Different Lattice Approximations for Høegh-Krohn's quantum field model

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Dedicated to Prof. Kiyosi Itô on the occasion of his 90th birthday

1 Introduction

It is a fundamental question whether one can give a rigorous meaning to quantum fields described heuristically by a "probability measure" on the space of Schwartz tempered distributions. Methods used include, for example, lattice approximation and wavelet approximation. We focus on the lattice approximation in this article.

Our main target of this article is Høegh-Krohn's quantum field model, which was first introduced in [8]. Simply speaking, Høegh-Krohn's quantum field model is the model with interaction of the form $: \exp \alpha\phi :$, here $: \cdot :$ means the Wick power.

Albeverio–Høegh-Krohn [6] showed that, for this model, when using the same cutoffs for the free and the interacting parts (see below for the precise meaning of these terms), the approximating probability measure converges as the lattice cutoff is removed. In this sense, they gave a rigorous meaning to Høegh-Krohn's quantum field model by using lattice approximation.

In this article, we consider lattice approximations with different lattice cutoffs in the free and the interacting parts.

2 Lattice approximation of the free field

First, let us recall the meaning of lattice approximation of the free field. Let $m_0 > 0$ be a fixed number and let μ_0 be the (Nelson or Euclidean) free field measure on $\mathbf{R}^2$ of mass m_0, *i.e.*, Gaussian measure on $\mathcal{S}'(\mathbf{R}^2)$ with the covariance $(-\Delta+m_0^2)^{-1}$. Here $\mathcal{S}'(\mathbf{R}^2)$ denotes the space of Schwartz tempered distributions, which is given as the dual of $\mathcal{S}(\mathbf{R}^2)$, the Schwartz space of rapidly decreasing smooth test functions, in $L^2(\mathbf{R}^2)$.

For any $a > 0$, let G_a be the free lattice measure of m_0 and lattice spacing a on $a\mathbf{Z}^2$, and let $C^{(a)}(x-y) = \langle \phi_x \phi_y \rangle_{G_a}$ for $x, y \in a\mathbf{Z}^2$, where $\langle \cdot \rangle_*$ denotes the expectation with respect to $*$. G_a is thus the lattice Gaussian measure with covariance $C^{(a)}$. One has by definition (see [9])

$$C^{(a)}(x-y) = (2\pi)^{-2} \int_{[-\frac{\pi}{a}, \frac{\pi}{a}]^2} e^{ik\cdot(x-y)} \mu_a(k)^{-2} dk,$$

where $\mu_a(k) := \left(m_0^2 + 2a^{-2} \sum_{j=1}^2 (1 - \cos(ak_j))\right)^{1/2}$ for $k = (k_1, k_2)$.

Notice that G_a is a probability measure on $a\mathbf{Z}^2$. So $\{G_a\}_{a>0}$ is a family of probability measures on different spaces. In order to consider their convergence, it is useful to convert them to probability measures on a common space. This is done in the following way. Let $\mu(k) := (m_0^2 + |k|^2)^{1/2}$, where $k = (k_1, k_2)$, $|k|^2 = k_1^2 + k_2^2$, and let $f_{a,x}(\cdot)$ be the function whose Fourier transform is

$$\mathcal{F}(f_{a,x})(k) = (2\pi)^{-1} e^{-ik\cdot x} \mu_a(k)^{-1} \mu(k) 1_{[-\frac{\pi}{a}, \frac{\pi}{a}]}(k_1) 1_{[-\frac{\pi}{a}, \frac{\pi}{a}]}(k_2).$$

Denote by ϕ the coordinate process associated with μ_0 (called Nelson's or Euclidean free field): ϕ is first defined as an element of $\mathcal{S}'(\mathbf{R}^2)$, so that $\phi(g)$ is the dualization of $g \in \mathcal{S}(\mathbf{R}^2)$ with $\phi \in \mathcal{S}'(\mathbf{R}^2)$. ϕ is then extended by continuity in $L^2(d\mu_0)$ to a linear process $\phi(g)$, with g belonging to a larger space than $\mathcal{S}(\mathbf{R}^2)$. In fact, this space contains functions of the form $f_{a,x}$, and it is easy to check that

$$\langle \phi(f_{a,x}) \phi(f_{a,y}) \rangle_{\mu_0} = \langle \phi_x \phi_y \rangle_{G_a}.$$

In this sense, we can realize the above Gaussian field ϕ_x on $a\mathbf{Z}^2$ by $\phi(f_{a,x})$ defined on $\mathcal{S}'(\mathbf{R}^2)$. (See, *e.g.*, [6], [9] for details).

