# The Invariant Distribution of a Diffusion: Some New Aspects

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## 1 Introduction

The subject is an old one, but the conventional discussion seems in one respect incomplete: *If you have an invariant distribution, what is it the distribution of?* M. Baldini and I have found an amusing answer to this question.

Fix 1) a standard $d$-dimensional Brownian motion with paths $b(t) : t \geq 0$, 2) a smooth, positive-definite diffusion coefficient $\sigma$, 3) a smooth drift coefficient $m$, and let $x^{\uparrow}(t, x) : t \geq 0$, $x \in \mathbb{R}^d$ be the flow determined by

1) $$dx^{\uparrow} = \sigma(x)db + m(x)dt \qquad \text{with } x^{\uparrow}(0, x) = x.$$

Here, "flow" means that $x^{\uparrow}(t, x) : t \geqslant 0, x \ominus \mathbb{R}^d$ is (implicity) a function of a single Brownian motion $b(t) : t \geqslant 0$. You solve 1) for $t \geqslant 0$, *simultaneously* for every *e.g.* terminating binary $x$ with the *same* Brownian motion b. Then Kolmogorov-Centsov is used to show that this, so to say "skeleton" is continuous in the past $(t, x)$ and so may be extended to the whole $[0, \infty]X\mathbb{R}^d$ so as to solve 1) identically in Cross $t$ & $x$ up to a possible explosion time, with probability $x$. This can be KONSTA [1990] together with the fact that if no explosion takes place, $x(t, \bullet)$ is a difflo=morphism of $\mathbb{R}^d$ (with probability $x$ of course. It is assumed that $x^{\uparrow}$ returns to every neighborhood of $\mathbb{R}^d$, over and over, and more: that it has a smooth invariant density $1/\psi^2$ of total mass $\int(1/\psi^2) = 1$. Then for nice functions $f$,

2′) $$\lim_{T\uparrow\infty} \frac{1}{T} \int_0^T f(x^{\uparrow})dt = \int \frac{f}{\psi^2}$$

with probability 1 for each $x^{\uparrow}(0) = x$, separately, and also

2″) $$\lim_{T\uparrow\infty} e^{T\mathfrak{g}} f = \int \frac{f}{\psi^2}$$

pointwise, of being the infinitesimal operator of the diffusion

3) $$\mathfrak{g} = \frac{1}{2}\sigma^2\partial^2/\partial x^2 + m\partial/\partial x.$$

Actually, it will be best to interpret 1) in Stratonovich's way, *i.e.* in Itô's language

1′) $$dx^{\uparrow} = \sigma db + mdt + \frac{1}{2}\sigma'\sigma dt \quad \text{with } (\sigma'\sigma)_i = \sum_{1\le j,k\le d} \frac{\partial \sigma_{ij}}{\partial x_k}\sigma_{kj},$$

and to take

3′) $$\mathfrak{g} = \frac{1}{2}\sigma^2\partial^2/\partial x^2 + (m + \frac{1}{2}\sigma'\sigma)\partial/\partial x$$

in accord with that. Itô's language is used everywhere below.

Two allied diffusions or flows are now introduced.

Fix a time $T > 0$, recompute $x^{\uparrow}(T, x)$ by solving 1′) not with the original Brownian motion $b(t) : t \le T$, but with the reversed $b^{\downarrow}(t) = b(T - t) \to (T) : t \le T$, and record only the final position $\equiv x^{\downarrow}(T, x)$. The motion $x^{\downarrow}(t, x) : t \ge 0$ is not quite a diffusion: as a diffeomorphism $x^{\downarrow}(t, \bullet) : t \ge 0$ is Markovian, but for fixed $x$, you have only

4) $$dx^{\downarrow}(t, x) = \mathfrak{g}x^{\downarrow}(t, x) - \frac{\partial x^{\downarrow}(t, x)}{\partial x}\sigma db$$

in which $\sigma$ is paired with the lower variable $x$ and with $db$. Obviously, $x^{\downarrow}(t, \bullet)$ is identical in law to $x^{\uparrow}(t, \bullet)$ for each fixed $t \ge 0$, separately, but their motion in time is very different, as will be seen: $x^{\uparrow}(T)$ is driven by the innovation $db(T)$, but for $x^{\downarrow}(T)$ the latter is buried in the past and its influence washes out. This is the first allied flow.

