Quantum and Classical Conserved Quantities: Martingales, Conservation Laws and Constants of Motion

Torbjörn Kolsrud

Department of Mathematics
Royal Institute of Technology
SE-100 44 Stockholm, Sweden
kolsrud@math.kth.se

Summary. We study a class of diffusions, conjugate Brownian motion, related to Brownian motion in Riemannian manifolds. Mappings that, up to a change of time scale, carry these processes into each other, are characterised. The characterisation involves conformality and a space-time version of harmonicity. Infinitesimal descriptions are given and used to produce martingales and conservation laws. The relation to classical constants of motion is presented, as well as the relation to Noether's theorem in classical mechanics and field theory.

To K. Itô on the occasion of his 90th birthday.

Introduction

The theme of the present article is *invariance properties* of a wide class of diffusions, termed *conjugate Brownian motion*. Much of the inspiration comes from the interplay between classical and quantum mechanics as expressed in the ideas of R.P. Feynman [FH] and early approaches to quantum mechanics. (See also Nelson [N].) Feynman's ideas are based on using the classical variational principle (in the free case)

$$\delta \int \frac{1}{2}|\dot{q}|^2 \, dt = 0 \tag{1}$$

to explain, e.g., its quantum counterpart

$$\delta \int \frac{1}{2}|du(q)|^2 \, dq = 0. \tag{2}$$

The latter case is related to *harmonic morphisms*, which, under pull back, preserve harmonic functions, and therefore Brownian motion. The basic result, due to Fuglede and Ishihara, is a characterisation of harmonic morphisms in terms of harmonicity (harmonic mappings Eells-Lemaire [EL]) and conformality. See [F1], [F2] and [Ish]. These concepts correspond to preservation of martingales, and conformality (Darling [D] and Emery [Em]). A non-geometric treatment of related problems can be found in Øksendal [Ok].

The heat equation can obtained from the variational principle ([BK2], [Gol], [IK], [MF])

$$\delta \iint \frac{1}{2} \left(\theta \dot{\theta}^* - \dot{\theta} \theta^* + \langle d\theta, d\theta^* \rangle \right) = 0, \tag{3}$$

where integration is carried out over space-time. The result is a pair of equations, viz., $\dot{\theta} + \frac{1}{2}\Delta\theta = 0$ and $\dot{\theta}^* - \frac{1}{2}\Delta\theta^* = 0$. Time-symmetry is built in: we have one backward and one forward heat equation.

The intimate relations between the classical Newton equation $\ddot{q} = 0$ and the (backward) heat equation $\dot{\theta} + \frac{1}{2}\theta'' = 0$, in one space dimension for simplicity, are fundamental. They are apparent when looking at the classical constants of motion $1, p, pt - q$, the Heisenberg algebra, and $p^2, p(pt-q), (pt-q)^2$, the Lie algebra sl_2, and comparing with the heat Lie algebra (Lie [Lie], Anderson-Ibragimov [AI], Ibragimov [Ibr2], [Ibr3], Olver [Or1]). The latter consists of linear differential operators of order at most one, viz., the Heisenberg algebra $\langle 1, \partial_q, t\partial_q - q \rangle$, and

$$\mathfrak{a} = \langle \partial_t, t\partial_t + \tfrac{1}{2}q\partial_q, \tfrac{1}{2}t^2\partial_t + \tfrac{1}{2}tq\partial_q - \tfrac{1}{4}(q^2 - t) \rangle. \tag{4}$$

This is another representation of sl_2. Whereas the first five elements in the classical and the heat Lie algebras correspond via the symbol map—a kind of Laplace transform, see Sect. 6—the sixth elements differ. In the classical case, we have the function $-\frac{1}{4}t^2p^2 + \frac{t}{2}pq - \frac{1}{4}q^2$, the symbol of the PDO $\frac{1}{2}t^2\partial_t + \frac{1}{2}tq\partial_q - \frac{1}{4}q^2$. In the heat Lie algebra, there is an additional term: instead of q^2 we have $q^2 - t$. The former function satisfies the equation $\dot{u} = 0$, the second satisfies $\dot{u} + \frac{1}{2}u'' = 0$. What we observe is *Itô's formula:* $q^2 - t \equiv :q^2:$ is the *renormalised second power*, corrected to fit the heat equation. In essence, this is the difference between the equations.

The main results presented below are

- A characterisation of the mappings that preserve ordinary and *conjugate* Brownian motion; (Sect. 4)
- The corresponding infinitesimal description in terms of Lie algebras, and the identification, in the free case, of the Lie algebra in terms of classical Lie algebras; (Sect. 5)
- Analysis of conservation laws and stochastic constants of motion (martingales), and their relation to the heat Lie algebra. Relations to the classical Lie algebra. Comparison via two Noether theorems. (Sect. 6)

In Sect. 2, we provide background on diffusions in manifolds, as well as the dynamical aspects of conjugate Brownian motion. In Sect. 3 we present the background for the mappings and Lie algebras needed. Cf. [Ibr2] or [Or1].

Among other related treatments we mention Djehiche–Kolsrud [DK], Ibragimov [Ibr2], Kolsrud [K2], Kolsrud–Loubeau [KL], Loubeau [Lob] and Thieullen–Zambrini [TZ1], [TZ2].

General references are [AI], [Arn], [AG], [AKN], [DKN], [DNF], [Ga], [Go], [Gol], [Gorb], [Ibr2], [Ibr3], [IW], [Ko], [LL], [M], [MF], [Or1].

1 Preliminaries

We shall consider a connected manifold N of dimension n. When N is given a Riemannian structure, the metric $g = (g_{ij})$ will also be written $\langle \cdot, \cdot \rangle$. ∇ shall denote covariant (Levi-Civita) differentiation with Christoffel symbols Γ^k_{ij}, and d the outer derivative. Recall that ∇ is completely determined by being *metric*: $\nabla g = 0$, and *symmetric*: $\Gamma^k_{(ij)} := \frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji}) = \Gamma^k_{ij}$. μ_g, or just vol, is the volume form determined by g. We will need two Laplacians: The *Laplace-Beltrami operator*

$$\Delta = \Delta_g = g^{ij}(\partial_i \partial_j - \Gamma^k_{ij}\partial_k) = \nabla^\dagger \nabla = g^{ij}\nabla_i \nabla_j = \mathrm{Tr}_g\, \nabla^2, \tag{5}$$

and the *de Rham-Hodge Laplacian*

$$\Box = -(dd^\dagger + d^\dagger d), \tag{6}$$

where $d^\dagger$ is the formal $L^2(\mu_g)$-adjoint of d. Here and below we use Einstein's summation convention with respect to repeated indices, one up, one down. When acting on functions, Δ and $\Box$ coincide. The Ricci tensor will be denoted by Ric. We shall often make use of Weitzenböck's identity

$$\Delta\alpha = \Box\alpha + \mathrm{Ric}\,\alpha \tag{7}$$

for one-forms α, and the fact that d and $\Box$ commute:

$$[d, \Box] = 0. \tag{8}$$

2 Basics on diffusions in manifolds

2.1 Connections, geodesics and scalar 2nd order elliptic PDOs

Consider a differential operator on N of the form $Q := \frac{1}{2}g^{ij}\partial_i\partial_j + \bar{b}_k\partial_k$. We assume that for all points x in N, $g^{ij}(x)\xi_i\xi_j > 0$ whenever some $\xi_i \neq 0$. Then (g^{ij}) is the inverse of a metric $g = (g_{ij})$ on N. Let $\Delta = \Delta_g$ be the corresponding Laplace-Beltrami operator. Q may be written $Q = \frac{1}{2}\Delta + b$, where b is a vector field. Up to a sign, this is the general expression for a scalar linear second order elliptic differential operator in N, satisfying $Q1 = 0$. It is formally self-adjoint (w.r.t. to $L^2(e^{2F}\mu_g)$) precisely when b is a gradient: $b = \mathrm{grad}\, F$. Let

$$\overline{\varGamma}_{ij}^k := \varGamma_{ij}^k + \frac{2}{n-1}(\delta_i^k b_j - g_{ij}b^k), \qquad n \neq 1, \tag{9}$$

where $b_k = g_{ik}b^i$. Then $(\overline{\varGamma}_{ij}^k)$ defines a metric connection $\overline{\nabla}$ such that $Q = \frac{1}{2}\mathrm{Tr}_g\,(\overline{\nabla}^2)$. $\overline{\nabla}$ is unique for $n = 2$, but not otherwise (Ikeda-Watanabe [IW], Prop. V.4.3). In general, the difference of two connections is a (1,2)-tensor. We may write

$$(\overline{\varGamma} - \varGamma)_{ij}^k = \overline{A}_{ij}^k + \overline{S}_{ij}^k, \tag{10}$$

where, for each k, $\overline{A}_{ij}^k$ is antisymmetric and $\overline{S}_{ij}^k$ symmetric in the lower indices. The $\overline{\nabla}$-geodesics are given by

$$\frac{\overline{D}^2 x^k}{dt^2} = \frac{D^2 x^k}{dt^2} + \overline{S}_{ij}^k \dot{x}^i \dot{x}^j = \ddot{x}^k + (\varGamma_{ij}^k + \overline{S}_{ij}^k)\dot{x}^i \dot{x}^j = 0, \quad 1 \leq k \leq n, \tag{11}$$

independently of $\overline{A}$. $\overline{A} \neq 0$ if and only if $\overline{\nabla} \neq \nabla$, the connections being metric. Q however, only depends on $\overline{\nabla}$ through the trace of its symmetric part.