In this way, we say that μ_0 can be approximated by G_a, as $a \to 0$.

3 Høegh-Krohn's model and its lattice approximations

Let us first give the heuristic "definition" of Høegh-Krohn's quantum field model. Let ν be any even positive measure with finite total mass and with support supp$(\nu) \subset [-\alpha_0, \alpha_0]$ for some $\alpha_0 < \frac{4}{\sqrt{\pi}}$, and let Λ be any compact subset of $\mathbf{R}^2$. Høegh-Krohn's quantum field model is heuristically given by

$$Z^{-1} e^{-\lambda \int_\Lambda dx \left(\int :e^{\alpha\phi_x}: \nu(d\alpha) \right)} \mu_0(d\phi). \tag{1}$$

Here λ is a positive constant, Z is the normalizing constant (depending on λ, Λ and α), and $: e^{\alpha\phi_x} :$ means the Wick exponential of $\alpha\phi_x$, *i.e.*, $: e^{\alpha\phi_x} : =$

$\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} : \phi_x^k :$, where $: \phi_x^k :$ is the k-th Wick power of ϕ_x with respect to μ_0 (see [9] for the precise definition of the latter).

As declared, we are interested in the lattice approximation of it. We have already given the lattice approximation G_a for the free part μ_0, and we still need to consider the lattice approximation of the interacting part. More precisely, we need to approximate the integral $\int_\Lambda$ on a lattice. We can do so by using either of the following approximation:

1. $a^2 \sum_{\ell \in \mathbf{Z}^2 : a\ell \in \Lambda}$,
2. $a'^2 \sum_{\ell \in \mathbf{Z}^2 : a'\ell \in \Lambda}$ (which will be written as $\int_{a'\mathbf{Z}^2 \cap \Lambda} dx$ for the sake of simplicity), with $a' = a'(a) \geq a$ satisfying $\lim_{a \to 0} a'(a) = 0$ and $a'\mathbf{Z}^2 \subset a\mathbf{Z}^2$.

The first one means that we use same "lattice cutoff" for both the free part and the interacting part. This approximation has been discussed by [6]. However, one can also use the latter approximation given above, which corresponds to the case of different lattice cutoffs for the free and the interacting part.

Correspondingly, we can consider the probability measure $\mu_{\lambda,a,a'}$ on $\mathcal{S}'(\mathbf{R}^2)$ given by

$$\mu_{\lambda,a,a'}(d\phi) \equiv Z_{\lambda,a,a'}^{-1} e^{-\lambda \int_{a'\mathbf{Z}^2 \cap \Lambda} dx \left(\int :e^{\alpha \phi(f_{a,x})}: \nu(d\alpha) \right)} \mu_0(d\phi),$$

where $Z_{\lambda,a,a'}$ is the normalizing constant. This is the object of our present article, and we want to know whether $\mu_{\lambda,a,a'}$ converges as $a \to 0$.

The corresponding problem for the ϕ_2^4-quantum field model has been discussed in [1].

4 Motivations

The study of this "different lattice cutoffs" problem was first motivated by the attempt to understand better the 3 space-time dimensional case, which is believed to have totally different properties from the 2 dimensional case. For example, although the ϕ_2^4-quantum field model with interaction in a bounded region Λ is equivalent to the 2 dimensional free field, it is believed that the ϕ_3^4-quantum field model with a corresponding interaction is singular to the 3 dimensional free field.

It is well-known that for the free field in d space-time dimensions, the Schwinger function $S(x, y) \equiv \langle \phi_x \phi_y \rangle_{\mu_0}$ has order

$$S_2(x, y) \sim \begin{cases} |\log |x-y||, & \text{if } d = 2, \\ \frac{1}{|x-y|}, & \text{if } d = 3, \end{cases}$$

as $|x - y| \to 0$. Here $a \sim b$ stands for that $\frac{a}{b}$ converges to a constant in $(0, \infty)$.

In other words, when using the same cutoff for both the free part and the interacting part, the situations in 2 space-time dimensional case and in 3 space-time dimensional case are actually different: we are using the lattice cutoff for the interacting part of the same order as for the Schwinger functions of the free field in 3 space-time dimensional case, and are using a lattice cutoff for the interacting part of a different order from the one of the Schwinger functions of the free field in 2 space-time dimensional case.

Therefore, it will be interesting to discuss what will happen when we use the lattice cutoff for the interacting part of the order of the Schwinger functions of the free field in 2 space-time dimensional case also. This is the first motivation of our present research.