The second is the bona fide diffusion $x^{\sharp}(t, x) : t \ge 0, x \in R^d$ determined by

5) $$dx^{\sharp} = \sigma db + (-m + \frac{1}{2}\sigma'\sigma)dt$$

with reversed drift $-m$ in place of $m$. It is intimately related to $x^{\downarrow}$: for each $t \ge 0$, $x^{\sharp}(t, \bullet)$ is the diffeomorphism inverse to $x^{\downarrow}(t, \bullet)$, assuming that $x^{\sharp}$ does not run out to $\infty$ in finite time.

It is the inter-relation of these three processes; $x^{\uparrow}$, $x^{\downarrow}$, and $x^{\sharp}$, that I will talk about. For the matter sketched above, I refer you to Kunita [1990]; it is the best presentation.

## 2 Mostly dimension 1

I will explain what happens here: With

$$\psi^2 = Z\sigma \exp\left[-\int_0^x 2m/\sigma^2\right],$$

you have

$$\mathfrak{g} = \frac{\psi^2}{2} D \frac{\sigma^2}{\psi^2} D \text{ with scale } \int_0^x \frac{\psi^2}{\sigma^2} \text{ and speed measure } \frac{2dx}{\psi^2},$$

$$\mathfrak{g}^\sharp = \frac{1}{2}\,\frac{\sigma^2}{\psi^2} D\psi^2 D \text{ with scale and speed measure reversed },$$

and

$$\int \frac{1}{\psi^2} = 1 \qquad \text{by choice of } Z.$$

Here, $1/\psi^2$ is the invariant density for $x^\uparrow$. I take

$$s[-\infty, 0] = \int_{-\infty}^0 \frac{\psi^2}{\sigma^2} \quad \text{and} \quad s[0, +\infty) = \int_0^\infty \frac{\psi^2}{\sigma^2} \quad \text{both } = +\infty,$$

as is automatic if $\sigma = 1$. Then for fixed $x, x^\sharp(t,x)$ tends almost surely to $\pm\infty$ as $t \uparrow \infty$ (but not before). $\mathfrak{g}$ can also be written $\frac{1}{2}\bar{D}^2 + (m/\sigma)\bar{D}$ in the scale $\bar{x} = \int_0^x (1/\sigma)$; this will be useful in section 5. Note that $m$ cannot vanish: otherwise, $\psi^2$ is effectively $\sigma$ and you cannot have both $\int 1/\psi^2 < \infty$ and $\int \psi^2/\sigma^2 = \infty$.

Now the chief facts in dimension 1 are these:

1) $\lim_{t\uparrow\infty} x^\downarrow(t,x) = x^\downarrow(\infty)$ exists, independently of $x$; it is distributed with density $1/\psi^2$. (No surprise.)

2′) $x^\sharp(t,x) \uparrow +\infty$ if $x > x^\downarrow(\infty)$.

2″) $x^\sharp(t,x) \downarrow -\infty$ if $x < x^\downarrow(\infty)$.

3) $x^\sharp(T, x^\downarrow(\infty)) = x^\downarrow(\infty)$ recomputed for the shifted Brownian motion $b^+(t) = b(t+T) - b(T) : t \geq 0$; as such it can be made stationary for $-\infty < T < +\infty$, and if its time is then reversed, you will see the stationary version of $x^\uparrow$ with initial distribution $dx/\psi^2$. For this reason, $x^\downarrow(\infty)$ is called the stagnation point.

4)

$$\int \frac{dx}{\psi^2} E|x^\downarrow(\infty) - x^\downarrow(t,x)|^2, = \lim_{T\uparrow\infty} \int \frac{dx}{\psi^2} E|x^\downarrow(T,x) - x^\downarrow(t,x)|^2$$
$$= \int \frac{da}{\psi^2} \int \frac{db}{\psi^2} E|x^\uparrow(t,b) - x^\uparrow(t,a)|^2,$$

and this quantity decreases to 0 provided $\int x^2/\psi^2 < \infty$, *i.e.* $x^\uparrow$ "focuses", as found by Hasminskii-Nevelson [1971] in a different form noted later on. The decay may exponentially fast or no, as you would think if $\mathfrak{g}$ has spectrum near the origin.

