aligned}
$$

We have the following isometry:

Proposition 37. *Let* $\eta \in M_G^2(0,T)$. *We have*

$$
\mathbb{E}[(\int_0^T \eta(s)dB_s)^2] = \mathbb{E}[\int_0^T \eta^2(s)d\langle B\rangle_s]. \tag{18}
$$

Proof. We first consider $\eta \in M_G^{2,0}(0,T)$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{I}_{[t_j,t_{j+1})}(t)$$

and thus $\int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j})$. By Proposition 22 we have

$$\mathbb{E}[X + 2\xi_j(B_{t_{j+1}} - B_{t_j})\xi_i(B_{t_{i+1}} - B_{t_i})] = \mathbb{E}[X], \text{ for } X \in L_G^1(\mathcal{F}),\ i \neq j.$$

Thus

$$\mathbb{E}[(\int_0^T \eta(s)dB_s)^2] = \mathbb{E}[\left(\sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j})\right)^2] = \mathbb{E}[\sum_{j=0}^{N-1} \xi_j^2(B_{t_{j+1}} - B_{t_j})^2].$$

This with Proposition 36, it follows that

$$\mathbb{E}[(\int_0^T \eta(s)dB_s)^2] = \mathbb{E}[\sum_{j=0}^{N-1} \xi_j^2(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j})] = \mathbb{E}[\int_0^T \eta^2(s)d\langle B\rangle_s].$$

Thus (18) holds for $\eta \in M_G^{2,0}(0,T)$. We thus can continuously extend the above equality to the case $\eta \in M_G^2(0,T)$ and prove (18).

5.4 Itô's formula for *G*-Brownian motion

We have the corresponding Itô's formula of $\Phi(X_t)$ for a "*G*-Itô process" X. For simplification, we only treat the case where the function Φ is sufficiently regular. We first consider a simple situation.

Lemma 38. *Let $\Phi \in C^2(\mathbb{R}^n)$ be bounded with bounded derivatives and $\{\partial^2_{x^\mu x^\nu}\Phi\}_{\mu,\nu=1}^n$ are uniformly Lipschitz. Let $s \in [0,T]$ be fixed and let $X = (X^1, \cdots, X^n)^T$ be an n-dimensional process on $[s,T]$ of the form*

$$X_t^\nu = X_s^\nu + \alpha^\nu(t-s) + \eta^\nu(\langle B\rangle_t - \langle B\rangle_s) + \beta^\nu(B_t - B_s),$$

where, for $\nu = 1, \cdots, n$, α^ν, η^ν and β^ν, are bounded elements of $L_G^2(\mathcal{F}_s)$ and $X_s = (X_s^1, \cdots, X_s^n)^T$ is a given $\mathbb{R}^n$-vector in $L_G^2(\mathcal{F}_s)$. Then we have

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu}\Phi(X_u)\beta^\nu dB_u + \int_s^t \partial_{x_\nu}\Phi(X_u)\alpha^\nu du \qquad (19)$$
$$+ \int_s^t [D_{x^\nu}\Phi(X_u)\eta^\nu + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta^\mu\beta^\nu]d\langle B\rangle_u .$$

Here we use the Einstein convention, i.e., each single term with repeated indices μ and/or ν implies the summation.

Proof. For each positive integer N we set $\delta=(t-s)/N$ and take the partition

$$\pi_{[s,t]}^N=\{t_0^N,t_1^N,\cdots,t_N^N\}=\{s,s+\delta,\cdots,s+N\delta=t\}.$$