Proposition 1. *Let ∇' and ∇'' be two g-metric connections with symmetric parts S' and S'', respectively. They have the same geodesics if and only if $S' = S''$. They have the same Laplacian if and only if $Tr_g(S')^k = Tr_g(S'')^k$ for each k. Moreover, this happens if and only if their corresponding torsion tensors satisfy $T'^k_{ik} = T''^k_{ik}$.*

The last characterisation can be found in [IW], Prop. V.4.3.

The important conclusion is

Observation 1. *Two 'quantum equivalent' connections need not be classically equivalent: Laplacians and geodesics do not correspond.*

Ground state transform

Consider now the case $Q1 \neq 0$. Write $Q = Q_0 - V$, where V is a smooth function (potential) on N, and $Q_0 = \frac{1}{2}\Delta + b$, as above. In general, multiplication by a function $\varOmega > 0$ is an isometry (unitary equivalence) between $L^2(\varOmega^{-2}\mu_g)$ and $L^2(\mu_g)$. Suppose also that $\varOmega$ solves the equation $Q\varOmega = 0$. (The case where $\varOmega$ corresponds to another eigenvalue can be handled by letting $V \to V + \mathrm{const.}$)

This is an implicit condition on V. Then (ground state transform or Doob's h-transform)

$$\overline{Q} := \varOmega^{-1} Q \varOmega = \varOmega^{-1}(Q_0 - V)\varOmega = Q_0 + \mathrm{grad}\,\log \varOmega, \tag{12}$$

independently of b. Conjugation by $\varOmega$ transforms Q to an operator without constant term. Let $b = 0$ (the symmetric case when b is a gradient can be handled similarly), so that Q_0 is the Laplace-Beltrami operator. Then, for $n \neq 2$, $\overline{Q}$ is a factor times the Laplace-Beltrami operator for a new, conformally equivalent, metric:

$$\Omega^{-1}\left(\frac{1}{2}\Delta_g - V\right)\Omega = \frac{1}{2}\Delta_g + \operatorname{grad}\log\Omega = \frac{1}{2}\rho^{-1}\Delta_{\rho g}, \quad \rho = \Omega^{\frac{4}{n-2}},\ n \neq 2. \tag{13}$$

There seems to be no general relations between the geodesics of ρg and the solutions curves of the classical Euclidean Newton equations

$$\frac{D^2 x}{dt^2} = \operatorname{grad} V(x). \tag{14}$$

In other words, there is no classical counterpart of the ground state transformation. The Maupertuis principle ([Arn], [Ga], [LL]) shows how, given a constant energy submanifold, one can remove a potential and instead introduce another conformally equivalent metric. This new metric is, however, not the one in (13).

2.2 $(g, \overline{\nabla})$-Brownian Motion and Conformal Martingales

Let (N, g) be Riemannian with a g-metric connection $\overline{\nabla}$ (not necessarily the Levi-Civita). The Christoffel symbols are denoted $\overline{\Gamma}^k_{ij}$. Let Y be a continuous N-valued semi-martingale, so that in local coordinates

$$dY^k = dA^k + dM^k, \quad 1 \leq k \leq n, \tag{15}$$

where the A^k are processes of finite variation, and the M^k are (ordinary) martingales. Define the covariant Itô differential d^cY of Y by

$$d^c Y^k_t := dY^k_t + \frac{1}{2}\overline{\Gamma}^k_{ij}(Y_t)d[Y^i, Y^j]_t, \quad 1 \leq k \leq n, \tag{16}$$

where the brackets indicate compensator. Given a connection, d^cY is well defined, as observed originally by Bismut.

Definition 2. *A semimartingale Y is a martingale w.r.t. $\overline{\nabla}$ if*

$$dA^k_t = -\frac{1}{2}\overline{\Gamma}^k_{ij}(Y_t)d[Y^i, Y^j]_t, \quad 1 \leq k \leq n. \tag{17}$$

Definition 3. *A semimartingale Y is conformal w.r.t. g if*

$$d[Y^i, Y^j]_t = g^{ij}(Y_t)dC_t, \quad 1 \leq i, j \leq n, \tag{18}$$

for some strictly positive and increasing continuous process C.

Itô's formula shows that if Y satisfies (17) and (18), the time-shifted process $Y(\gamma_t)$, where γ is the inverse of C, has generator $\frac{1}{2}\overline{\Delta} := \frac{1}{2}\operatorname{Tr}\overline{\nabla}^2$. $\overline{\Delta}$ is the *Laplacian* given by the connection.

Proposition 4. *With respect to $(g, \overline{\nabla})$, any conformal martingale is a time-shift of Brownian motion, and conversely.*

2.3 Conjugate BM

Let $N = (N, g)$ and $\overline{\nabla}$ be as in Sect. 2.2 with corresponding Laplacian $\overline{\Delta}$. Let $I = [-1, 1]$, and let $\theta : I \times N \to (0, \infty)$ be smooth. Write $V = (\dot{\theta} + \frac{1}{2}\overline{\Delta}\theta)/\theta$, and $H = -\frac{1}{2}\overline{\Delta} + V$, so that $\dot{\theta} = H\theta$. Hence θ is a solution of the (backward) heat equation with potential V. By *conjugating* the generator of $\mathrm{BM}(g, \nabla)$ with θ we obtain a new, in general non-stationary, diffusion $Z = (Z_t,\, t \in I)$, called *conjugate BM*. Its (forward) *generator* (regularised forward derivative of u along Z) is

$$Du := \frac{1}{\theta}\left(\frac{\partial}{\partial t} - H\right)(\theta \cdot u) = \left(\frac{\partial}{\partial t} + \frac{1}{2}\overline{\Delta}\right)u + \frac{1}{\theta}\langle d\theta, du\rangle. \tag{19}$$

When θ is independent of time, this is the ground state transform as in Sect. 2.1. The (forward) *Itô equation* is

$$d^c Z_t = \operatorname{grad} \log \theta(t, Z_t)\, dt + dM_t. \tag{20}$$

The compensator satisfies

$$d[Z^\alpha, Z^\beta]_t = g^{\alpha\beta}(Z_t)\, dt. \tag{21}$$

Note that (19) makes sense for tensor fields:

$$D\sigma = \frac{\partial \sigma}{\partial t} + \frac{1}{2}\overline{\nabla}^i \overline{\nabla}_i \sigma + \overline{\nabla}_\xi \sigma, \tag{22}$$

where ξ is the vector field dual to $d\theta/\theta$. The following variant of Itô's formula is useful:

$$D(\Phi(u)) = \Phi'(u)Du + \frac{1}{2}\Phi''(u)|du|^2, \qquad \Phi \in C^2(\mathbb{R}). \tag{23}$$

2.4 Schrödinger diffusions

Schrödinger (or Bernstein) diffusions are *time-symmetric* in the following sense. The forward description is a conjugate BM w.r.t. a positive solution of a backward heat equation, whereas the backward description is a conjugate backward BM w.r.t. a positive solution of the corresponding forward heat equation. (Cf. [KZ1] for details on the construction of Bernstein processes. See also [CZ].)

Consider again the situation in the preceding section, but with V given (and sufficiently smooth). For simplicity we assume that V does not depend on time. We assume that θ^* solves the usual, forward, heat equation with potential V: $\dot{\theta}^* = -H\theta^*$. The forward generator of the Bernstein diffusion Z is as in (19). The backward generator is

$$D^* u := \frac{1}{\theta^*}\left(\frac{\partial}{\partial t} + H\right)(\theta^* \cdot u) = \left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right)u - \frac{1}{\theta^*}\langle d\theta^*, du\rangle. \tag{24}$$

There is also a backward Itô equation similar to the one in Sect. 2.3.

Generally speaking, the backward heat evolution creates irregularities. We can produce smooth positive solutions by letting $\theta(t,\cdot) = \exp\{-(1-t)H\}\chi$, where $0 \not\equiv \chi \geq 0$ is given. Similarly, we obtain solutions to the forward heat equation with potential V by $\theta^*(t,\cdot) = \exp\{-(1+t)H\}\chi^*$. In particular we may choose $\theta^*(t,\cdot) = \theta(-t,\cdot)$.

The *probability density* of Z is the product of these two functions: $\mathbb{P}(Z_t \in C) = \int_C \theta(t,\cdot)\theta^*(t,\cdot)\, d\mu_g$, $C \in \mathrm{Borel}\,(N)$. This requires the normalisation

$$\int_N \chi^* \exp(-2H)\chi \, d\mathrm{vol} = 1. \tag{25}$$

2.5 Forward dynamics of conjugate BMs

Drift and momenta

If X is a continuous semi-martingale with values in a manifold with connection $\{\overline{\Gamma}^\alpha_{\beta\gamma}\}$, we define its *drift*, DX, by

$$d^c X_t = DX_t\, dt + dM_t, \tag{26}$$

where (M_t) is a martingale. DX_t is a tangent vector above X_t for each t. The drift measures the martingale deviation for X.

For conjugate BM, $DZ_t = \mathrm{grad}\, A(t, Z_t)$, where $A = \log\theta$. We let p be the corresponding one-form, i.e. $p = dA = d\log\theta$. This is the vector of *momenta* corresponding to our process.

$p(t, Z_t)$ is a regularised forward derivative along Z. For the Levi-Civita connection, one can calculate the drift directly, but along *harmonic coordinates*. Then $Dq^\alpha = \langle p, dq^\alpha\rangle = g^{\alpha\beta}p_\beta = p^\alpha$.

The Lagrangian

Using $D\theta = |d\theta/\theta|^2 + V = |p|^2 + V$, (23) yields

$$DA = D\log\theta = \frac{1}{2}|p|^2 + V = L(p,\cdot), \quad L(\omega, q) := \frac{1}{2}|\omega|^2 + V(q). \tag{27}$$

L is the *classical Euclidean Lagrangian*. By Dynkin's formula we get the *path integral formula*

$$\theta(t, Z_t) = \exp\Big\{ -\mathbb{E}\big[\int_t^1 L(DZ_{t'}, Z_{t'})\, dt' - \log\theta(1, Z_1)|\mathcal{F}_t\big]\Big\}. \tag{28}$$

We therefore look at A as the *forward action density*.