5) What I would *not* have thought is that focusing always takes place pathwise exponentially fast:

$$\lim_{t\uparrow\infty} \frac{1}{t}\ell n[sox^{\uparrow}(t,b) - so(x^{\uparrow}(t,a)] = -\gamma,$$

simultaneously for every $a < b$, in which you see the natural scale $s(x) = \int_0^x \psi^2/\sigma^2$ of $x^{\uparrow}$, and $\gamma$ is the (to me) mysterious number

$$0 < \gamma = 2\int \frac{m^2}{\sigma^2}\frac{1}{\psi^2} \leq \infty.$$

This $\gamma$ is always bigger than, and in special cases equal to, the spectral gap $g$ of $\mathfrak{g}$, but this gap can vanish, so that's not $\gamma$; it is also bigger than or equal to the ground state of $\mathfrak{g}^{\sharp}$, but that can vanish, too. I think that $\gamma$ should have *some* spectral meaning, but don't know what it is.

The proofs of 1)-5) occupy the rest of this report.

Dimension $d \geq 2$ is *much* harder. You can put on unattractive conditions to make 1)-4) and a crude version of 5) come out; see Baldini [2006] for this. I believe you need next to nothing but am now just as far from the proof as I was 2 years ago.

## 3 Ornstein–Uhlenbeck

This process with $\mathfrak{g} = \frac{1}{2}D^2 - xD$, $\psi^2 = \sqrt{\pi}e^{+x^2}$, and scale $\int_0^x \sqrt{\pi}e^{y^2}dy$ will illustrate all this. Here, $dx^{\uparrow} = db - x^{\uparrow}dt$, *i.e.* $x^{\uparrow}(t,x) = e^{-t}x + e^{-t}\int_0^t e^s db$, and you have

2.1) $x^{\downarrow}(t,1x) = e^{-t}x - \int_0^t e^{-s}db$, tending to $x^{\downarrow}(\infty) = -\int_0^{\infty} e^{-s}db$,

2.2) $x^{\sharp}(t,x) = e^t x + e^t\int_0^t e^{-s}db$ tending to $\pm\infty$ according as $x > x^{\downarrow}(\infty)$ or $x < x^{\downarrow}(\infty)$,

2.3) $x^{\sharp}(t,x^{\uparrow}(\infty)) = e^t\int_t^{\infty} e^{-s}db$, is identical in law to $e^t B\left(\frac{1}{2}(e^{-2t})\right)$ with a new Brownian motion $B$, *i.e.* it is $x^{\uparrow}$ = Ornstein-Uhlenbeck made stationary.

2.4) $\partial x^{\uparrow}/\partial x = e^{-t}$, so $\int \frac{dx}{\psi^2}E|x^{\downarrow}(\infty) - x^{\downarrow}(t,x)|^2 = 2\int \frac{x^2}{\psi^2}e^{-2t} = e^{-2t}$,

2.5) $\lim\limits_{t\uparrow\infty} \frac{1}{t}\ell n\int_{x^{\uparrow}(t,a)}^{x^{\uparrow}(t,b)} \sqrt{\pi}e^{x^2}dx = -\gamma = 2\int \frac{x^2}{\psi^2} = 1$, and this number is the actual spectral gap of $\mathfrak{g}$.

## 4 Proofs in dimension 1

Recall the general function

$$\psi^2 = Z\sigma\exp[-\int_0^x 2m/\sigma^2]$$

and the form of the infinitesimal operators

$$\mathfrak{g} = \frac{\sigma^2}{2^2}D^2 + (m + \frac{1}{2}\sigma'\sigma)D = \frac{\psi^2}{2}D\frac{\sigma^2}{\psi^2}D.$$

and

$$\mathfrak{g}^\sharp = \frac{\sigma^2}{2}D^2 + (-m + \frac{1}{2}\sigma'\sigma)D = \frac{1}{2}\frac{\sigma^2}{\psi^2}D\psi^2 D.$$