We have

$$\begin{aligned}\Phi(X_t)&=\Phi(X_s)+\sum_{k=0}^{N-1}[\Phi(X_{t_{k+1}^N})-\Phi(X_{t_k^N})]\\&=\Phi(X_s)+\sum_{k=0}^{N-1}[\partial_{x^\mu}\Phi(X_{t_k^N})(X_{t_{k+1}^N}^\mu-X_{t_k^N}^\mu)\\&+\frac{1}{2}[\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})(X_{t_{k+1}^N}^\mu-X_{t_k^N}^\mu)(X_{t_{k+1}^N}^\nu-X_{t_k^N}^\nu)+\eta_k^N]]\end{aligned}\tag{20}$$

where

$$\begin{aligned}\eta_k^N=[\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N}+\theta_k(X_{t_{k+1}^N}-X_{t_k^N}))-\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})]\\(X_{t_{k+1}^N}^\mu-X_{t_k^N}^\mu)(X_{t_{k+1}^N}^\nu-X_{t_k^N}^\nu)\end{aligned}$$

with $\theta_k\in[0,1]$. We have

$$\begin{aligned}\mathbb{E}[|\eta_k^N|]&=\mathbb{E}[|[\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N}+\theta_k(X_{t_{k+1}^N}-X_{t_k^N}))\\&\quad-\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})](X_{t_{k+1}^N}^\mu-X_{t_k^N}^\mu)(X_{t_{k+1}^N}^\nu-X_{t_k^N}^\nu)|]\\&\le c\mathbb{E}[|X_{t_{k+1}^N}-X_{t_k^N}|^3]\le C[\delta^3+\delta^{3/2}],\end{aligned}$$

where c is the Lipschitz constant of $\{\partial^2_{x^\mu x^\nu}\Phi\}_{\mu,\nu=1}^n$. Thus $\sum_k\mathbb{E}[|\eta_k^N|]\to 0$. The rest terms in the summation of the right side of (20) are $\xi_t^N+\zeta_t^N$, with

$$\begin{aligned}\xi_t^N=\sum_{k=0}^{N-1}\{\partial_{x^\mu}\Phi(X_{t_k^N})[\alpha^\mu(t_{k+1}^N-t_k^N)+\eta^\mu(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})+\\\beta^\mu(B_{t_{k+1}^N}-B_{t_k^N})]+\frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\beta^\mu\beta^\nu(B_{t_{k+1}^N}-B_{t_k^N})(B_{t_{k+1}^N}-B_{t_k^N})\}\end{aligned}$$

and

$$\begin{aligned}\zeta_t^N=\frac{1}{2}\sum_{k=0}^{N-1}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})[\alpha^\mu(t_{k+1}^N-t_k^N)+\eta^\mu(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})]\\\times[\alpha^\nu(t_{k+1}^N-t_k^N)+\eta^\nu(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})]\\+\beta^\nu[\alpha^\mu(t_{k+1}^N-t_k^N)+\eta^\mu(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})](B_{t_{k+1}^N}-B_{t_k^N}).\end{aligned}$$

We observe that, for each $u\in[t_k^N,t_{k+1}^N)$,

$$\begin{aligned}&\mathbb{E}[|\partial_{x^\mu}\Phi(X_u)-\sum_{k=0}^{N-1}\partial_{x^\mu}\Phi(X_{t_k^N})\mathbf{I}_{[t_k^N,t_{k+1}^N)}(u)|^2]\\&=\mathbb{E}[|\partial_{x^\mu}\Phi(X_u)-\partial_{x^\mu}\Phi(X_{t_k^N})|^2]\\&\le c^2\mathbb{E}[|X_u-X_{t_k^N}|^2]\le C[\delta+\delta^2].\end{aligned}$$

Thus $\sum_{k=0}^{N-1}\partial_{x^\mu}\Phi(X_{t_k^N})\mathbf{I}_{[t_k^N,t_{k+1}^N)}(\cdot)$ tends to $\partial_{x^\mu}\Phi(X_\cdot)$ in $M_G^2(0,T)$. Similarly,

$$\sum_{k=0}^{N-1}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\mathbf{I}_{[t_k^N,t_{k+1}^N)}(\cdot)\to\partial^2_{x^\mu x^\nu}\Phi(X_\cdot),\ \text{in}\ \ M_G^2(0,T).$$

Let $N\to\infty$, by the definitions of the integrations with respect to dt, dB_t and $d\langle B\rangle_t$ the limit of ξ_t^N in $L_G^2(\mathcal{F}_t)$ is just the right hand of (19). By the estimates of the next remark, we also have $\zeta_t^N\to 0$ in $L_G^1(\mathcal{F}_t)$. We then have proved (19).

