Formation of singularities in Madelung fluid: a nonconventional application of Itô calculus to foundations of Quantum Mechanics

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Summary. Stochastic Quantization is a procedure which provides the equation of motion of a Quantum System starting from its classical description and incorporating quantum effects into a stochastic kinematics. After the pioneering work by E.Nelson in 1966 the method has been developed in the eighties in various different ways. In this communication I summarize and systematize the results obtained within an approach based on a Lagrangian variational principle where 3/2 order contributions in Itô calculus are required, leading to a generalization of Madelung fluid equations where velocity fields with vorticity are allowed.

Such a vorticity induces dissipation of the energy so that the irrotational solutions, corresponding to the usual conservative solutions of Schroedinger equation, act as an attracting set. Recent numerical experiments show generation of zeroes of the density with concentration of vorticity and formation of isolated central vortex lines.

1 Introduction

This communication is concerned with an application of Itô calculus to the problem of describing the dynamical evolution of a quantum system once its classical description (which can be given in terms of forces, lagrangian or hamiltonian) is given. We know that, if the classical hamiltonian is given, the canonical quantization rules lead to Schrödinger equation, which beautifully describes the behavior of microscopical systems. But we also know that this procedure seems to fail when applied to microscopical systems interacting with a (macroscopic) measuring apparatus. This fact has been a motivation for investigating other quantization procedures.

In his pioneering work in 1966 E. Nelson proposed a Stochastic Quantization (often called Stochastic Mechanics) where, given the forces acting on the system, quantum effects are incorporated into a stochastic kinematics [18]. This approach was widely developed during the eighties, with the introduction of stochastic variational principles (see for example [19], [2] , [15] and references quoted therein). I present here a synthesis of the results obtained within an approach which leads to a dissipative generalization of Schrödinger equation, the usual conservative solutions being in fact dynamical equilibrium states which form an attracting set [13] [14][10]. The basic tool is Itô calculus where stochastic increments must be estimated to the order $\frac{3}{2}$.

For a quantum particle of mass m, subjected to a force which is the gradient of a scalar potential Φ, Schrödinger equation reads

$$\imath \hbar \partial_t \Psi = \left(-\frac{1}{2m} \hbar^2 \nabla^2 + \Phi \right) \Psi \tag{1}$$

ψ denoting the quantum mechanical wave function.

By a change of variables Schrödinger equation can be formally written in a fluidodynamical version, the so called Madelung fluid equations .

$$\begin{cases} \partial_t \rho & = -\nabla \cdot (\rho v) \\ \partial_t v & + (v \cdot \nabla) v - \frac{\hbar^2}{2m^2} \nabla \left(\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right) = -\frac{1}{m} \nabla \Phi \end{cases} \tag{2}$$

where

$$\rho = |\psi|^2$$

$$v = \nabla S$$

S being the phase of the wave function ψ.

The equivalence is only formal if the density ρ is not strictly positive at all times.

The velocity field of Madelung fluid is irrotational in all points where the density is different from zero. In many examples solutions of Schrödinger equation which exhibit nodes correspond to solutions of Madelung fluid equations with singular velocity and isolated vortex lines.

In our setting we are led to a dissipative generalization of such equations, which allow velocity fields with a distributed vorticity. It was conjectured that such a vorticity asymptotically can concentrate in the zeroes of the density, describing the formation of the singularities and in particular of isolated vortex lines. The problem is very difficult from the analytical point of view but recent numerical results seem to confirm this conjecture [3]. It is worth stressing that arrays of isolated vortex lines are observed in quantum fluids, as liquid Helium and Bose Einstein condensates (see [9], [11], [12], [1]), but the mechanism underlying their formation is still not well understood. Describing the formation of isolated vortex lines in Madelung fluid, from smooth initial data, could represent a contribution to the solution of this problem.