2.2) is obvious: $x^\sharp(t,x)$ is transient, tending to $\pm\infty$ with probability 1 for each $x$ separately, and since $x^\sharp(t,\bullet)$ is a diffeomorphism, $x^\sharp(t,b)$ tends to $+\infty$ as soon as $x^\sharp(t,a)$ does so for any $a < b$, and so forth, the self-evident conclusion being that there is a single (random) point $x^\downarrow(\infty)$ as in 2.2): $x^\sharp(t,x)$ tends to $+\infty$ if $x > x^\downarrow(\infty)$ and to $-\infty$ if $x < x^\downarrow(\infty)$. But then, for any $\varepsilon > 0$, large $L$, and sufficiently large $T$,

$$x^\sharp(T, x^\downarrow(\infty) - \varepsilon) < -L < +L < x^\sharp(T, x^\downarrow(\infty) + \varepsilon),$$

with the implication 2.1):

$$x^\downarrow(\infty) - \varepsilon < x^\downarrow(T, -L) < x^\downarrow(T, +L) < x^\downarrow(\infty) + \varepsilon.$$

Besides, for nice $f$,

$$Efox^\downarrow(\infty) = \lim_{T\uparrow\infty} Efox^\downarrow(T,x) = \lim_{T\uparrow\infty} Efox^\uparrow(T,x) = \int \frac{f}{\psi^2},$$

by 1.2″), so $x^\downarrow(\infty)$ is distributed by the invariant density $1/\psi^2$.

2.3) is next: With a self-evident notation, $x^\downarrow(T, x|\mathbb{B}_0^T) = x^\downarrow(t,\bullet|\mathbb{B}_0^t)ox^\downarrow(T-t, x|\mathbb{B}_t^\infty)$ for $T > t$, so

$$x^\sharp(t, x^\downarrow(T,x)) = x^\downarrow(T-t, x|\mathbb{B}_t^\infty)$$

which produces

$$x^\sharp(t, x^\downarrow(\infty)) = x^\downarrow(\infty|\mathbb{B}_t^\infty)$$

at $T = \infty$, showing that $x^\sharp(t, x^\downarrow(\infty))$ is (or rather can be made) stationary. But, $dx^\sharp = \sigma db + (-m + \frac{1}{2}\sigma'\sigma)dt$, and reversing the time as in $x^\sharp(t) \to x^\flat(t) = x^\sharp(-t)$, produces $dx^\flat(t) = \sigma db + (m + \frac{1}{2}\sigma'\sigma)dt$, which is to say that the stationary $x^\sharp(\bullet, x^\downarrow(\infty))$ reversed is a copy of the stationary $x^\uparrow(t,x)$ with $x$ distributed by $1/\psi^2$.

2.4): For fixed $T > t$, $x^\downarrow(T,x) = x^\downarrow(t,\bullet|\mathbb{B}_0^t)ox^\downarrow(T-t, x|\mathbb{B}_t^\infty)$ is identical in law to $x^\downarrow(t,\bullet|\mathbb{B}_0^t)ox^\uparrow(T-t, x|\mathbb{B}_t^\infty)$, by the independence of the fields $\mathbb{B}_0^t$ and $\mathbb{B}_t^\infty$, so you have

$$E[x^\downarrow(T,x)|\mathbb{B}_0^t] = e^{(T-t)\mathfrak{g}}x^\downarrow(t,x)$$

with $e^{(T-t)\mathfrak{g}}$ applied to the variable $x$, provided $\int x^2/\psi^2 < \infty$. Now the same rule applies if $f$ is any nice (*e.g.* smooth, compact) function:

$$E[fox^{\downarrow}(T,x)|\mathbb{B}_0^t] = e^{(T-t)\mathfrak{g}} fox^{\downarrow}(t,x),$$

so

$$\int \frac{dx}{\psi^2} E|fox^{\downarrow}(T,x) - fox^{\downarrow}(t,x)|^2$$

$$= \int \frac{dx}{\psi^2} E[f^2 ox^{\downarrow}(T,x) - 2e^{T-t)\mathfrak{g}} fox^{\downarrow}(t,x) \times fox^{\downarrow}(t,x) + f^2 ox^{\downarrow}(t,x)]$$

in which all the arrows can be turned up, producing

$$2\int \frac{f^2}{\psi^2} - 2\int \frac{dx}{\psi^2} E[e^{(T-t)\mathfrak{g}} fox^{\uparrow}(t,x) \times fox^{\uparrow}(t,x)]$$