Remark 39. We have the following estimates: for $\psi^N\in M_G^{1,0}(0,T)$ such that $\psi_t^N=\sum_{k=0}^{N-1}\xi_{t_k}^N\mathbf{I}_{[t_k^N,t_{k+1}^N)}(t)$, and $\pi_T^N=\{0\le t_0,\cdots,t_N=T\}$ with $\lim_{N\to\infty}\mu(\pi_T^N)=0$ and $\sum_{k=0}^{N-1}\mathbb{E}[|\xi_{t_k}^N|](t_{k+1}^N-t_k^N)\le C$, for all $N=1,2,\ldots$, we have

$$\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(t_{k+1}^N-t_k^N)^2|]\to 0,$$

and, thanks to Lemma 35,

$$\begin{aligned}\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})^2|]&\le\sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|\cdot\mathbb{E}[(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})^2|\mathcal{F}_{t_k^N}]]\\&=\sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|](t_{k+1}^N-t_k^N)^2\to 0,\end{aligned}$$

as well as

$$\begin{aligned}&\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})(B_{t_{k+1}^N}-B_{t_k^N})|]\\&\le\sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|]\mathbb{E}[(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})|B_{t_{k+1}^N}-B_{t_k^N}|]\\&\le\sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|]\mathbb{E}[(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})^2]^{1/2}\mathbb{E}[|B_{t_{k+1}^N}-B_{t_k^N}|^2]^{1/2}\\&=\sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|](t_{k+1}^N-t_k^N)^{3/2}\to 0.\end{aligned}$$

We also have

$$\begin{aligned}
&\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})(t_{k+1}^N-t_k^N)|] \\
&\leq \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|(t_{k+1}^N-t_k^N)\cdot\mathbb{E}[(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})|\mathcal{F}_{t_k^N}]] \\
&=\sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|](t_{k+1}^N-t_k^N)^2\to 0
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[|\sum_{k=0}^{N-1}\xi_k^N(t_{k+1}^N-t_k^N)(B_{t_{k+1}^N}-B_{t_k^N})|] \\
&\leq \sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|](t_{k+1}^N-t_k^N)\mathbb{E}[|B_{t_{k+1}^N}-B_{t_k^N}|] \\
&=\sqrt{\frac{2}{\pi}}\sum_{k=0}^{N-1}\mathbb{E}[|\xi_k^N|](t_{k+1}^N-t_k^N)^{3/2}\to 0.
\end{aligned}$$

We now consider a more general form of Itô's formula. Consider

$$X_t^\nu=X_0^\nu+\int_0^t\alpha_s^\nu ds+\int_0^t\eta_s^\nu d\langle B\rangle_s+\int_0^t\beta_s^\nu dB_s.$$

Proposition 40. *Let α^ν, β^ν and η^ν, $\nu=1,\cdots,n$, are bounded processes of $M_G^2(0,T)$. Then for each $t\geq 0$ and in $L_G^2(\mathcal{F}_t)$ we have*

$$\begin{aligned}
\Phi(X_t)-\Phi(X_s)=&\int_s^t\partial_{x^\nu}\Phi(X_u)\beta_u^\nu dB_u+\int_s^t\partial_{x_\nu}\Phi(X_u)\alpha_u^\nu du \qquad (21)\\
&+\int_s^t[\partial_{x^\nu}\Phi(X_u)\eta_u^\nu+\frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^\mu\beta_u^\nu]d\langle B\rangle_u
\end{aligned}$$

Proof. We first consider the case where α, η and β are step processes of the form

$$\eta_t(\omega)=\sum_{k=0}^{N-1}\xi_k(\omega)\mathbf{I}_{[t_k,t_{k+1})}(t).$$

From the above Lemma, it is clear that (21) holds true. Now let

$$X_t^{\nu,N}=X_0^\nu+\int_0^t\alpha_s^{\nu,N}ds+\int_0^t\eta_s^{\nu,N}d\langle B\rangle_s+\int_0^t\beta_s^{\nu,N}dB_s$$

where α^N, η^N and β^N are uniformly bounded step processes that converge to α, η and β in $M_G^2(0,T)$ as $N\to\infty$. From Lemma 38