We can now deduce the *(forward) regularised Newton's equations*:

Proposition 5. *For the Levi-Civita connection,*

$$Dp - \frac{1}{2}\mathrm{Ric}\, p = dV. \tag{29}$$

Proof. On forms, the term in the generator involving the logarithmic derivatives of θ has to be replaced by the corresponding covariant differentiation. By Weitzenböck's formula $Dp - \frac{1}{2}\mathrm{Ric}\, p = \partial_t p + \frac{1}{2}\Box p + \nabla_\xi p := D_0 p$, where ξ is the vector field dual to p. The right-hand side equals $D_0 dA = dD_0 A + [D_0, d]A = dDA + [D_0, d]A =: I + II$. Using $\nabla g = 0$ and $\xi^\alpha = g^{\alpha\gamma}\nabla_\gamma A$, we find $\nabla_\xi dA = p^\alpha \nabla_\alpha \nabla_\beta A dq^\beta = g^{\alpha\gamma}\nabla_\gamma A \nabla_\alpha \nabla_\beta A dq^\beta = \frac{1}{2} d|dA|^2 = \frac{1}{2} d|p|^2$. Since ∂_t and $\Box$ commute with d, $II = [\nabla_\xi, d]A$. Obviously $d\nabla_\xi A = d(dA(\xi)) = d|dA|^2 = d|p|^2$, so $II = -\frac{1}{2} d|p|^2$. Finally, $dDA = d(\frac{1}{2}|p|^2 + V)$ according to (27) above.

Energy

The space-time differential

$$\bar{d}A := \dot{A}\, dt + \partial_\alpha A\, dq^\alpha = E\, dt + p\, dq, \tag{30}$$

of the forward action density A plays a similar role as the Poincaré invariant in classical mechanics. The second term is (forward) momentum. The first function E is the (forward) *energy* $\dot{A} = H\theta/\theta$. We have

$$E = -\frac{1}{2}|p|^2 - \frac{1}{2}\nabla^\dagger p + V. \tag{31}$$

The energy is a *stochastic constant of motion* in that $DE = 0$ whenever $\dot{V} = 0$. See Sect. 6.

Time reflection

For Bernstein diffusions, all that has been said has a backward counterpart. We do not go into details here but refer to to [KZ1] and references therein.

3 Groups of mappings and their Lie algebras

3.1 Extensions of Diff and Vect

Let M_0 and M_1 be differentiable manifolds, without any additional structure. Let $f : M_1 \to M_0$ and $a : M_0 \to \mathbb{R}$ be C^∞. The pair $\gamma := (f, a)$ induces, by pullback and multiplication, a mapping $C^\infty(M_0) \to C^\infty(M_1)$ by

$$u \to u \cdot \gamma := a \cdot u \circ f, \tag{32}$$

cf. [DK].

If we have several manifolds and mappings: $f_2 : M_2 \to M_1$ and $f_1 : M_1 \to M_0$ and a_j are functions on M_j, we get

$$\gamma_1 \cdot \gamma_2 = (f_1 \circ f_2, a_2 \cdot a_1 \circ f_2). \tag{33}$$

Now suppose that all manifolds are one and the same, M, and denote by $\mathcal{D} := \mathrm{Diff}\,(M)$ all diffeomorphisms $M \to M$. If $f_i \in \mathcal{D}$, and the a_i are never zero, the previous identity is the composition in the group

$$\widetilde{\mathcal{D}} := \mathcal{D} \ltimes C^\infty(M)^\times, \tag{34}$$

where $\ltimes$ indicates semi-direct product. $\widetilde{\mathcal{D}}$ is an extension of $\mathcal{D}$, and $u \to u \cdot \gamma$ is a right-action of $\widetilde{\mathcal{D}}$ on $C^\infty(M)$.

The infinitesimal version of this is as follows. Let $\mathcal{V} := \mathrm{Vect}\,(M)$ denote all vector fields on M, and consider all first-order differential operators

$$u \to \Lambda u := X(u) + U \cdot u, \tag{35}$$

where $X \in \mathcal{V}$ and U is a function on M. Equipped with the natural commutator, this is the Lie algebra

$$\widetilde{\mathcal{V}} := \mathcal{V} \oplus C^\infty(M), \tag{36}$$

where the sum is semi-direct. In analogy with the preceding case, $\widetilde{\mathcal{V}}$ is a central extension of $\mathcal{V}$.

We shall write the elements of $\widetilde{\mathcal{V}}$ as (X, U) or simply $X + U$. By definition $X + U = 0$ if and only if both components X and U vanish. Explicitly, the bracket in $\widetilde{\mathcal{V}}$ is

$$[\Lambda_1, \Lambda_2] = [(X_1, U_1), (X_2, U_2)] := ([X_1, X_2], X_1(U_2) - X_2(U_1)), \tag{37}$$

where the first term on the right is the usual commutator of vector fields.

The relation between $\gamma = (a, f)$ and $\Lambda = (X, U)$ can be described thus: Let $f_\varepsilon := \exp \varepsilon X$ denote the (local) flow of X, and assume $f = f_1$. Then, by Lie's formula,

$$a = a_1 = \exp \int_0^1 U \circ f_\varepsilon \, d\varepsilon, \tag{38}$$

and $\gamma = \exp \Lambda$ (cf. the Feynman-Kac formula).

Conversely, differentiating a local one-parameter group $\gamma_\varepsilon = (f_\varepsilon, a_\varepsilon)$ at $\varepsilon = 0$, one obtains a first order differential operator $\Lambda = X + U$.

3.2 Orbits for PDOs

Consider again the situation in (32). Let K_i be linear partial differential operators on $C^\infty(M_i)$, $i = 0, 1$. Assume $\gamma = (f, a)$ satisfies

$$K_1(u \cdot \gamma) = \phi \cdot (K_0 u) \cdot \gamma, \tag{39}$$

where in general $\phi = \phi_\gamma \in C^\infty(M_1)^\times$ will depend on γ.

This identity implies that the pullback under γ of solutions to $K_0 u = 0$ yield solutions to $K_1 v = 0$. Except for the 'conformal' factor ϕ, this is an intertwining relation.

Composing, as in Sect. 3.1, we obtain the *cocycle identities*

$$\phi_{\gamma_1\gamma_2} = \phi_{\gamma_2}(\phi_{\gamma_1} \cdot \gamma_2) = \phi_{\gamma_2} \cdot \phi_{\gamma_1} \circ f_2, \tag{40}$$
$$K_2(u \cdot \gamma_1\gamma_2) = \phi_{\gamma_1\gamma_2} \cdot (K_1 u) \cdot \gamma_1\gamma_2. \tag{41}$$

In the group case $\gamma = (f, a) \in \widetilde{\mathcal{D}}$, with $M_i = M$ and $K_i = K$, (39) may be written

$$(\gamma^{-1} K \gamma) u = \phi_\gamma \circ f^{-1} \cdot K u. \tag{42}$$

Thus, except for the cocycle, K and its conjugation under the inner automorphism given by γ are equal. Clearly, this identity defines a group. It may be seen as a deformed (by the cocycle ϕ) subgroup of $\widetilde{\mathcal{D}}$.

Suppose now that in (42) we have a (local) one-parameter group (γ_ε) with generator $\Lambda = (X, U)$ as above, and with corresponding cocycle $\phi^\varepsilon = 1 + \varepsilon\Phi + o(\varepsilon)$. Then, upon differentiating w.r.t. ε at 0 we get

$$[K, \Lambda] = \Phi \cdot K, \tag{43}$$

where the function Φ depends on Λ. The relation between ϕ and Φ is as in (38) for a and U. In particular,

$$K e^\Lambda = \exp\{\int_0^1 \Phi \circ f_\varepsilon \, d\varepsilon\} \cdot e^\Lambda K. \tag{44}$$

For fixed K, the relation $[K, \Lambda] = \Phi \cdot K$ defines a Lie algebra:

Proposition 6. *Let K be a linear differential operator on M. Then all the first order linear differential operators $\Lambda = (X, U)$ such that $[K, \Lambda] = \Phi \cdot K$ for some function $\Phi = \Phi_\Lambda$, form a Lie algebra with the commutator in* (38)*. We have*

$$\Phi_{[\Lambda_1, \Lambda_2]} = X_1(\Phi_2) - X_2(\Phi_1). \tag{45}$$

Remark 7. Henceforth this Lie algebra will be denoted Lie (K).

Proof (Proposition 6). Assuming (45) holds for Λ_1 and Λ_2, we must show that it also holds for their commutator. By the Jacobi identity, and with obvious notation,

$$[K, [\Lambda_1, \Lambda_2]] = [\Lambda_1, [K, \Lambda_2]] - [\Lambda_2, [K, \Lambda_1,]] = [\Lambda_1, \Phi_2 \cdot K] - [\Lambda_2, \Phi_1 \cdot K]. \tag{46}$$

We always have

$$[\Lambda, \Phi \cdot K] = X(\Phi) \cdot K + \Phi \cdot [\Lambda, K], \tag{47}$$

from which the conclusion is immediate.

Remark 8. It is a very general fact that the relation $K, \Lambda] = \Phi \cdot K$ defines a Lie algebra. We have only used that the bracket is the natural one, and that we have a derivation plus a multiplication.