$$\simeq 2\int \frac{f^2}{\psi^2} - 2\int \frac{dx}{\psi^2} E[\int fox^{\uparrow}(t,x')\frac{dx'}{\psi^2} \times fox^{\uparrow}(t,x)]$$

for $T \uparrow \infty$, by 1.2″), *i.e*

$$\int \frac{dx}{\psi^2} E|fox^{\downarrow}(\infty) - fox^{\downarrow}(t,x)|^2$$

$$= \int \frac{dx}{\psi^2} \int \frac{db}{\psi^2} E|fox^{\uparrow}(t,b) - fox^{\uparrow}(t,x)|^2$$

as in 2.4). Besides,

$$\int \frac{dx}{\psi^2} E|fox^{\downarrow}(\infty) - fox^{\downarrow}(t,x)|^2$$

$$= 2\int \frac{f^2}{\psi^2} - 2E[fox^{\downarrow}(\infty)\int \frac{fox^{\downarrow}(t,x)}{\psi^2} dx],$$

and here

$$\int fox^{\downarrow}(t,x)\frac{dx}{\psi^2} = E[fox^{\downarrow}(\infty)|\mathbb{B}_0^t]$$

is a martingale and also a projection, which is to say

$$\int \frac{dx}{\psi^2} E|fox^{\downarrow}(\infty) - fox^{\downarrow}(t,x)|^2 \downarrow 0 \quad \text{as } t \uparrow \infty.$$

The rest, which is to carry all this over to $f(x) = x$ is easy: if $f(x^{\downarrow}) = x^{\downarrow} \times$ the indicator of $|x^{\downarrow}| \le R$, then

$$\int \frac{dx}{\psi^2} E|fox^{\downarrow}(t,x) - x^{\downarrow}(t,x)|^2 = \int\limits_{|x|>R} x^2/\psi^2$$

is small for large $R$, independently of $t \ge 0$.

2.5) is surprising, but its pretty easy, too. It states that, in the natural scale $s(x) = \int_0^x \psi^2/\sigma^2$, $x^{\uparrow}$ focuses pathwise, exponentially fast, at rate $\gamma = 2\int m^2/\sigma^2\psi^2$. Fix $a < b$ and write $A$ for $sox^{\uparrow}(t,x)$ and $B$ for $sox^{\uparrow}(t,b)$.

*Step 1.*

The role of the scale is to make $B - A$ a (positive) super-martingale ($\mathfrak{g}s = 0$). As such, it has a limit $0 \leq C < \infty$. Now

$$\lim_{T\uparrow\infty} \frac{1}{T}\int_0^T tan^{-1}(B)dt = \int tan^{-1} os(x)\frac{dx}{\psi^2} \qquad \text{by } 1.2'),$$

and

$$tan^{-1}(B) \simeq tan^{-1}(A + C) \text{ for } t \uparrow \infty,$$

so also

$$\lim_{T\uparrow\infty} \frac{1}{T}\int_0^T tan^{-1}(B)dt = \int tan^{-1} o[s(x) + C]\frac{dx}{\psi^2}.$$

This is not possible unless $C = 0$, *i.e.* $B - A = o(1)$. Thus far Hasminski-Nevelson [1971: lemma 2, Part I].

*Step 2*

is to compute the differential of $B - A$: with

$$F = \left[\frac{\psi^2}{\sigma} ox^{\uparrow}(t, b) - \frac{\psi^2}{\sigma} ox^{\uparrow}(t, a)\right] \times (B - A)^{-1}$$

you find

$$d(B - A) = (B - A) \neq x$$

the differential db of the Brownian motion and so you may write

$$B - A[B(0) - A(0)]x^{\int_0^t F db - \frac{1}{2}\int_0^t F^2 dt}.$$

*Step 3*

is an over-estimate. The mean-value theorem is applied to $F$ as follows:

$$\left(\frac{\psi^2}{\sigma} os^{-1}\right)' = -\frac{2m}{\sigma^2}\,\frac{\psi^2}{\sigma} os^{-1} \times \left(\frac{\psi^2}{\sigma^2} o(s^{-1})\right)^{-1} = -\frac{2m}{\sigma} o(s^{-1}),$$

so

$$F = \left(\frac{\psi^2}{\sigma} os^{-1}\right)(B) - \left(\frac{\psi^2}{\sigma} os^{-1}\right)(A) \qquad \text{over } B - A$$

$$= \left(-\frac{2m}{\sigma} os^{-1}\right)(C) \quad \text{with } C \text{ between } A \text{ and } B.$$