$$\Phi(X_t^{\nu,N}) - \Phi(X_0) = \int_s^t \partial_{x^\nu}\Phi(X_u^N)\beta_u^{\nu,N}dB_u + \int_s^t \partial_{x_\nu}\Phi(X_u^N)\alpha_u^{\nu,N}du \qquad (22)$$
$$+ \int_s^t [\partial_{x^\nu}\Phi(X_u^N)\eta_u^{\nu,N} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u^N)\beta_u^{\mu,N}\beta_u^{\nu,N}]d\langle B\rangle_u$$

Since

$$\mathbb{E}[|X_t^{\nu,N} - X_t^\nu|^2] \le 3\mathbb{E}[|\int_0^t (\alpha_s^N - \alpha_s)ds|^2] + 3\mathbb{E}[|\int_0^t (\eta_s^{\nu,N} - \eta_s^\nu)d\langle B\rangle_s|^2]$$
$$+3\mathbb{E}[|\int_0^t (\beta_s^{\nu,N} - \beta_s^\nu)dB_s|^2] \le 3\int_0^T \mathbb{E}[(\alpha_s^{\nu,N} - \alpha_s^\nu)^2]ds + 3\int_0^T \mathbb{E}[|\eta_s^{\nu,N} - \eta_s^\nu|^2]ds$$
$$+ 3\int_0^T \mathbb{E}[(\beta_s^{\nu,N} - \beta_s^\nu)^2]ds,$$

we then can prove that, in $M_G^2(0,T)$, we have (21). Furthermore

$$\partial_{x^\nu}\Phi(X_\cdot^N)\eta_\cdot^{\nu,N} + \partial^2_{x^\mu x^\nu}\Phi(X_\cdot^N)\beta_\cdot^{\mu,N}\beta_\cdot^{\nu,N} \to \partial_{x^\nu}\Phi(X_\cdot)\eta_\cdot^\nu + \partial^2_{x^\mu x^\nu}\Phi(X_\cdot)\beta_\cdot^\mu\beta_\cdot^\nu$$
$$\partial_{x_\nu}\Phi(X_\cdot^N)\alpha_\cdot^{\nu,N} \to \partial_{x_\nu}\Phi(X_\cdot)\alpha_\cdot^\nu$$
$$\partial_{x^\nu}\Phi(X_\cdot^N)\beta_\cdot^{\nu,N} \to \partial_{x^\nu}\Phi(X_\cdot)\beta_\cdot^\nu$$

We then can pass limit in both sides of (22) and get (21).

6 Stochastic differential equations

We consider the following SDE defined on $M_G^2(0,T;\mathbb{R}^n)$:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t h(X_s)d\langle B\rangle_s + \int_0^t \sigma(X_s)dB_s,\ t \in [0,T]. \qquad (23)$$

where the initial condition $X_0 \in \mathbb{R}^n$ is given and $b, h, \sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$ are given Lipschitz functions, i.e., $|\phi(x) - \phi(x')| \le K|x - x'|$, for each x, $x' \in \mathbb{R}^n$, $\phi = b$, h and σ. Here the horizon $[0,T]$ can be arbitrarily large. The solution is a process $X \in M_G^2(0,T;\mathbb{R}^n)$ satisfying the above SDE. We first introduce the following mapping on a fixed interval $[0,T]$:

$$\Lambda_\cdot(Y) := Y \in M_G^2(0,T;\mathbb{R}^n) \longmapsto M_G^2(0,T;\mathbb{R}^n)$$

by setting Λ_t with

$$\Lambda_t(Y) = X_0 + \int_0^t b(Y_s)ds + \int_0^t h(Y_s)d\langle B\rangle_s + \int_0^t \sigma(Y_s)dB_s,\ t \in [0,T].$$

We immediately have

Lemma 41. *For each $Y, Y' \in M_G^2(0,T;\mathbb{R}^n)$, we have the following estimate:*

$$\mathbb{E}[|\Lambda_t(Y) - \Lambda_t(Y')|^2] \le C \int_0^t \mathbb{E}[|Y_s - Y_s'|^2] ds, \ \ t \in [0,T],$$

where $C = 3K^2$.