Intertwining of Lie(K_i)

It is obvious that (39) relates the Lie algebras of K_0 and K_1 to one another. To makes this more clear, suppose $\gamma = (f, a)$ satisfies $K_1\gamma = \phi \cdot \gamma K_0$, and suppose Λ_i are related by

$$\Lambda_1\gamma = \gamma\Lambda_0, \tag{48}$$

i.e.,

$$X_1(a)u \circ f + aX_1(u \circ f) + a(U_1 u) \circ f = aX_0(u) \circ f + a(U_0 u) \circ f. \tag{49}$$

One finds

$$K_1\Lambda_1\gamma = K_1\gamma\Lambda_0 = \phi\gamma K_0\Lambda_0, \tag{50}$$

and

$$\Lambda_1 K_1\gamma = \Lambda_1(\phi \cdot \gamma K_0) = X_1(\phi)\gamma K_0 + \phi\gamma\Lambda_0 K_0. \tag{51}$$

Hence

$$[K_1, \Lambda_1]\gamma = \phi\gamma[K_0, \Lambda_0] + X_1(\phi)\gamma K_0. \tag{52}$$

Since ϕ is never zero, we see that, on the appropriate domains, Λ_i preserve the kernels of K_i simultaneously. If $\Lambda_0 \in \operatorname{Lie} K_0$, then

$$[K_1, \Lambda_1]\gamma = \phi\gamma\Phi_0 \cdot K_0 + X_1(\phi)\gamma K_0 = (\Phi_0 \circ f + X_1(\log\phi))K_1\gamma. \tag{53}$$

Similarly, $\Lambda_1 \in \operatorname{Lie} K_1$ implies

$$\gamma[K_0, \Lambda_0] = (\Phi_1 - X_1(\log\phi))\gamma K_0. \tag{54}$$

4 Heat and Harmonic Morphisms

4.1 Basic characterisations

Let $\widetilde{M}$ and M be two manifolds, and f a map of $\widetilde{M}$ into M. Let P and K be two (scalar) linear differential operators on $C^\infty(\widetilde{M})$ and $C^\infty(M)$, respectively.

Definition 9. *f is a morphism for P and K if for each open set $\Omega \subset M$ and each $u \in C^\infty(\Omega)$,*

$$Ku = 0 \quad \text{on} \quad \Omega \qquad \Longrightarrow \qquad P(u \circ f) = 0 \quad \text{on} \quad f^{-1}\Omega. \tag{55}$$

Suppose now that $\widetilde{K}$ and K are differential operators on $\widetilde{M}$ and M, and $a : \widetilde{M} \to \mathbb{R} \setminus 0$. Suppose also that $\gamma = (f, a)$ satisfies (cf. (39))

$$\widetilde{K}\gamma = \phi \cdot \gamma K, \tag{56}$$

i.e.,

$$\widetilde{K}(a \cdot u \circ f) = \phi a \cdot Ku \circ f, \tag{57}$$

where $\phi \neq 0$. Writing

$$Pv := a^{-1}\widetilde{K}(av), \tag{58}$$

we obtain

$$P(u \circ f) = \phi \cdot Ku \circ f. \tag{59}$$

Let now $\widetilde{N}$ and N be Riemannian manifolds with metric connections $\widetilde{\nabla}$ and ∇, and corresponding Laplacians $\widetilde{\Delta}$ $(= \mathrm{Tr}\,(\widetilde{\nabla}^2))$ and Δ, respectively. Let $\widetilde{I}$ and I be time intervals with variables s and t. We shall consider the following two situations:

(i) Harmonic morphisms: $\widetilde{M} = \widetilde{N}$, $M = N$, $\widetilde{K} = \widetilde{\Delta}$, and $K = \Delta$.
(ii) Heat morphisms: $\widetilde{M} = \widetilde{I} \times \widetilde{N}$, $M = I \times N$, $\widetilde{K} = \partial_s + \frac{1}{2}\widetilde{\Delta}$, and $K = \partial_t + \frac{1}{2}\Delta$.

In order not to get a zero-order term we must require

$$Ka = 0. \tag{G2}$$

When (G2) holds, we have (assuming without loss of generality $a > 0$)

$$P = \widetilde{K} + \mathrm{grad}\,\log a \tag{60}$$

in both cases.

The associated diffusion $\widetilde{X}$ with generator P is a conjugate of $\mathrm{BM}(\widetilde{N}, \widetilde{g}, \widetilde{\nabla})$ satisfying

$$\begin{aligned} d^c\widetilde{X}_s &= \mathrm{grad}\,\log a(s, \widetilde{X}_s)\,ds + d\widetilde{\xi}_s, \\ d[\widetilde{X}^i, \widetilde{X}^j]_s &= \widetilde{g}^{ij}(\widetilde{X}_s)\,ds, \end{aligned} \tag{61}$$

where only in case (ii) a will depend explicitly on s. The process X corresponding to Q is in both cases $\mathrm{BM}(N, g, \nabla)$. Let us write

$$f = (f^0, f^\alpha,\, 1 \leq \alpha \leq n), \quad t = f^0,$$

where f^0 does not appear in case (i).

Proposition 10. *Let P be as in* (58)*. Then f is a morphism for P and K if and only if $P(u \circ f) = \phi \cdot Ku \circ f$.*

Proof. Clearly the condition implies that f is a morphism. The converse follows from Fuglede's beatiful argument in [F1], Remark 1, p. 129. The only thing needed is a function w satisfying $Qw > 0$ on some neighbourhood, arbitrarily small, of a given point in M, and then we may take

$$\phi := \frac{P(w \circ f)}{Kw \circ f}. \tag{62}$$

This proves our assertion.

We now normalise so that f is *time-preserving*:

$$\frac{dt}{ds} = \dot{f}^0 > 0. \tag{63}$$

As in [DK] we shall use the following further conditions on $\gamma = (f, a)$:

$$df^0 = 0; \tag{G1}$$

$$\frac{1}{2}\widetilde{\Delta} f^\alpha + a^{-1}\langle da, df^\alpha\rangle + \frac{1}{2}\Gamma^\alpha_{\beta\gamma} \circ f \langle df^\beta, df^\gamma\rangle = 0, \quad 1 \le \alpha \le n; \tag{G3i}$$

$$\dot{f}^\alpha + \frac{1}{2}\widetilde{\Delta} f^\alpha + a^{-1}\langle da, df^\alpha\rangle + \frac{1}{2}\Gamma^\alpha_{\beta\gamma} \circ f \langle df^\beta, df^\gamma\rangle = 0, \quad 1 \le \alpha \le n; \tag{G3ii}$$

$$\langle df^\beta, df^\gamma\rangle = \lambda^2 g^{\beta\gamma} \circ f; \tag{G4i}$$

$$\langle df^\beta, df^\gamma\rangle = 2\frac{dt}{ds} g^{\beta\gamma} \circ f. \tag{G4ii}$$

The roman numerals of course refer to the cases defined above.

In [DK] we showed for the heat case that (G1)–(G4) are sufficient for f to be a morphism for P and K. Following [F1], [F2], [Ish], in the case of harmonic morphisms, we now show that these conditions are also necessary in the heat case. We assume the normalisation (63). To this end, let $\widetilde{X}$ be the process in (61), and put $Y^\alpha_s := f^\alpha(s, \widetilde{X}_s)$, $1 \le \alpha \le n$. By Itô's formula,

$$dY^\alpha_s = (\dot{f}^\alpha + \frac{1}{2}\widetilde{\Delta} f^\alpha + a^{-1}\langle da, df^\alpha\rangle)(s, \widetilde{X}_s)\, ds + \partial_i f^\alpha(s, \widetilde{X}_s) d\widetilde{\xi}^i_s. \tag{64}$$

Thus

$$d[Y^\beta, Y^\gamma]_s = \langle df^\beta, df^\gamma\rangle(s, \widetilde{X}_s)\, ds. \tag{65}$$

We may write this as

$$\begin{aligned} d^c Y^\alpha_s = (\dot{f}^\alpha + \frac{1}{2}\widetilde{\Delta} f^\alpha + a^{-1}\langle da, df^\alpha\rangle + \frac{1}{2}\Gamma^\alpha_{\beta\gamma} \circ f \langle df^\beta, df^\gamma\rangle)(s, \widetilde{X}_s)\, ds \\ + \partial_i f^\alpha(s, \widetilde{X}_s) d\widetilde{\xi}^i_s. \end{aligned} \tag{66}$$

To start with, Y must be a time shift of X (Sect. 2.2). Since X is a martingale, we see that condition (G3) is necessary, and then

$$d^c Y_s^\alpha = \partial_i f^\alpha(s, \widetilde{X}_s) d\widetilde{\xi}_s^i. \tag{67}$$

The conformality of X requires condition (G4i).

Up to now we have only considered the 'harmonic' case under a time-dependent transformation. To reach the heat equation, and condition (G4ii), we must study the time-space process $f(s, \widetilde{X}_s) = (f^0(s), Y_s)$. If u is a (local) function on M, then

$$d(u \circ f(s, \widetilde{X}_s)) = \left(\frac{dt}{ds} \dot{u} \circ f(s, \widetilde{X}_s) + \frac{1}{2} \lambda^2 \Delta u \circ f(s, \widetilde{X}_s) \right) ds + d\xi, \tag{68}$$

where ξ is a martingale. The coefficient of the first (ds-)term on the right-hand side must be proportional to $(\dot{u} + (1/2)\Delta u) \circ f(s, \widetilde{X}_s)$. Clearly this requires (G4ii).

We collect our findings in

Theorem 11. *Given $\gamma = (f, a)$, suppose* (63) *holds and $a > 0$. The following are equivalent*
– in case (i):

a) $\widetilde{\Delta}(a \cdot u \circ f) = \phi a \Delta u \circ f$, with $\phi = \lambda^2 = n^{-1} \, Tr \, df \otimes df$;
b) the process $f(\widetilde{X}_s)$ is a time shift of $\mathrm{BM}(N, g, \nabla)$*;*
c) Eqs. (G2)–(G4) hold.