The peculiar instance on $s^{-1}$ pays off as follows. Take $G < 2(m^2/\sigma^2)os^{-1}$ with bounded slope. Then $G(C) \simeq G(B)$ for $t \uparrow \infty$, by step 1, and

$$\lim_{T\uparrow\infty} \frac{1}{T}\,\frac{1}{2}\int_0^T F^2 dt \geq \lim_{T\uparrow\infty}\frac{1}{T}\int_0^T G(B)dt$$
$$= \int Gos(x)\frac{dx}{\psi^2}$$
$$> \gamma'$$

for any number $\gamma' < \gamma$, by choice of $G$, *i.e.* by step 2,

$$\lim_{t\uparrow\infty}\frac{1}{t}\ell n(B-A) \leq -\gamma$$

in view of

$$\left|\int_0^t F db\right| \leq \sqrt{(2+)\int_0^t F^2\ \ell n\ell n\int_0^t F^2}.$$

*Step 4*

is the final under-estimate: Now write

$$B - A = \int_a^b \frac{\partial X}{\partial x}dx \qquad \text{with } X = sox^{\uparrow}(t,x).$$

You have $dX = (\psi^2/\sigma)(x^{\uparrow})db$, so

$$d\frac{\partial X}{\partial x} = -\frac{2m}{\sigma}\psi^2(x^{\uparrow})\frac{\partial X}{\partial x}db$$

and

$$\frac{\partial X}{\partial x} = e^{-2\int_0^t \frac{m}{\sigma}(x^{\uparrow})db - 2\int_0^t \frac{m^2}{\sigma^2}(x^{\uparrow})dt'}.$$

Now if $\gamma = \infty$, there is nothing to do, while if $\gamma < \infty$ then, for any $\gamma' > \gamma$, Fatou's lemma implies

$$\underline{\lim}_{t\uparrow\infty} e^{\gamma' t}(B-A)$$
$$\geq \int_a^b \underline{\lim}_{t\uparrow\infty} e^{\gamma' t}e^{-2\int_0^t \frac{m}{\sigma}(x^{\uparrow})db - 2\int_0^t \frac{m^2}{\sigma^2}(x^{\uparrow})dt'}dx$$
$$= +\infty$$

in view of

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T 2\frac{m^2}{\sigma^2}(x^{\uparrow})dt \quad = 2\int\frac{m^2}{\sigma^2}\frac{1}{\psi^2} = \gamma,$$

*i.e.*

$$\underline{\lim}_{t\uparrow\infty}\frac{1}{t}\ell n(B-A) \geq -\gamma.$$

## 5 More about $\gamma$

The proof of 2.1)–2.5) is finished, but what is $\gamma$? Surely, it has some spectral meaning, but I don't know what. It has a little to do with the spectral gap of $\mathfrak{g}$, which is the distance $g$ from its ground state ($= 0$ since $\mathfrak{g}1 = 0$) to the rest of its spectrum. This is the infimum of the quadratic form

$$Q \equiv -\int f\mathfrak{g}f\frac{1}{\psi^2} = \frac{1}{2}\int \frac{{f'}^2\sigma^2}{\psi^2} \text{ for nice } f \text{ with } \int \frac{f^2}{\psi^2} < \infty \text{ and } \int \frac{f}{\psi^2} = 0.$$

*Item 1:*

$g \leq \gamma$. Take $f = A(\int_0^x \frac{1}{\sigma} - B)$ on a big interval $I = [-a, b]$ and extend it to the right/left by the constant values $f(b)/f(-a)$, with $B$ taken to make $\int f/\psi^2 = 0$ and $A > 0$ to make $\int f^2/\psi^2 = 1$. Then, with $N = \int_I 1/\psi^2$,