2 A stochastic quantization procedure

For a quantum particle of mass m in a scalar potential Φ we denote its configuration at time t by $q(t)$. We model the evolution in time of the configuration by a "smooth diffusion", in the following sense:

Definition 1. *A diffusion q is a "smooth diffusion" if*

1) Its drift v_+ is a smooth (i.e. infinitely differentiable) time dependent vector field and its diffusion coefficient is constant (in this setting equal to $\frac{\hbar}{m}$, $\hbar$ denoting Planck's constant divided by 2π)

2) There exists a probability space $(\Omega, \mathcal{F}, P)$ and a standard Brownian Motion W s.t., for $t \in [0,T]$, $T > 0$,

$$q(t) = q(0) + \int_0^t v_+ \left(q(s), s\right) ds + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} W(t) \tag{3}$$

3) There exists a reversed standard Brownian Motion W^* on $(\Omega, \mathcal{F}, P)$ and v_- s.t., for any $t \in [0,T]$,

$$q(t) = q(0) + \int_0^t v_- \left(q(s), s\right) ds + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} \left(W^*(t) - W^*(0)\right) \tag{4}$$

I recall that a reversed standard Brownian Motion W^* on the finite time interval $[0.T]$ is defined by the equality

$$W^*(t) = \hat{W}(T-t), \quad t \in [0,T] \tag{5}$$

$\hat{W}$ still denoting a standard Brownian Motion.

The finite energy condition is sufficient for property 3) (See [5]. An extension to the infinite dimensional case is given in [6]). We also recall that if ρ is the (time dependent) density of a smooth diffusion one has, in particular

$$\frac{v_+ - v_-}{2} = \frac{\hbar}{2m} \nabla \ln \rho \tag{6}$$

$$\partial_t \rho = -\nabla \cdot (\rho v) \tag{7}$$

were v is the "current velocity", defined as

$$v := \frac{v_+ + v_-}{2} \tag{8}$$

For any finite time interval $[t_a, t_b]$ and positive integer N we fix the notations

$$\Delta := \tfrac{t_b - t_a}{N}$$

$\Delta^+ q(t_i) := q(t_{i+1}) - q(t_i)$ future increment

$\Delta^- q(t_i) := q(t_i) - q(t_{i-1})$ past increment

We now consider the following mean discretized version of the classical action functional

$$A^N_{[t_a,t_b]}[q] := \mathcal{E} \sum_{i=1}^{N} \left[\frac{1}{2} m \frac{\Delta^+ q(t_i) \cdot \Delta^+ q(t_i)}{\Delta^2} - \Phi(q(t_i)) \right] \Delta \tag{9}$$

were q is uniquely determined by the triple $[W, v_+, q_o]$ and $\mathcal{E}$ denotes the expectation.

By exploiting the backward representation and estimating $\Delta^+ q(t_i)$ to the order $\Delta^{\frac{3}{2}}$, which gives

$$\begin{aligned} \Delta^+ q(t) = & \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} \Delta^+ W(t) + v_+ \left(q(t), t\right) \Delta + \\ & + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} \sum_{k=1}^{3} \left[\partial_k v_+ \left(q(t), t\right) \int_t^{t+\Delta} \left(W_k(s) - W_k(t)\right) ds \right] + \\ & + o(\Delta^{\frac{3}{2}}) \end{aligned} \tag{10}$$

we find

$$\begin{aligned} A^N_{[t_a,t_b]}[q] = \ & \mathcal{E} \sum_{i=1}^{N} \left[\frac{1}{2} m \frac{\Delta^+ q(t_i) \cdot \Delta^- q(t_i)}{\Delta^2} \right. \\ & \left. + \frac{3}{2} \frac{\hbar}{\Delta} + o(\Delta) - \Phi(q(t_i)) \right] \Delta \end{aligned} \tag{11}$$

In order to generalize the classical action principle, starting from the above defined functional, two methods have been considered. The former, that will be called Eulerian or Stochastic Control approach, consists in eliminating the divergent term in the discretized action and then take the limit for N going to infinity. After simple manipulations one can see that such a limit can be expressed as a simple functional of the drift field v_+. This allows to exploit stochastic control like techniques [8]. The latter, that will be called Lagrangian or path-wise approach, consists in taking pathwise variations of q for fixed W in $A^N_{[t_a,t_b]}[q]$. This eliminates the divergent term .The limit for N going to infinity is taken only at the end of the calculus of variations (see [13], [14] and [10]).This is the approach considered in the following.