To see another motivation more clearly, let us use the ϕ_2^4-model to explain. The ϕ_2^4-quantum field model is the probability measure on $\mathcal{S}'(\mathbf{R}^2)$ heuristically described by

$$Z^{-1}e^{-\lambda\int_{\Lambda}:\phi_x^4:dx}\mu_0(d\phi),$$

with the notations same as before. The lattice approximation of it with different cutoffs a and $a'(a)$ in the free and the interacting parts, respectively, is given by

$$\mu_{a,a'} = Z_{a,a'}^{-1}e^{-\lambda\int_{a'\mathbf{Z}^2\cap\Lambda}dx:\phi(f_{a,x})^4:}\mu_0(d\phi),$$

where $Z_{a,a'}$ is the normalizing constant.

Since $\int_\Lambda\int_\Lambda(\log|x-y|)^4dxdy<\infty$, by a simple calculation, we have that

$$E\Big[\Big(\int_{a'\mathbf{Z}^2\cap\Lambda}dx:\phi(f_{a,x})^4:\Big)^2\Big]\sim\begin{cases}a'^2|\log a|^4, & \text{if } \lim_{a\to0}a'|\log a|^2=+\infty,\\ 1, & \text{if } \lim_{a\to0}a'|\log a|^2<+\infty,\end{cases}$$

as $a\to0$. Therefore, the order of the interaction changes dramatically according to whether $a'|\log a|^2$ converges or diverges as $a\to0$. More precisely, [5] showed that if $\lim_{a\to0}a'|\log a|^2=+\infty$, then $\frac{1}{a'|\log a|^2}\int_{a'\mathbf{Z}^2\cap\Lambda}dx:\phi(f_{a,x})^4:$ under μ_0 converges to a Gaussian measure. (The corresponding central limit theory problem for Høegh-Krohn's quantum field model has been discussed by [4]). Therefore, when $\lim_{a\to0}a'|\log a|^2=+\infty$, the density of the probability measure $\mu_{a,a'}$ with respect to μ_0, which is given by

$$Z_{a,a'}^{-1}\exp\Big(-\lambda a'|\log a|^2\Big\{\frac{1}{a'|\log a|^2}\int_{a'\mathbf{Z}^2\cap\Lambda}dx:\phi(f_{a,x})^4:\Big\}\Big),$$

is getting more and more singular as $a\to0$. This gives us hope to find the limit of $\mu_{a,a'}$ as a new probability measure, different from the "classical" ϕ_2^4-field in Λ (given as the limit of $\mu_{a,a'}$ with $a'=a$), which is well-known to be equivalent with respect to μ_0 (see [3] for some related results).

In other words, by using this different cutoff approximation procedure, it seems to be hopeful to find two (singular with respect to each other) probability measures, which are given by the same heuristic one.

5 Results and the sketch of proof

Let us come back to our Høegh-Krohn's quantum field model in $\mathbf{R}^2$ described in Section 3. For any $m\in\mathbf{N}$, let $S_{2m}^{\lambda,a,a'}$ be the $2m$-point-function given by

$$S_{2m}^{\lambda,a,a'}(x_1,\cdots,x_{2m}) \equiv \langle \phi(f_{a,x_1})\cdots\phi(f_{a,x_{2m}})\rangle_{\mu_{\lambda,a,a'}}.$$

Then we have the following result about the existence of the limit probability measure (see [2]).

Theorem 1 (Albeverio–Liang [2]). *Assume that* $\lim_{a\to 0} a'(a)|\log a| < \infty$. *Then there exists a* $\lambda_0 > 0$ *such that for any* $\lambda \in [0,\lambda_0]$, *there exists a sequence* $\{a_n\}_{n\in\mathbf{N}}$ *with* $\lim_{n\to\infty} a_n = 0$ (*and writing* $a'(a_n) = a'_n$) *such that for any given* $m \in \mathbf{N}$ *and* $f_1,\cdots,f_{2m} \in \mathcal{S}(\mathbf{R}^2)$, *the following limit exists*

$$S_{2m}^{\lambda}(f_1,\cdots,f_{2m}) := \lim_{n\to\infty} \sum_{x_1,\cdots,x_{2m}} S_{2m}^{\lambda,a_n,a'_n}(x_1,\cdots,x_{2m}) \prod_{i=1}^{2m} a_n^2 f_i(x_i).$$

Moreover, there exists a probability measure μ_λ *on* $\mathcal{S}'(\mathbf{R}^2)$ (*which may depend on* Λ)*such that*

$$S_{2m}^{\lambda}(f_1,\cdots,f_{2m}) = \int_{\mathcal{S}'(\mathbf{R}^2)} \phi(f_1)\cdots\phi(f_{2m})\mu_\lambda(d\phi)$$

for $m \in \mathbf{N}$ *and* $f_1,\cdots,f_{2m} \in \mathcal{S}(\mathbf{R}^2)$.