– in case (ii):

a) $(\partial_s + (1/2)\widetilde{\Delta})(a \cdot u \circ f) = \phi a (\partial_t + (1/2) a \Delta u) \circ f$, with $\phi = dt/ds$;
b) the process $f(s, \widetilde{X}_s)$ is a time shift of (t, X_t)*, where X is* $\mathrm{BM}(N, g, \nabla)$*;*
c) Eqs. (G1)–(G4) hold.

Remark 12. Condition (G2) means that martingales are preserved ([D], [Em]). In case (i), f is a *harmonic mapping* ([F1], [F2], [Ish]) when $a \equiv 1$. See also the discussion on affine maps in [Em]. Condition (G4) means that the map is *horizontally conformal*, see [F1]. Together with (G2) we have (in case (ii)) a characterisation of maps preserving conformal martingales, cf. Proposition 4.

Condition (G4ii) links the time and space scales. It implies that the dilation w.r.t. the space variable is independent of the space variable, as opposed to the case of harmonic morphisms.

Remark 13. If ϕ is a harmonic morphism with constant dilation, and $dt/ds = \frac{1}{2}\lambda^2$, then (t, ϕ) is a heat harmonic morphism. This case was treated independently by Loubeau [Lob].

4.2 Morphisms for conjugate BM

We consider case (ii) of the preceding section, and assume that (f, a) satisfy (G1)–(G4). Let $\theta > 0$ and V be as in Sect. 2.3 and put

$$\widetilde{V} := (dt/ds)V \circ f. \tag{69}$$

We write $\widetilde{D}_0 := \partial_s + \frac{1}{2}\widetilde{\Delta} - \widetilde{V}$ and $D_0 := \partial_t + \frac{1}{2}\Delta - V$, so that $D_0\theta = 0$. By Theorem 11 and the definition of $\widetilde{V}$ $\widetilde{D}_0(au \circ f) = a(dt/ds)D_0 u \circ f$. Defining $\widetilde{\theta} := a \cdot \theta \circ f$, also $\widetilde{D}_0\widetilde{\theta} = 0$.

We introduce the forward generators $\widetilde{D} := \widetilde{D}_{\widetilde{\theta}}$ and $D := D_\theta$ as in (19):

$$\widetilde{D}\widetilde{u} := \widetilde{D}_0(\widetilde{\theta}\widetilde{u})/\widetilde{\theta} \quad \text{and} \quad Du = D_0(\theta u)/\theta. \tag{70}$$

The following result from [DK] shows that heat harmonic morphisms also preserve conjugate BM.

Theorem 14. *Suppose (f, a) satisfy (G1)–(G4) and the normalisation* (63) *in Sect. 4.1. Then f is a morphism for $\widetilde{D}$ and D:*

$$\widetilde{D}(u \circ f) = \frac{dt}{ds} Du \circ f \tag{71}$$

for any smooth function u on M.

Proof. By what we have just seen,

$$\begin{aligned}\widetilde{D}(u \circ f) = \frac{1}{\widetilde{\theta}}\widetilde{D}_0(\widetilde{\theta} \cdot u \circ f) = \frac{1}{a\theta \circ f}\widetilde{D}_0(a \cdot (\theta u) \circ f)& \\ = \frac{1}{\theta \circ f}\frac{dt}{ds} D_0(\theta u) \circ f = \frac{dt}{ds} Du \circ f.&\end{aligned} \tag{72}$$

Let $\widetilde{Z}$ denote the process associated with $\widetilde{\theta}$. Then $(f(s, \widetilde{Z}_s))_{s \in \widetilde{I}}$ and $(t, Z_t)_{t \in I}$, after a time-change, have the same distribution (cf. [Ok] and Sect. 2.2–2.3).

4.3 Dynamical invariance for conjugate BM

We now give a result already stated in [DK] describing the transformation properties of the (forward) energy, momenta, Lagrangian and equations of motion. We only consider the situation when ∇ is the Levi-Civita connection. We assume that (f, a) satisfy (G1)–(G4) and the normalisation $\dot{f}^0 > 0$.

In the sequal T^*f denotes pullback via f w.r.t. the space variables.

Theorem 15. *Let $\Psi := \log a$ and $\widetilde{E}_0 := \dot{a}/a$. Then*

$$\widetilde{E} = \widetilde{E}_0 + \frac{dt}{ds} E \circ f + \dot{f}^\alpha p_\alpha \circ f, \tag{73}$$

$$\widetilde{p} = T^* f p + d\Psi, \tag{74}$$

$$\widetilde{L} = \frac{dt}{ds} L \circ f + \widetilde{D}\Psi, \tag{75}$$

$$\left(\widetilde{D} - \frac{1}{2}\widetilde{\mathrm{Ric}}\right)\widetilde{p} = \frac{dt}{ds} T^* f \left(Dp - \frac{1}{2}\mathrm{Ric}\, p\right). \tag{76}$$

Proof. (74) is obvious since $\widetilde{p}_i = \partial_i \log \widetilde{\theta} = \partial_i \log a + \partial_\alpha \theta \circ f \partial_i f^\alpha = \partial_i \log a + (T^* f p)_i$. (73) is obtained similarly.

Squaring and summing (74), (G4) yields

$$\begin{aligned}\frac{1}{2}\frac{dt}{ds}|p|^2 \circ f &= \frac{1}{2}\lambda^2 (g^{\alpha\beta} p_\alpha p_\beta) \circ f = \frac{1}{2}\widetilde{g}^{ij} p_\alpha \circ f \partial_i f^\alpha p_\beta \circ f \partial_i f^\beta \\ &= \frac{1}{2}\widetilde{g}^{ij}\left(\widetilde{p}_i - \frac{\partial_i a}{a}\right)\left(\widetilde{p}_j - \frac{\partial_j a}{a}\right) = \frac{1}{2}|\widetilde{p}|^2 + \frac{1}{2}\left|\frac{da}{a}\right|^2 - \frac{1}{a}\langle \widetilde{p}, da\rangle. \qquad (77)\end{aligned}$$

Now $\widetilde{D}a = \partial_s a + \frac{1}{2}\Delta a + \langle \widetilde{p}, da/a\rangle = \langle \widetilde{p}, da/a\rangle$, so by (23) $\widetilde{D}\log a = \langle \widetilde{p}, da/a\rangle - \frac{1}{2}|da/a|^2$. Hence, using the definition of $\widetilde{V}$ in Sect. 4.2 we obtain (75):

$$\frac{dt}{ds}L \circ f = \frac{dt}{ds}\left(\frac{1}{2}|p|^2 \circ f + V \circ f\right) = \frac{1}{2}|\widetilde{p}|^2 - \widetilde{D}\log a + \widetilde{V} = \widetilde{L} - \widetilde{D}\Psi. \qquad (78)$$

(76) follows from the Newton equations (29) for the two processes, together with the definition of $\widetilde{V}$.

Remark 16. Concerning the extra terms on the right, e.g., in (75), one should recall that already in the classical case, the equations of motion are not altered when a total time differential is added to the Lagrangian. (See, e.g., [DNF], Sect. 31, p. 305.) (75) states the invariance of the Lagrangian time-differential up to a regularised time derivative: $\widetilde{L}\, ds - L \circ f\, dt = \widetilde{D}\Psi\, ds$.

The transformation of the energy in (73) may seem strange. In the example $f = (t, \phi)$ of Remark 13, however, it transforms in a less exotic way: $\widetilde{E} = \widetilde{E}_0 + \frac{dt}{ds} E \circ f$.

5 The heat and Laplace Lie algebras

Throughout this chapter, the underlying manifold N is Riemannian with metric g. We shall only consider the Levi-Civita connection.

For a (possibly time-dependent) vector field Q on N, we shall denote by ω its dual one-form: $\omega_k = g_{ik} Q^i$, i.e. $Q = \omega^\sharp$.

5.1 Conformal groups and Lie algebras

For any vector field Q we have

$$(\mathcal{L}_Q g)_{ij} = \nabla_j \omega_i + \nabla_j \omega_i = 2(\nabla\omega)_{(ij)}, \qquad (79)$$

where $\mathcal{L}_Q$ denotes the Lie derivative along Q, and the parentheses indicate symmetrisation.

Definition 17. *The conformal Lie algebra* $\mathfrak{conf}$ *consists of all vector fields* Q *on* N *such that*

$$\mathcal{L}_Q g = \mu \cdot g, \qquad (80)$$

for some function μ *depending on* Q.

The corresponding situation for maps, i.e. local flows, is

$$g \circ f(df(\xi_1), df(\xi_2)) = \kappa \cdot g(\xi_1, \xi_2), \tag{81}$$

for some function κ. That $\mathfrak{conf}$ is a Lie algebra follows from

$$[\mathcal{L}_{Q_1}, \mathcal{L}_{Q_2}]g = (Q_1(\mu_2) - Q_2(\mu_1)) \cdot g. \tag{82}$$

Equation (79) implies

$$\mu = -\frac{2}{n} d^\dagger \omega = \frac{2}{n} \nabla^\dagger \omega, \quad Q \in \mathfrak{conf}. \tag{83}$$

By, e.g., [Go], Eqs. (3.7.4) and (3.8.4) or [Ko],

$$\frac{1}{2}\Box\omega + \mathrm{Ric}\,\omega - \big(\frac{1}{2} - \frac{1}{n}\big) dd^\dagger\omega = \frac{1}{2}\Box\omega + \mathrm{Ric}\,\omega + \frac{n-2}{4} d\mu = 0. \tag{84}$$

Define three Lie algebras:

$$\mathfrak{k} := \{Q \in \mathfrak{conf} : \mu = 0\}, \tag{85}$$

$$\mathfrak{h} := \{Q \in \mathfrak{conf} : \mu = \text{const.}\}, \tag{86}$$

$$\mathfrak{conf}_h := \{Q \in \mathfrak{conf} : \varDelta\mu = 0\}. \tag{87}$$

Clearly,

$$\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{conf}_h \subset \mathfrak{conf}. \tag{88}$$

Here $\mathfrak{k}$ stands for *Killing vector fields*, i.e., infinitesimal isometries, and $\mathfrak{h}$ indicates that the corresponding flows are *homothetic transformations*. The inclusions are obvious, and 5.1.4 shows that $\mathfrak{k}$ and $\mathfrak{h}$ are Lie algebras. Clearly $\mathfrak{h} = \mathfrak{k} \oplus \mathbb{R}$ globally.