Definition 2. *The set of admissible test diffusions for a given W is constituted by the set of all smooth diffusions associated to W according to the previous definition.*

For the test diffusion $q(t)$ at time t let $q'(t) := q(t) + \delta q(t)$ denote the varied diffusion. We require that this is still a smooth diffusion with the same W. Therefore there must exist a smooth drift field v'_+ such that

$$q(t) = q(0) + \int_0^t v_+ \left(q(s), s\right) ds + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} W(t) \tag{12}$$

$$q'(t) = q(0) + \int_0^t v'_+ \left(q'(s), s\right) ds + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} W(t) \tag{13}$$

We introduce the variation process h and the variation of the drift f by putting, for $\epsilon > 0$,

$$\begin{cases} \varepsilon h(t) := \delta q(t) \quad \varepsilon > 0 \\ \varepsilon f := v'_+ - v_+ \end{cases} \tag{14}$$

Then one finds

$$\dot{h}(t) = \sum_{j=1}^{3} \partial_j v_+ \left(q(t), t\right) h_j(t) + f\left(q(t), t\right) \tag{15}$$

so that $h(t)$ is a differentiable stochastic process. It satisfies a first order ODE for every realization of q. As a consequence h cannot be fixed both in t_a and t_b. This fact, which has no counterpart in the classical case, comes to be a typical quantum peculiarity.

Definition 3. *A process h will be said "admissible variation" for the test diffusion q if it is solution of* (15) *for a smooth f.*

We want now to characterize the motions which are represented by "critical diffusions" :

Definition 4. *A smooth diffusion q^* is critical with fixed initial position if, $\forall\, h$ admissible,*

$$\lim_{N\uparrow\infty} \left\{ A^N_{[t_a,t_b]}\left[q^* + \varepsilon h\right] - A^N_{[t_a,t_b]}\left[q^*\right] - \varepsilon p_{t_b} h_{t_b} \right\} = o(\varepsilon) \tag{16}$$

$h(t_a) = 0$ and a smooth diffusion q^* is critical with fixed final position if, $\forall\, h$ admissible,

$$\lim_{N\uparrow\infty} \left\{ A^N_{[t_a,t_b]}\left[q^* + \varepsilon h\right] - A^N_{[t_a,t_b]}\left[q^*\right] + \varepsilon p_{t_a} h_{t_a} \right\} = o(\varepsilon) \tag{17}$$

$h(t_b) = 0$ p_{t_a} and p_{t_b} are fixed random variables playing the role of the classical initial and final "momentum".

We can prove the following

Theorem 5. *A sufficient condition in order a smooth diffusion q^* to be critical with fixed initial condition is*

$$q^*(t) = q^*(0) + \int_0^t v_+ \left(q^*(s), s\right) ds + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} W(t)$$

where:

$$v_+ = v + \frac{\hbar}{2m} \nabla \ln \rho$$

and, if the initial position is fixed,

$$\begin{aligned} &\partial_t \rho = -\nabla \cdot (\rho v) &(18)\\ &\partial_t v + (v \cdot \nabla) v - \frac{\hbar^2}{2m^2} \nabla \left(\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{\hbar}{m} \left(\nabla \ln \rho + \nabla\right) \wedge \left(\nabla \wedge v\right) = -\frac{1}{m} \nabla \Phi \end{aligned}$$

with the boundary constraint

$$mv(q_{t_b}, t_b) = p_{t_b}$$

or, if the final position is fixed,

$$\begin{aligned} &\partial_t \rho = -\nabla \cdot (\rho v) &(19)\\ &\partial_t v + (v \cdot \nabla) v - \frac{\hbar^2}{2m^2} \nabla \left(\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}\right) + \frac{\hbar}{m} \left(\nabla \ln \rho + \nabla\right) \wedge \left(\nabla \wedge v\right) = -\frac{1}{m} \nabla \Phi \end{aligned}$$

with the boundary constraint

$$mv(q_{t_a}, t_a) = p_{t_a}$$

Proof. (Outline)