The idea of the proof (see [2]) is learned from Brydges–Fröhlich–Sokal [7]. We give the sketch of the proof in the following (for details see [2]). First, we have the following skeleton inequality:

Lemma 2.

$$S_{2m}^{\lambda,a,a'}(x_1,\cdots,x_{2m}) \geq \frac{1}{(2m-1)!} \sum_{\pi\in Q_{2m}} \prod_{i=1}^{m} S_2^{\lambda,a,a'}(x_{\pi(2i-1)},x_{\pi(2i)}),$$

$$S_{2m}^{\lambda,a,a'}(x_1,\cdots,x_{2m}) \leq \sum_{\pi\in Q_{2m}} \prod_{i=1}^{m} S_2^{\lambda,a,a'}(x_{\pi(2i-1)},x_{\pi(2i)}),$$

with Q_{2m} *denoting the set of all pair-partitions of* $\{1,2,\cdots,2m\}$.

By Lemma 2, we have that the behavior of $S_{2m}^{\lambda,a,a'}$ is dominated by $S_2^{\lambda,a,a'}$. Therefore, we only need to estimate $S_2^{\lambda,a,a'}$.

For $S_2^{\lambda,a,a'}$, we first show the following relation between $S_2^{\lambda,a,a'}$ and $C^{(a)}$, the covariance of the lattice free field G_a, by using the integration by parts formula for the lattice free field.

Lemma 3.

$$\begin{aligned} &S_2^{\lambda,a,a'}(x,y) - C^{(a)}(x-y) \\ &= \lambda \int_{a'\mathbf{Z}^2\cap\Lambda} dz \langle \Big(\int \nu(d\alpha)\alpha : e^{\alpha\phi(f_{a,z})} : \Big) \phi(f_{a,y})\rangle_{\mu_{\lambda,a,a'}} C^{(a)}(x-z). \end{aligned}$$

It is well-known that

1. $C^{(a)}(z) \le C(1+|\log|z||), z \in a\mathbf{Z}^2 \setminus \{0\}$,
2. $C^{(a)}(0) \le C|\log a|$ as $a \to 0$.

Therefore, it is sufficient to show that the difference between $S_2^{\lambda,a,a'}(x,y)$ and $C^{(a)}(x-y)$ is small enough. Let

$$X_{\lambda,a} = \sup_{x,y \in a'\mathbf{Z}^2} |S_2^{\lambda,a,a'}(x,y) - C^{(a)}(x-y)|.$$

Then by using Lemma 2 (ii), Lemma 3 and the Taylor expansion for $e^{\alpha\phi(f_{a,z})}$, we can show the following.

Lemma 4. *Suppose that* $\lim_{a\to 0} a'|\log a| < \infty$. *Then there exists a constant* $c_1 > 0$ *independent of* a *such that*

$$X_{\lambda,a} \le \lambda c_1 (X_{\lambda,a}+1) e^{\frac{1}{2}\alpha_0^2 X_{\lambda,a}}, \tag{2}$$

where α_0 *is the constant given in Section 3.*

Let $\lambda_0 = (3C_1 e^{\alpha_0^2})^{-1}$. Then for any $\lambda \in [0,\lambda_0]$, we have by (2) that if $X_{\lambda,a} \le 2$ then $X_{\lambda,a} \le 1$. In other words, $X_{\lambda,a}$ can never take values in $(1,2]$. This combined with the fact that $X_{0,a} = 0$ and that $X_{\lambda,a}$ is continuous with respec to λ gives us that $X_{\lambda,a} \le 1$. Substituting this into (2) again, we get the following: There exist constants $\lambda_0, C_2 > 0$ independent of a such that

$$X_{\lambda,a} \le C_2\lambda, \qquad \text{for any } \lambda \in [0,\lambda_0], a > 0.$$

Therefore, $S_2^{\lambda,a,a'}(x,y)$ behaves like $|\log|x-y||$ as $|x-y| \to 0$. This completes the proof of our theorem.

6 Remarks and open problems

This article is just a starting of the study of "different cutoffs problem", and there are still a lot of open problems left. For example, we do not know whether the condition $\lim_{a\to 0} a'|\log a| < \infty$ is optimal for the convergence of the sequence $\mu_{\lambda,a,a'}$ as $a \to 0$; also, the uniqueness of the limit is not proved. Moreover, the following conjecture, which is one of the main motivation of the present study, is not solved yet, either.

Conjecture. The probability measure μ_λ is singular with respect to μ_0 if a' is big enough compared with a.

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