Proof. This is a direct consequence of the inequalities (9), (11) and (16).

We now prove that SDE (23) has a unique solution. By multiplying e^{-2Ct} on both sides of the above inequality and then integrate them on $[0,T]$. It follows that

$$\begin{aligned}\int_0^T \mathbb{E}[|\Lambda_t(Y) - \Lambda_t(Y')|^2] e^{-2Ct} dt \\ &\le C \int_0^T e^{-2Ct} \int_0^t \mathbb{E}[|Y_s - Y_s'|^2] ds dt \\ &= C \int_0^T \int_s^T e^{-2Ct} dt \mathbb{E}[|Y_s - Y_s'|^2] ds \\ &= (2C)^{-1} C \int_0^T (e^{-2Cs} - e^{-2CT}) \mathbb{E}[|Y_s - Y_s'|^2] ds.\end{aligned}$$

We then have

$$\int_0^T \mathbb{E}[|\Lambda_t(Y) - \Lambda_t(Y')|^2] e^{-2Ct} dt \le \frac{1}{2} \int_0^T \mathbb{E}[|Y_t - Y_t'|^2] e^{-2Ct} dt.$$

We observe that the following two norms are equivalent in $M_G^2(0,T;\mathbb{R}^n)$:

$$\int_0^T \mathbb{E}[|Y_t|^2] dt \sim \int_0^T \mathbb{E}[|Y_t|^2] e^{-2Ct} dt.$$

From this estimate we can obtain that $\Lambda(Y)$ is a contract mapping. Consequently, we have

Theorem 42. *There exists a unique solution $X \in M_G^2(0,T;\mathbb{R}^n)$ of the stochastic differential equation (23).*

7 Appendix

For $r > 0$, $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$|a+b|^r \le \max\{1, 2^{r-1}\}(|a|^r + |b|^r), \quad \forall a, b \in \mathbb{R} \tag{24}$$

$$|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}. \tag{25}$$

Proposition 43.

$$\mathbb{E}[|X+Y|^r] \leq C_r(\mathbb{E}[|X|^r] + \mathbb{E}[|Y|^r]), \tag{26}$$

$$\mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{1/p} \cdot \mathbb{E}[|Y|^q]^{1/q}, \tag{27}$$

$$\mathbb{E}[|X+Y|^p]^{1/p} \leq \mathbb{E}[|X|^p]^{1/p} + \mathbb{E}[|Y|^p]^{1/p}. \tag{28}$$

In particular, for $1 \leq p < p'$, *we have* $\mathbb{E}[|X|^p]^{1/p} \leq \mathbb{E}[|X|^{p'}]^{1/p'}$.

Proof. (26) follows from (24). We set

$$\xi = \frac{X}{\mathbb{E}[|X|^p]^{1/p}}, \quad \eta = \frac{Y}{\mathbb{E}[|Y|^q]^{1/q}}.$$

By (25) we have

$$\begin{aligned}\mathbb{E}[|\xi\eta|] &\leq \mathbb{E}[\frac{|\xi|^p}{p} + \frac{|\eta|^q}{q}] \leq \mathbb{E}[\frac{|\xi|^p}{p}] + \mathbb{E}[\frac{|\eta|^q}{q}] \\ &= \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

Thus (27) follows. We now prove (28):

$$\begin{aligned}\mathbb{E}[|X+Y|^p] &= \mathbb{E}[|X+Y| \cdot |X+Y|^{p-1}] \\ &\leq \mathbb{E}[|X| \cdot |X+Y|^{p-1}] + \mathbb{E}[|Y| \cdot |X+Y|^{p-1}] \\ &\leq \mathbb{E}[|X|^p]^{1/p} \cdot \mathbb{E}[|X+Y|^{(p-1)q}]^{1/q} \\ &\quad + \mathbb{E}[|Y|^p]^{1/p} \cdot \mathbb{E}[|X+Y|^{(p-1)q}]^{1/q}\end{aligned}$$

This with $(p-1)q = p$ implies (28).

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