Considering, without loss of generality, the first case, we have

$$\begin{aligned} \delta A^N_{[t_a, t_b]} \left[q\right] =& \epsilon \sum_{i=1}^N \frac{m}{2} \mathcal{E} \left(\frac{\Delta^+ q(t_i) \cdot \Delta^- h(t_i)}{\Delta^2} + \frac{\Delta^- q(t_i) \cdot \Delta^+ h(t_i)}{\Delta^2} + o(\Delta) \right) \Delta \\ & - \epsilon \sum_{i=1}^N \mathcal{E} (\nabla \Phi (q(t_i, t_i) \cdot h(t_i) \Delta - \epsilon \mathcal{E} \left(p_{t_b} \cdot h(t_b)\right)) + o(\epsilon) \qquad (20) \end{aligned}$$

The analysis to the order $\Delta^{\frac{3}{2}}$ of the finite forward and backward increments in the kinetic terms gives

$$\mathcal{E}\left(\frac{\Delta^{+}q(t_i)\cdot\Delta^{-}h(t_i)}{\Delta^2}\right)=\frac{1}{\Delta}\mathcal{E}\left(v^{+}(q(t),t)\cdot\Delta^{-}h(t)+o(\Delta)\right) \tag{21}$$

and

$$\begin{aligned}&\mathcal{E}\left(\frac{\Delta^{-}q(t_i)\cdot\Delta^{+}h(t_i)}{\Delta^2}\right)=\\ &=\frac{1}{\Delta}\mathcal{E}\left(v^{-}(q(t),t)\cdot\Delta^{+}h(t)+\left(\frac{\hbar}{m}\right)^{\frac{1}{2}}\Delta^{-}W_{*}(t)\cdot\dot{h}+o(\Delta)\right)\end{aligned} \tag{22}$$

The difference between the two kinetic terms comes from the fact that the variation process h is measurable with respect to the σ algebra generated by the past of q and not by the future, if the initial position is fixed. The proof then exploits a discrete "integration by parts" and the equality

$$\Delta^{-}W^{*}(t)=2(\frac{m}{\hbar})^{\frac{1}{2}}\nabla\ln\rho\Delta+\Delta^{+}W(t-\Delta)+o(\Delta) \tag{23}$$

Going to the limit at the end we get, exploiting (6) and (8)(see [10] for the details)

$$\begin{aligned}\lim_{N\to\infty}\delta A^{N}_{[t_a,t_b]}[q]=\ &\epsilon\,\mathcal{E}\int_{t_a}^{t_b}\left[-\partial_t v-(v\cdot\nabla)v+\frac{\hbar^2}{2m^2}\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)+\right.\\ &\left.+\frac{\hbar}{m}(\nabla\ln\rho+\nabla)\wedge(\nabla\wedge v)-\frac{1}{m}\nabla\Phi\right](q(t),t)\cdot h(t)dt+\\ &+\ \epsilon\,\mathcal{E}\left[mv(q_{t_b},t_b)-p_{t_b}\right]\end{aligned} \tag{24}$$

The assertion immediately follows recalling that the continuity equation (7) always holds if (ρ,v) are the density and the current velocity field, respectively, of a smooth diffusion.

In the case with final fixed condition the two kinetic terms read

$$\begin{aligned}&\mathcal{E}\left(\frac{\Delta^{+}q(t_i)\cdot\Delta^{-}h(t_i)}{\Delta^2}\right)=\\ &=\frac{1}{\Delta}\mathcal{E}\left(v^{+}(q(t),t)\cdot\Delta^{-}h(t)+\left(\frac{\hbar}{m}\right)^{\frac{1}{2}}\Delta^{+}W(t)\cdot\dot{h}+o(\Delta)\right)\end{aligned} \tag{25}$$

and

$$\mathcal{E}\left(\frac{\Delta^{-}q(t_i)\cdot\Delta^{+}h(t_i)}{\Delta^2}\right)=\frac{1}{\Delta}\,\mathcal{E}\left(v^{-}(q(t),t)\cdot\Delta^{+}h(t)+o(\Delta)\right) \tag{26}$$

Then (23), with t replaced by $t+\Delta$, is exploited to estimate $\Delta^{+}W(t)$. This turns to change the sign in front of the term of first order in $\frac{\hbar}{m}$.