We shall need

Lemma 18. *If $Q_1 \in \mathfrak{conf}$ and $Q_2 \in \mathfrak{conf}_h$, then*

$$\varDelta(Q_1(\mu_2)) = -\frac{n-2}{2} \langle d\mu_1, d\mu_2 \rangle. \tag{89}$$

We assume the lemma momentarily. For $Q \in \mathfrak{conf}$, let

$$U := \frac{n-2}{4}\mu = \frac{n-2}{2n}\nabla^\dagger\omega, \tag{90}$$

and

$$\overline{Q} := (Q, U) := Q + U = Q + \frac{n-2}{2n}\nabla^\dagger\omega \in \widetilde{\mathcal{V}}, \tag{91}$$

where $\widetilde{\mathcal{V}}$ was defined in Sect. 3.1. The bracket is (by (38))

$$[\overline{Q}_1, \overline{Q}_2] = [Q_1, Q_2] + Q_1(U_2) - Q_2(U_1) = [Q_1, Q_2] + \frac{n-2}{4n} \mathrm{Tr}\,(\mathcal{L}_{[Q_1,Q_2]}g). \tag{92}$$

This way we get a map

$$\mathcal{V} \supset \mathfrak{conf} \ni Q \to \overline{Q} \in \widetilde{\mathcal{V}}. \tag{93}$$

We note that in dimension $n = 2$ on $\mathcal{V}$, or on $\mathfrak{k}$ in any dimension, (93) is the identity map: $\overline{Q} = (Q, 0)$.

Theorem 19. $\mathfrak{conf}_h$ *is a Lie algebra. The restriction of (10) to* $\mathfrak{conf}_h$ *is a Lie algebra isomorphism onto its image:*

$$[\overline{Q}_1, \overline{Q}_2] = [Q_1, Q_2] + \frac{n-2}{2n} \nabla^\dagger [\omega_1, \omega_2], \qquad X \in \mathfrak{conf}_h, \tag{94}$$

where $[\omega_1, \omega_2]$ *is the one-form dual to* $[Q_1, Q_2]$*.*

Proof. The first statement follows from Lemma 18 which clearly implies that $\Delta(Q_1(\mu_2) - Q_2(\mu_1)) = 0$ for $Q_i \in \mathfrak{conf}_h$.

We now show the displayed identity. Up to a multiplicative constant $Q_1(\mu_2)$ is equal to $\langle \omega_1, (\Box + \mathrm{Ric}\,)\omega_2 \rangle$ by (84). Hence, by Weitzenböck, the difference $Q_1(\mu_2) - Q_2(\mu_1)$ is a constant times

$$\langle \omega_1, \Delta\omega_2 \rangle - \langle \omega_2, \Delta\omega_1 \rangle, \tag{95}$$

since the Ricci tensor is symmetric. This equals

$$g^{ij} \nabla_j (\langle \omega_1, \nabla_i \omega_2 \rangle - \langle \omega_2, \nabla_i \omega_1 \rangle). \tag{96}$$

Using that ∇ commutes with the duality $T^*N \cong TN$ given by g, the expression in the parentheses is dual to

$$\nabla_{Q_1} Q_2 - \nabla_{Q_2} Q_1 = [Q_1, Q_2], \tag{97}$$

the connection being torsion free. Checking the constant, one finds (94).

Proof (Lemma 18). Using $\nabla g = 0$ we get

$$\nabla^2 Q_1(\mu_2) = \langle \nabla^2 \omega_1, d\mu_2 \rangle + 2\langle \nabla \omega_1, \nabla d\mu_2 \rangle + \langle \omega_1, \nabla^2 d\mu_2 \rangle.$$

To obtain the Laplacian, we must take the trace. Having done this, the second term on the right is proportional to $\mu_1 \Delta \mu_2$, by (79), (80). By assumption it vanishes. We get, using the Weitzenböck formula, the identity $[d, \Box] = 0$ and (84),

$$\begin{aligned} \Delta Q_1(\mu_2) =& \langle \Delta\omega_1, d\mu_2 \rangle + \langle \omega_1, \Delta d\mu_2 \rangle \\ =& \langle (\Box + \mathrm{Ric}\,)\omega_1, d\mu_2 \rangle + \langle \omega_1, \mathrm{Ric}\, d\mu_2 \rangle \\ =& \langle (\Box + 2\mathrm{Ric}\,)\omega_1, d\mu_2 \rangle = -\frac{n-2}{2} \langle d\mu_1, d\mu_2 \rangle. \end{aligned}$$

The claim follows.

5.2 Characterisation of the heat Lie algebra.

We consider the situation in (61), case (ii), with the further requirement that Δ be the Laplace-Beltrami operator. For $\Lambda = (X, U)$ we write

$$X = T\frac{\partial}{\partial t} + Q^i\frac{\partial}{\partial x^i} = T\frac{\partial}{\partial t} + Q, \tag{98}$$

where, at this point, T and the Q^i are functions of $t \in I$ and $x \in N$.

We now characterise the heat Lie algebra in terms of PDEs. In general, the system obtained is overdetermined. It should be no surprise that the equations obtained are completely analogous to Eqs. (G1-G4) in Sect. 4.1.

Theorem 20. $\Lambda = (T, Q, U) \in \mathrm{Lie}\,(\partial_t + \frac{1}{2}\Delta - V)$ *if and only if the following equations are satisfied:*

$$dT = 0; \tag{A1}$$

$$\dot{U} + \frac{1}{2}\Delta U + \frac{\partial}{\partial t}\big(TV\big) + Q(V) = 0; \tag{A2}$$

$$\dot{\omega} + \frac{1}{2}\Box\omega + Ric\,\omega + dU = 0; \tag{A3}$$

$$(\nabla\omega)^{(ij)} = \frac{1}{2}\dot{T}g^{ij}. \tag{A4}$$

The associated cocycle is $\Phi = \Phi_\Lambda = \dot{T}$.

Proof. We start with the free case $V \equiv 0$. With $K := \partial_t + \frac{1}{2}\Delta$ we easily find

$$[K, U]u = (\dot{U} + \frac{1}{2}\Delta U)u + \langle dU, du\rangle, \tag{99}$$

and

$$[\partial_t, X]u = \dot{X}u = \dot{T}\dot{u} + \langle\dot{\omega}, du\rangle. \tag{100}$$

Furthermore,

$$[\Delta, X]u = \Delta T\cdot\dot{u} + 2\langle dT, d\dot{u}\rangle + T\Delta\dot{u} + \langle\Box\omega, du\rangle + 2\mathrm{Ric}\,(\omega, du) + 2\langle\nabla\omega, \nabla du\rangle. \tag{101}$$

To see this, note that the left-hand side is

$$\Delta(T\dot{u} + \langle\omega, du\rangle) - T\Delta\dot{u} - \langle\omega, d\Delta u\rangle = \Delta T\cdot\dot{u} + 2\langle dT, d\dot{u}\rangle + T\Delta\dot{u} + \langle\Delta\omega, du\rangle) + \langle\omega, \Delta du\rangle + 2\langle\nabla\omega, \nabla du\rangle - T\Delta\dot{u} - \langle\omega, d\Delta u\rangle, \tag{102}$$

and use the identities for Laplacians stated in Sect. 1.

¿From the above equations we now get

$$[Q, \Lambda]u = (\dot{U} + \frac{1}{2}\Delta U)u + (\dot{T} + \frac{1}{2}\Delta T)\dot{u} + \langle dT, d\dot{u}\rangle + \langle\dot{\omega} + \frac{1}{2}\Box\omega + \mathrm{Ric}\,\omega + dU, du\rangle + \langle\nabla\omega, \nabla^2 u\rangle. \tag{103}$$

We want the left-hand side to be a function times Ku. First, no constant term is allowed, so that U has to satisfy (A2). To avoid terms with mixed time and space derivatives we must also require that T only depends on time, i.e., that (A1) holds. The first-order space derivatives disappear if and only if (A3) holds.

We now have

$$[K,\Lambda]u = \dot{T}\dot{u} + \langle \nabla\omega, \nabla^2 u\rangle. \tag{104}$$

To get a Laplacian out of the second term, we must require that the symmetric part of $\nabla\omega$ is proportional to the inverse metric (g^{ij}), $\nabla^2 u$ being symmetric. (A4), finally, is needed to adjust the scales between time and space derivatives.

The general case, including a potential V, follows from $[V,\Lambda]u = -X(V)\cdot u$ and $[K-V,\Lambda] = [K,\Lambda] - [V,\Lambda] = \Phi(K-V) = \dot{T}(K-V)$.

Remark 21. Combining Eqs. (A2), (A4) and (A5), we see that for each t, we must have $\omega(t,\cdot)$ in $\mathfrak{h}$ (Sect. 5.1). Hence the heat Lie algebra is trivial (i.e., consists only of constants) whenever $\mathfrak{k} = 0$. This is different from the Laplace case where elements in $\mathfrak{conf} \setminus \mathfrak{h}$ may occur, e.g. in one space-dimension. See also Sect. 5.3 below.