$$Q = \frac{1}{2}A^2 N,$$

$$A/N = \int\limits_I \frac{f'\sigma}{\psi^2},$$

and so

$$Q = \frac{1}{2N}\left(\int\limits_I \frac{f'\sigma}{\psi^2}\right)^2.$$

I want to integrate by parts for which I need $\underline{\lim}_{x\uparrow\infty}\, \sigma/\psi^2 = 0$ and likewise at $-\infty$. But if, for example, $\underline{\lim}_{x\uparrow\infty}\, \sigma/\psi^2 = 2$, then $1/\psi^2 \geq 1/\sigma \geq \psi^2/\sigma^2$ far out, contradicting $\int \psi^2/\sigma^2 = \infty$. Now you can write

$$\begin{aligned} g \leq Q &= \frac{1}{2N}\left(\int_{-\infty}^{+\infty} f\left(\frac{\sigma}{\psi^2}\right)'\right)^2 \\ &= \frac{1}{2N}\left(\int 2f\frac{m}{\sigma}\cdot\frac{1}{\psi^2}\right)^2 \\ &\leq \frac{2}{N}\int\frac{f^2}{\psi^2}\int\frac{m^2}{\sigma^2}\,\frac{1}{\psi^2} \\ &= \frac{\gamma}{N}, \end{aligned}$$

and making $I$ increase to the whole line makes $N \uparrow 1$, confirming $g \leq \gamma$.

*Item 2:*

$g$ can vanish so that is not the meaning of $\gamma$. Take $\sigma = 1$ and $\psi^2 = \pi(1+x^2)$. Then $m = -\psi'/\psi = -x \times (1+x^2)^{-1}$, and $\gamma = 1$, while if $f$ is the odd function $x/h$ for $0 \le x \le h$ and 1 beyond, then

$$\int f/\psi^2 = 0 \quad \text{and} \quad \frac{1}{2}\frac{\int f'^2/\psi^2}{\int f^2/\psi^2} \le \frac{h^{-2}\int\limits_0^h \frac{1}{\pi(1+x^2)}}{2\int_h^\infty \frac{1}{\pi(1+x^2)}} \simeq \frac{h^{-2}}{2/\pi h} = o(1)$$

for $h \uparrow \infty$.

*Item 3.*

$g = \gamma$ only if $\int\limits_{-\infty}^{0} 1/\sigma = \int_0^\infty 1/\sigma = +\infty$ and

$$\bar{x}(t,x) = \int_0^{x^\uparrow(t,x)} \frac{1}{\sigma(y)} dy$$

is the standard Ornstein-Uhlenbeck process, up to scalings $x \to ax+b$ and $t \to ct$. The proof uses the second display of item 1 in the form

$$\frac{\gamma}{2} = \int \frac{m^2}{\sigma^2}\frac{1}{\psi^2} \le \frac{1}{N}\left(\int \frac{fm}{\sigma}\frac{1}{\psi^2}\right)^2 = \text{ with } N \text{ and } f \text{ as before,}$$

which is to say

$$\int f\frac{m}{\sigma}\frac{1}{\psi^2} < -\sqrt{\frac{N\gamma}{2}}$$

in view of $A/N = \int f'\sigma/\psi^2 > 0$. This permits you to estimate

$$\begin{aligned}\int (f + C\frac{m}{\sigma})^2 \frac{1}{\psi^2} &\le 1 + 2C\int f\frac{m}{\sigma}\frac{1}{\psi^2} + C^2\int \frac{m^2}{\sigma^2}\frac{1}{\psi^2} \\ &< 1 - 2C\sqrt{\frac{N\gamma}{2}} + \frac{C^2}{2}\gamma \\ &= o(1)\end{aligned}$$

for $I$ increasing to the whole line, by choice of $C = \sqrt{2/\gamma}$, so that

$$f = A\left(\int_0^x \frac{1}{\sigma} - B\right) = -\sqrt{\frac{2}{\gamma}}\frac{m}{\sigma}.$$

with an error which is small in mean-square. It follows easily that, in the limit $I = \mathbb{R}$,

$$\frac{m}{\sigma} = -A\int_0^x \frac{1}{\sigma} + B$$

with new constants $A \geq 0$ and $-\infty < B < \infty$. Now

$$\mathfrak{g} = \frac{1}{2}\sigma D \sigma D + \frac{m}{\sigma}\sigma D = \frac{1}{2}\bar{D}^2 + (-A\bar{x} + B)\bar{D}$$