The sufficient conditions as proved in the theorem are also necessary in the following sense:

Corollary 6. *Let q be critical with fixed initial position and let ρ and v be its density and current velocity respectively. Let also $p_t = mv(q(t),t)$ for all $t \in [t_a, t_b]$. Then the equality*

$$\Big[\partial_t v + (v\cdot\nabla)\, v - \frac{\hbar^2}{2m^2}\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) - \\ -\frac{\hbar}{m}\left(\nabla \ln\rho + \nabla\right)\wedge(\nabla\wedge v) + \frac{1}{m}\nabla\Phi\Big](q(t),t) = 0 \qquad (27)$$

holds a.s. for all t in $[t_a, t_b]$. The analogous necessary condition holds in order q to be critical with final fixed position.

Proof. (see [14] pag. 1986)

Let q be critical with initial fixed position and let (ρ, v) be its density and current velocity, respectively. Let us also denote by $F(q(t),t) = 0$ the equality (27) and put $\delta A_{[t^*,t]}[q] := \lim_{N\to\infty} A^N_{[t_a,t_b]}$. Then if h is the admissible variation of q which solves (15) for $f = F$ with $h(t^*) = 0$ we get by (15)

$$\begin{aligned} \delta A_{[t^*,t]}|_{t=t^*} &= 0\ , \\ \frac{d}{dt}\delta A_{[t^*,t]}|_{t=t^*} &= 0\ , \\ \frac{d^2}{dt^2}\delta A_{[t^*,t]}|_{t=t^*} &= \mathcal{E}\left[F^2(q(t^*),t^*)\right]\ . \end{aligned}$$

Thus if at time t^* (27) does not hold with probability one then q is not critical.

So we find a generalization of Madelung fluid equations, where in particular the velocity field v is not necessarily the gradient of some scalar field.

The two systems of PDE.s (18) and (19) represent two dynamical evolutions which are one the time reversal of the other.

The second one, with the + sign in front of the term of the first order in $\frac{\hbar}{m}$, turns out to be dissipative.

In fact if (ρ, v) is a smooth solution of (19) and ρ has a good behavior at infinity, we have, with $u := \frac{\hbar}{2m}\nabla\ln\rho$ (osmotic velocity) and introducing the energy functional

$$E[\rho, v] = \int_{\mathbb{R}^3}\left(\frac{1}{2}mv^2 + \frac{1}{2}mu^2 + \Phi\right)\rho d^3x \qquad (28)$$

the following equality

$$\frac{dE}{dt} = -\frac{\hbar}{2}\int_{\mathbb{R}^3} (\nabla \wedge v)^2 \rho d^3x \tag{29}$$

This **Energy Theorem** was proved in [10] by a purely analytical method, exploiting the equivalence of the new system of dynamical equations with a nonlinear Schrödinger equation of electromagnetic type.

Thus we consider (19) as physical equations. They are related to the variational principle with final fixed position, while (18) are interpreted as their time reversed picture.

Concluding, (19) is a dissipative generalization of Madelung fluid equations and such a dissipation is caused by the vorticity of the velocity field.

Notice that the domain of definition of the two systems of PDEs (18) and (19) is by construction C^∞ since the admissible test diffusions are smooth diffusions according to definition 1. The global existence for the linear Gaussian solutions of the bidimensional harmonic oscillator was proved in [16]. The general existence and uniqueness problem is still open. If a solution (ρ, v) of (18) or (19) is irrotational dx-a.s. then the energy is conserved and it solves Madelung equation.