A part of Lie (Δ) is always contained in Lie $(\partial_t + \frac{1}{2}\Delta)$, viz., when non-void, the one corresponding to constant (in time) elements in $\mathfrak{h}$. Comparing with (83) we see that $2c = \mu = -(2/n)d^\dagger\omega$, so we may take U constant and $T(t) = \mu t + \mu_0$. Note that when $\omega \in \mathfrak{k}$, the Killing algebra, time does not enter explicitly, i.e., T is constant.

Our next result clarifies the relation between Lie $(\partial_t + \frac{1}{2}\Delta)$ and $\mathfrak{h}$.

Theorem 22. Lie $(\partial_t + \frac{1}{2}\Delta)$ *consists of all* $\Lambda = (T,Q,U)$ *with* $Q \in C^\infty(I \to \mathfrak{h})$ *such that its dual one-form* ω *satisfies*

$$\frac{\partial^2}{\partial t^2} d^\dagger\omega = 0, \tag{105}$$

$$\frac{\partial}{\partial t} d\omega = 0. \tag{106}$$

T is a polynomial in t of degree at most 2 with $\dot{T} = -(2/n)d^\dagger\omega$. $U = \frac{1}{2}\alpha - \dot{\alpha}U_0$, where $\alpha = d^\dagger\omega = -(n/2)\dot{T}$, and U_0 satisfies $\Delta U_0 = 1$.

Proof. Note first that if ω does not depend on time, (105) and (106) are trivially fulfilled, and we are back in the case discussed in Remark 21.

Let us start from $Q \in C^\infty(I \to \mathfrak{h})$ satisfying (105) and (106). It is clear how to obtain T from ω. We shall show how to find U.

We know from (84) that $\frac{1}{2}\Box\omega + \mathrm{Ric}\,\omega = 0$ for each fixed time. Hence, (A3) becomes

$$\dot{\omega} + dU = 0. \tag{107}$$

By (106), this is fulfilled for some U. We must show that we can arrange so that U satisfies the anti-heat equation (A1). From $\dot{\omega} + dU = 0$ follows, invoking (105),

$$\varDelta U = d^\dagger \dot\omega = \text{ const.} = \varkappa. \tag{108}$$

If this constant vanishes, we are again back to Remark 21, and may take U constant.

If not, write $U_0 = U/\varkappa$, so that $\varDelta U_0 = 1$. (Note that, locally, there are always such functions.) Without altering this, we may put $U = \frac{1}{2}\alpha - \dot\alpha U_0$, and then U satisfies (A2).

This proves that (105) and (106) are sufficient. Clearly condition (106), as well as the expression for T is necessary. It remains to deduce (105), and the explicit form for U. We may assume that U is non-constant. As above, we get $U(t,\cdot) = \psi(t) + \varphi(t)U_0$, for some functions φ and ψ, where U_0 is independent of time and $\varDelta U_0 = 1$. Then (A2) holds if and only if $\dot\varphi = 0$ and $-\frac{1}{2}\dot\psi = \varphi$. Since also (A4) must hold we deduce $\varphi = d^\dagger\dot\omega$, which implies (105).

5.3 Characterisation of the Laplace Lie algebra

We now consider the Lie algebra of the Laplace-Beltrami operator on (N, g), in which case $X = Q$ (cf. (98)).

As in Sect. 5.2 we get

Theorem 23. $\varLambda = (Q, U) \in \mathrm{Lie}\,(\varDelta)$ *if and only if the following equations are satisfied:*

$$\varDelta U = 0; \tag{a2}$$

$$\frac{1}{2}\Box\omega + \mathrm{Ric}\,\omega + dU = 0; \tag{a3}$$

$$(\nabla\omega)^{(ij)} = c \cdot g^{ij}. \tag{a4}$$

The cocycle is $\varPhi = \varPhi_\varLambda = c$.

¿From Theorem 19 we immediately get

Theorem 24. *As Lie algebras* $\mathrm{Lie}\,(\varDelta) = \mathfrak{conf}_h$, *through the map* (93).

6 Constants of Motion, Conservation Laws and Martingales

6.1 Quantum picture

We start from a self-adjoint Hamiltonian $H = -\frac{1}{2}\varDelta + V$, where $V = V(t, q)$, and q is the coordinate in N. Let $\varLambda = T\partial_t + Q^i\partial_i + U = X + U$, where X is a (smooth) vector field on space-time M and U is a (smooth) function on M. This is the general form for a linear PDO of order ≤ 1 on M. We write $\mathfrak{D}^1(M)$ for all such operators.

Denote by

$$K := \frac{\partial}{\partial t} - H \tag{109}$$

the (backward) *heat operator*. If $\Lambda \in \mathrm{Lie}\,(K)$, i.e., $[K, \Lambda] = \Phi \cdot K$ for some function Φ, then

$$K\Lambda u = ([K, \Lambda] + \Lambda K)u = (\Phi + \Lambda)Ku. \tag{110}$$

Hence,

$$K\Lambda u = 0 \quad \text{if} \quad Ku = 0 \quad \text{and} \quad \Lambda \in \mathrm{Lie}\,(K), \tag{111}$$

i.e., the Lie algebra preserves the kernel of K.

The operator $1 : u \to u$ always belongs to $\mathrm{Lie}\,(K)$. It expresses the *conservation law*

$$\frac{d}{dt} \int_N \theta\theta^* \,\mathrm{dvol} = 0. \tag{112}$$

This is a direct consequence of H being self-adjoint: the left-hand side is

$$\langle \dot{\theta}, \theta^* \rangle + \langle \theta, \dot{\theta}^* \rangle = \langle H\theta, \theta^* \rangle + \langle \theta, -H\theta^* \rangle = \langle H\theta, \theta^* \rangle - \langle H\theta, \theta^* \rangle = 0. \tag{113}$$

Suppose f is a smooth function on M with appropriate growth conditions. Then

$$\frac{d}{dt} \int_N f \cdot \theta\theta^* \,\mathrm{dvol} = \int_N Df \cdot \theta\theta^* \,\mathrm{dvol} = \int_N D^* f \cdot \theta\theta^* \,\mathrm{dvol} \tag{114}$$

Suppose now that Λ is any element the heat Lie algebra. By definition $D = \theta^{-1} K \theta$, so that

$$D(\theta^{-1}\Lambda\theta) = \theta^{-1} K \Lambda \theta = \theta^{-1}(\Phi + \Lambda)K\theta = 0, \tag{115}$$

i.e., along the process, $\theta^{-1}\Lambda\theta$ is a martingale. This can also be expressed as the conservation law

$$\frac{d}{dt} \int_N \Lambda\theta \cdot \theta^* \,\mathrm{dvol} = 0, \tag{116}$$

because the LHS is $(d/dt) \int_N \theta^{-1}\Lambda\theta \cdot \theta\theta^* \,\mathrm{dvol} = \int_N D(\theta^{-1}\Lambda\theta) \cdot \theta\theta^* \,\mathrm{dvol} = 0$, as we just saw.

By repetition of these arguments,

Theorem 25. *If $\Lambda_j \in \mathrm{Lie}\,(K)$, and if $s_j \geq 0$ are integers, $1 \leq j \leq k$, then, along the process, $\theta^{-1}\Lambda_1^{s_1} \cdots \Lambda_k^{s_k}\theta$ is a martingale, and we have the conservation law*

$$\frac{d}{dt} \int_N \Lambda_1^{s_1} \cdots \Lambda_k^{s_k} \theta \cdot \theta^* \,\mathrm{dvol} = 0. \tag{117}$$

We remark that by time-reflection and duality we get corresponding statements for θ^* and $\Lambda_1^{\dagger s_1} \cdots \Lambda_k^{\dagger s_k}$.

Let us now return to the function $\theta^{-1}\Lambda\theta$. With the notation $\widehat{p}_i = \partial_i\theta/\theta$ and $\widehat{E} = \dot{\theta}/\theta$, it becomes

$$\frac{\Lambda\theta}{\theta} = \widehat{E}T + \widehat{p}_i Q^i + U. \tag{118}$$

$\widehat{p}_i$ are the *momentum densities* and $\widehat{E}$ the *energy density* from Sect. 2.5. (We have added hats in this chapter to distinguish between classical and quantum objects.)

Hence, by the theorem,

Corollary 26.

$$D\left(\widehat{E}T + \langle \widehat{p}, Q\rangle + U\right) = 0 \tag{119}$$

whenever $\Lambda = (T, Q, U) \in \mathrm{Lie}\,(K)$.

We recall that with $A = \log\theta$ the coefficients $\widehat{p}_i = \partial_i\theta/\theta$ and $\widehat{E} = \dot{\theta}/\theta$ are given by

$$\overline{d}A = \widehat{E}\,dt + \widehat{p}_i\,dq^i = \widehat{E}\,dt + \widehat{p}dq, \tag{120}$$

where $\overline{d}A$ signifies space-time differential. The right-hand side is the restriction of the first fundamental form $\omega = Edt + pdq$ to a Lagrangian manifold: the second fundamental form $\Omega = \overline{d}\omega = dE \wedge dt + dp \wedge dq$ vanishes there, because $\overline{d}^2 = 0$.

6.2 Classical picture

M is the *configuration space*, and the cotangent bundle T^*M is the (extended) *phase space*. The fibre coordinates are (E, p).