in the new scale $\bar{x} = \int_0^x 1/\sigma$, which is to say that $\bar{x}(t,x) = \bar{x} o x^{\uparrow}(t,x)$ solves $d\bar{x} = db + (-A\bar{x} + B)dt$, *i.e.* it is a sort of Ornstein-Uhlenbeck process if $A > 0$, or a Brownian motion with drift *if* $A = 0$, up to the first time it comes to $\bar{x}(-\infty) = -\int_{-\infty}^0 1/\sigma$ or $\bar{x}(+\infty) = \int_0^\infty 1/\sigma$, which must be finite if either of $\bar{x}(\pm\infty)$ is finite. But this never happens since $x^{\uparrow}$ never comes to $\pm\infty$, so $\int_{-\infty}^0 1/\sigma = \int\limits_0^\infty 1/\sigma = \infty$, and $\bar{x}$ reduces to standard Ornstein-Uhlenbeck by scaling; in particular, if $\sigma = 1$, $x^{\uparrow}$ itself may be so reduced.

*Item 4.*

The best interpretation of $\gamma$ I have found is in terms of Fisher's information $\int (f')^2/f$. Write $\mathfrak{g} = \frac{1}{2}\sigma D \sigma D + mD = \frac{1}{2}\bar{D}^2 + \frac{m}{\sigma}\bar{D}$ in the scale $\bar{x} = \int_0^x 1/\sigma$. The invariant density relative to the new scale is $f(\bar{x}) = (\sigma/\psi^2)(x)$, and from $2m/\sigma = (\sigma/\psi^2)'\psi^2 = f'/f$, you find

$$\int \frac{f'^2}{f}\, d\bar{x} = \int \sigma \left(\frac{\sigma}{\psi^2}\right)'^2 \frac{\psi^2}{\sigma}\frac{dx}{\sigma} = \psi \int \frac{m^2}{\sigma^2}\frac{1}{\psi^2} = 2\gamma$$

Note that

$$\int \frac{f'^2}{f}\, d\bar{x} \int \bar{x}^2 f d\bar{x} \geq \left(\int f'\bar{x} d\bar{x}\right)^2 = \left(\int f d\bar{x}\right)^2 = 1$$

provided $\int \bar{x}^2 f d\bar{x} < \infty$, so

$$\gamma \geq \frac{1}{2}\left(\int \frac{\bar{x}^2}{\psi^2}\right)^{-1} \text{ which is } \geq \frac{1}{2}\left(\int \frac{x^2}{\psi^2}\right)^{-1} \text{ if } \sigma \geq 1.$$

*Item 5.*

Fisher's information *does* have a spectral meaning of sorts, as Varadhan suggested to me. Express $\mathfrak{g}$ in the scale $\bar{x}$ as $\frac{1}{2}D^2 + mD$ where, for simplicity, $\bar{x}$ has been replaced by $x$, plain, and $m/\sigma$ by $m$. The invariant density is $f = \exp(\int 2m)/\mathbb{Z}$, and $-2\sqrt{f}\mathfrak{g}/\sqrt{f}$, which is similar to $-2\mathfrak{g}$, turns out to be $-D^2 + v$ with $v = m' + m^2$. The latter has ground state $e = \sqrt{f}$ with $\int e^2 = \int f = 1$, as is obvious from $\mathfrak{g}1 = 0$. Now, for general $v$, if $e$ is the ground state of $-D^2+v$ with eigenvalue $\lambda(v)$ and $\int e^2 = 1$, then the (convex) dual $\lambda^*(u)$ of the (convex) function $\lambda(v)$ is the minimum in respect to $v$ of the form $-\int uv + \lambda(v)$. Take $u = f = e^2$. Then from $\text{grad}[-\int uv + \lambda(v)] = -u + e^2$, you see that

$$\lambda^*(f) = -\int e^2 v + \lambda(v) \quad \text{with} \quad v = m' + m^2 \quad \text{as above}$$
$$= -\int e'' e = \int e'^2 = \frac{1}{4}\int f'^2/f.$$

In this way,$\gamma = 2\times$ Fisher's information is related to the ground state eigenvalue of $\mathfrak{g}$, which is amusing, but, for me, $\gamma$ lies still in some obscurity.

## References

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