To be more precise, if (ρ, v) satisfy equations of motion and there exists an open set $Q \in \mathbb{R}^3$ s.t.

$$\begin{aligned} (\nabla \wedge v)(x,t) &= 0 \quad \forall x \in Q \quad ,\forall t \geq 0 \\ \rho(x,t) &> 0 \quad \forall x \in Q \quad ,\forall t \geq 0 \end{aligned}$$

then $\exists\, S$ s.t.

$$v(x,t) = \frac{1}{m}\nabla S(x,t) \quad \forall x \in Q \quad ,\forall t \geq 0$$

Then putting

$$\Psi = \rho^{\frac{1}{2}} \mathrm{e}^{\frac{\imath}{\hbar}S}, \quad (\Psi \colon Q \times [0,\infty) \to \mathbb{C})$$

we have

$$\imath\hbar\partial_t \Psi = \left(-\frac{1}{2m}\hbar^2 \Delta + \Phi\right)\Psi$$

These solutions conserve the energy (which turns to be the usual quantum mechanical expectation of the observable energy) and work as an attracting set. The case of Gaussian and linear solution for the bidimensional harmonic oscillator was studied in [16] and [17]. In particular it was proved that Schrödinger solutions constitute a center manifold and that the convergence is in the sense of the relative entropy.

We also quote that a version of the Lagrangian variational principle leading to (18) and (19) with a free parameter multiplying the term of the first order in $\frac{\hbar}{m}$ is proposed in [7].

3 Concentration of vorticity

As an example we consider a bidimensional symmetric harmonic oscillator. To be more precise we put, denoting by (r, θ) polar coordinates in the (x, y) plane and by z the third spatial coordinate,

$$\Phi := \Phi(r) = \frac{1}{2} r^2 \ , \quad r = \sqrt{x^2 + y^2}$$

We consider the simultaneous eigenfunctions of the Hamiltonian $\mathcal{H}$ and of the angular momentum L_z with respect to the z axis

$$\chi_{n_d,n_g} = |\chi_{n_d,n_g}| \exp[\imath(n_d - n_g)]\theta, \qquad n_d, n_g = 0, 1, 2, \ldots$$

These can be easily computed recursively (see for example [4]). The eigenvalues of the Hamiltonian turn to be

$$E_{n_d,n_g} = 2(n_d + n_g + 1)$$

while those of the angular momentum read

$$\ell_{n_d,n_g} = n_d - n_g$$

The eigenfunctions χ_{n_d,n_g} correspond to the following time invariant solutions of Madelung equations, and of course of our new equations (19), on the open set $\mathbb{R}^2 \setminus \{0\}$

$$\rho_{n_d,n_g}(r) = \left|\chi_{n_d,n_g}(r)\right|^2$$
$$\boldsymbol{v}_{n_d,n_g}(r) = \frac{\hbar}{m} \nabla((n_d - n_g)\,\theta) = \frac{\hbar}{m} \frac{n_d - n_g}{r} \hat{\theta}$$

The vorticity of the velocity field $\boldsymbol{v}_{n_d,n_g}, \quad n_d, n_g = 0, 1, 2, \ldots$ is at every time equal to zero in $\mathbb{R}^2 \setminus \{0\}$ but, if $n_d - n_g$ is different from zero, the circulation around $\{0\}$ is equal to $\frac{h}{m}(n_d - n_g)$.

Indeed in this case a vortex line is present in $\{0\}$ (roughly we have an "infinite vorticity" in $\{0\}$). Notice that for all n_d and n_g, except the case $n_d + n_g = 0$, corresponding to the ground state with bivariate symmetric gaussian density centered in $\{0\}$, $\left|\chi_{n_d,n_g}(r)\right|^2$ exhibits systems of rings of zeroes (see Figure 1).

Let now (ρ_o, v_o) be initial data for (19), smooth on the whole plane and with distributed vorticity. We choose, for a_o, A_o and Ω_o positive constants

$$v_o(r) := a_o r \hat{r} - \Omega_o r \hat{\theta}, \qquad \rho_o(r) := \frac{A_o}{2\pi} \exp\left[-\frac{A_o}{2} r^2\right]$$

The results of numerical computations with finite elements method in finite circular domains (for adimensional variables) show formation of rings of

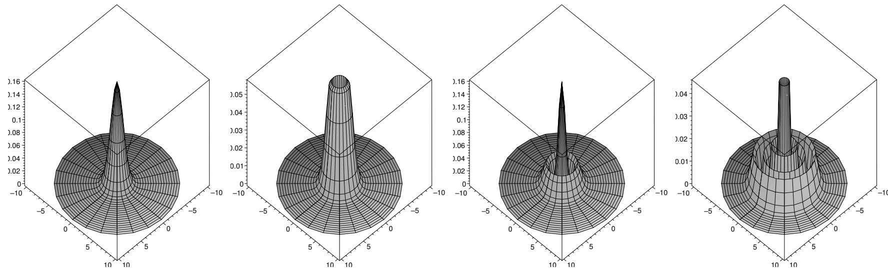

Fig. 1. Squared absolute value of some simultaneous eigenfunctions of energy and momentum operators

zeroes for the density and concentration of vorticity near such zeroes, with the approximation of an isolated vortex line in $\{0\}$ [3].

An example is given in the Figures $2-7$.

We can see that vorticity tends to take oscillating relative maxima and minima in correspondence of the zeroes and maxima of the density, respectively. In particular the periodic maximum in the origin increases in time, approaching a central vortex line.

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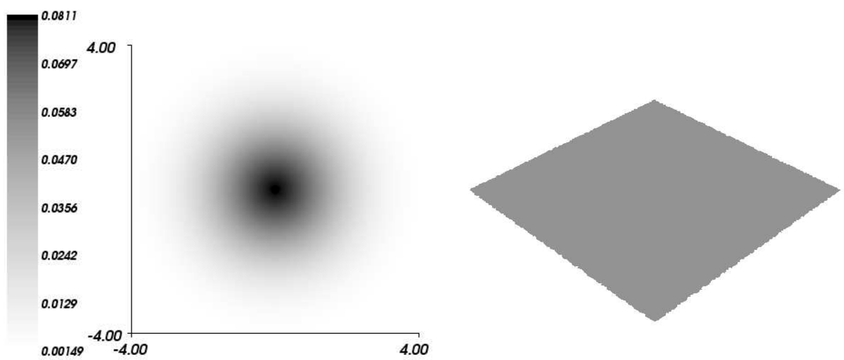

Fig. 2. ρ and $-\nabla \wedge v = 2\Omega_o$ at time $t = 0$, $E = 195$

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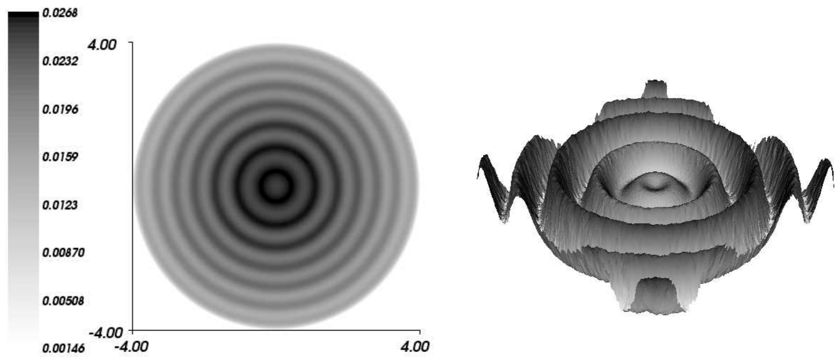

Fig. 3. ρ and $-\nabla \wedge v$ at time $t = 0.08$, $E = 43$

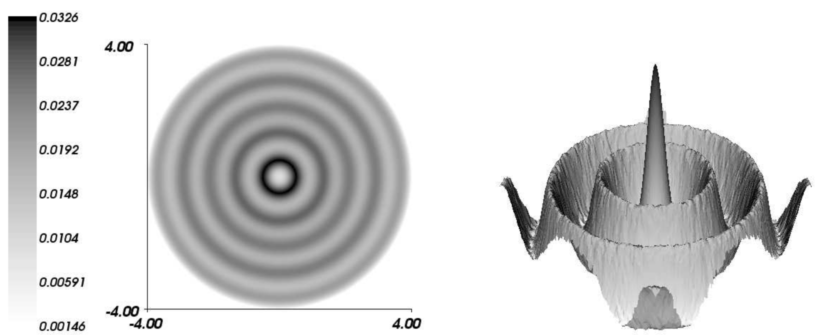

Fig. 4. ρ and $-\nabla \wedge v$ at time $t = 0.14$, $E = 17$

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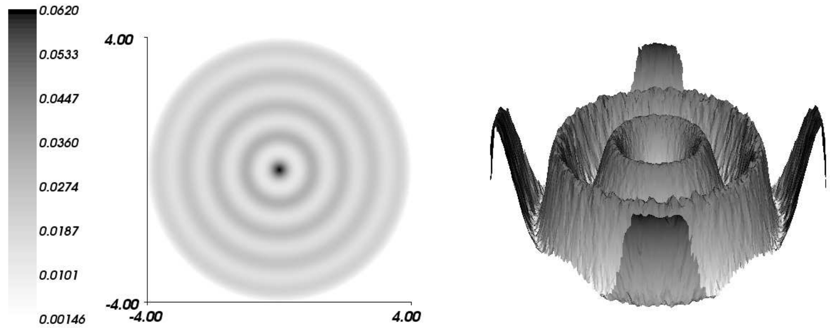

Fig. 5. ρ and $-\nabla \wedge v$ at time $t = 0.16$, $E = 15$

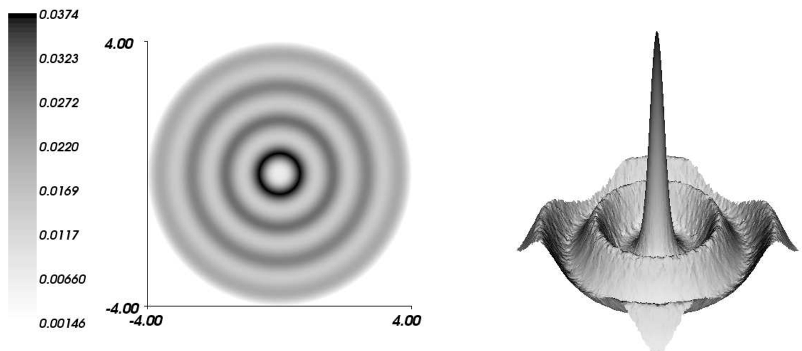

Fig. 6. ρ and $-\nabla \wedge v$ at time $t = 0.19$, $E = 11$

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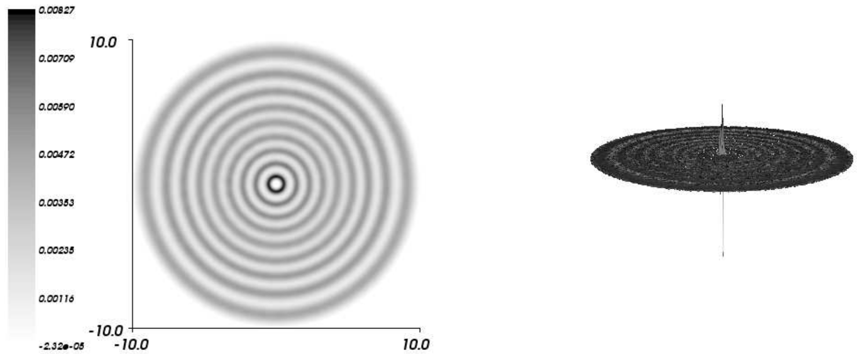

Fig. 7. ρ and $-\nabla \wedge v$ at time $t = 0.23$, $E = 9$

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