Definition 27. *The symbol map takes* $\Lambda \in \mathfrak{D}^1(M)$ *to the function* $F_\Lambda \in C^\infty(T^*M)$ *defined by*

$$F_\Lambda(t, q, E, p) := ET(t, q) + \langle p, Q(t, q)\rangle + U(t, q). \tag{121}$$

Using the first fundamental form $\omega = E\,dt + p\,dq$, $F = \omega(\pi_* X) + U$, where $X = (T, Q)$ and π is the projection $TM \to M$. The second fundamental form $\Omega = d\omega = dE \wedge dt + dp \wedge dq$ on T^*M gives rise to the *Poisson bracket*

$$\{\phi, \psi\} = \frac{\partial\phi}{\partial E}\frac{\partial\psi}{\partial t} - \frac{\partial\phi}{\partial t}\frac{\partial\psi}{\partial E} + \frac{\partial\phi}{\partial p_i}\frac{\partial\psi}{\partial q^i} - \frac{\partial\phi}{\partial q^i}\frac{\partial\psi}{\partial p_i}, \quad \phi,\, \psi \in C^\infty(T^*M). \tag{122}$$

Theorem 28. *The symbol map is 1–1 onto the space of functions in* $C^\infty(T^*M)$ *which are of order at most one in* E *and* p*. It is a Lie algebra morphism:*

$$\{F_{\Lambda_1}, F_{\Lambda_2}\} = F_{[\Lambda_1, \Lambda_2]}. \tag{123}$$

Given an 'ordinary' Hamiltonian $H = H(t, q, p)$, we get an *extended Hamiltonian* K defined by

$$K = K(t, q, E, p) := E - H(t, q, p). \tag{124}$$

For any differentiable function F on T^*M we define

$$\frac{dF}{dt} := \{K, F\}. \tag{125}$$

This is equal to

$$\frac{\partial F}{\partial t} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q^i} + \frac{\partial H}{\partial q^i}\frac{\partial F}{\partial p_i} + \frac{\partial H}{\partial t}\frac{\partial F}{\partial E} \tag{126}$$

and leads to

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + p_i\frac{\partial F}{\partial q^i} + \frac{\partial V}{\partial q^i}\frac{\partial F}{\partial p_i} + \frac{\partial V}{\partial t}\frac{\partial F}{\partial E} \tag{127}$$

if we choose the Euclidean Hamiltonian

$$H = -\frac{1}{2}|p|^2 + V, \quad \text{where} \quad V = V(t, q). \tag{128}$$

¿From now on, this is our choice for H. The *equations of motion* become

$$\dot{q} = p^\sharp, \quad \frac{Dp}{dt} = dV, \quad \dot{E} = \dot{V}, \quad \dot{t} = 1. \tag{129}$$

Here, D/dt denotes the covariant derivative.

Definition 29. *The classical Lie algebra,* $\mathrm{Lie}_c(K)$*, consists of all* $F \in C^\infty(T^*M)$ *of order at most one in* (E, p) *which satisfy*

$$\{K, F\} = a \cdot K \tag{130}$$

for some (local) function $a = a_F$*.*

By the implicit function theorem, this is equivalent to requiring that $dF/dt = 0$ on the set where $E = H(t, q, p)$.

The next point is to determine the classical Lie algebra. We calculate dF_Λ/dt and substitute $E = H(t, q, p)$. The result is a polynomial of order three in p, with coefficients depending on (t, q). All these coefficients must vanish.

$$\begin{aligned}
&\frac{d}{dt}(ET + \langle p, Q\rangle + U) \\
=&\dot{V}T + (-\tfrac{1}{2}|p|^2 + V)(\dot{T} + \langle p, dT\rangle) + \langle dV, Q\rangle \\
&+ \langle p, \dot{Q}\rangle + \nabla Q(p, p) + \dot{U} + \langle p, dU\rangle \\
=&-\tfrac{1}{2}|p|^2\langle p, dT\rangle + \left(\nabla Q^{(ij)} - \tfrac{1}{2}\dot{T}g^{ij}\right)p_i p_j \\
&+ \langle p, V dT + \dot{\omega} + dU\rangle + \dot{U} + \dot{T}V + T\dot{V} + Q(V).
\end{aligned} \tag{131}$$

Here, ω is the 1-form dual to Q.

Theorem 30. *$\Lambda = (T, Q, U)$ belongs to the classical Lie algebra for $K = E - H$ precisely when the following equations hold:*

$$dT = 0; \tag{C1}$$

$$\dot{U} + \frac{\partial}{\partial t}(TV) + Q(V) = 0; \tag{C2}$$

$$\dot{\omega} + dU = 0; \tag{C3}$$

$$\nabla\omega^{(ij)} = \tfrac{1}{2}\dot{T}g^{ij}. \tag{C4}$$

We shall now connect our findings with the Noether theorem. We refer to the presentation in Ibragimov [Ibr2], pp. 236–239.

6.3 The classical case

In general, a classical Lagrangian $L = L(t, q, \dot{q})$ and a Hamiltonian $H = H(t, q, p)$ are related by (Euclidean conventions)

$$L = \dot{q}^i p_i + H. \tag{132}$$

For $H = -\frac{1}{2}|p|^2 + V$, the Lagrangian becomes $L = \frac{1}{2}|\dot{q}|^2 + V$. The Euler-Lagrange equations are

$$\frac{\delta L}{\delta q^i} = 0, \quad i = 1, \ldots, n, \tag{133}$$

where

$$\frac{\delta L}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}. \tag{134}$$

Given a vector field $X = T\partial/\partial t + Q^i\partial/\partial q^i$, we make an infinitesimal variation of the action $\int L\,dt$:

$$\delta_X \int L\,dt = \int \left(X^{(1)}(L) + L\frac{dT}{dt}\right) dt. \tag{135}$$

Here, the vector field X is *prolonged* ([Ibr2], [Or1]) to $X^{(1)}$, so that it can act also on the variable $\dot{q}$:

$$X^{(1)} = X + \left(\frac{DQ^i}{dt} - \dot{q}^i\frac{dT}{dt}\right)\frac{\partial}{\partial \dot{q}^i} = X + W^i\frac{\partial}{\partial \dot{q}^i}. \tag{136}$$

After some manipulation we get

$$X^{(1)}(L) + L\frac{dT}{dt} = \frac{d}{dt}\left(TL + W^i\frac{\partial L}{\partial \dot{q}^i}\right) + W^i\frac{\delta L}{\delta q^i}. \tag{137}$$

Thus, *Noether's invariance condition* (invariance of the Lagrangian differential $L\,dt$ modulo exact differentials)

$$X^{(1)}(L) + L\frac{dT}{dt} = -\frac{dU}{dt}, \tag{138}$$

for some function $U(t,q)$, implies that

$$\frac{d}{dt}\left(TL + W^i\frac{\partial L}{\partial \dot{q}^i} + U\right) = 0, \tag{139}$$

i.e.,

$$\frac{d}{dt}\left(TE + \langle p, Q\rangle + U\right) = 0. \tag{140}$$

This is exactly the case in Sect. 6.2 and should be compared with Corollary 6.1.7. In both cases we have what perhaps should be termed a Poisson-Noether algebra. It is a Lie algebra of conserved quantities; in general, only a part of the Lie algebra of a differential equation gives rise to conserved quantities.

We remark that this case of Noether's theorem was proved by M. Lévy already in 1878.

6.4 Quantum case

The Hamiltonian is $H = -\frac{1}{2}\Delta + V$, and the *Hamiltonian field density* is $\mathsf{H} = \frac{1}{2}(H\theta\cdot\theta^* + \theta H\theta^*)$. This is an equivalent form of $\frac{1}{2}\langle d\theta, d\theta^*\rangle + V\theta\theta^*$.

The *Lagrangian field density* is

$$\mathsf{L} = \frac{1}{2}(\theta\dot{\theta}^* - \dot{\theta}\theta^*) + \mathsf{H}. \tag{141}$$

Write $(\theta,\theta^*) = (\theta^0,\theta^1)$, and $\theta^a_\mu = \partial_\mu\theta^a$. In general, the Euler-Lagrange equations, obtained from the space-time variational principle

$$\delta\iint \mathsf{L}\,dt\,d\mathrm{vol} = 0, \tag{142}$$

are

$$\frac{\delta\mathsf{L}}{\delta\theta^a} = 0, \quad a = 0,1, \tag{143}$$

where

$$\frac{\delta\mathsf{L}}{\delta\theta^a} := \frac{\partial\mathsf{L}}{\partial\theta^a} - D_\mu\frac{\partial\mathsf{L}}{\partial\theta^a_\mu}. \tag{144}$$

In the present case we get

$$\dot{\theta} - H\theta = 0, \qquad \dot{\theta}^* + H\theta^* = 0. \tag{145}$$

Write $x = (x^0, x^1, ..., x^n) = (t, q^1, .., q^n)$. Consider a vector field Λ on the first order jet bundle over M with variables x^μ and θ^a, $0\le\mu\le n$, $a = 0,1$:

$$\Lambda = X^\mu\frac{\partial}{\partial x^\mu} + \eta^a\frac{\partial}{\partial\theta^a} = X^\mu\partial_\mu + \eta^a\frac{\partial}{\partial\theta^a}. \tag{146}$$

Write $dx := dt\, d\mathrm{vol}$. We get

$$\delta_{\Lambda} \int_M \mathsf{L}\, dx = \int_M (\Lambda^{(2)}(\mathsf{L}) + \mathsf{L} D_\mu X^\mu)\, dx, \tag{147}$$

where $D_\mu := d/dx^\mu$ denotes the total derivative w.r.t. x_μ. Λ is prolonged to the second-order jet bundle. One finds

$$\Lambda^{(2)}(\mathsf{L}) + \mathsf{L} D_\mu X^\mu = D_\mu \left(\mathsf{L} X^\mu + W^a \frac{\partial \mathsf{L}}{\partial \theta^a_\mu} \right) + W^a \frac{\delta \mathsf{L}}{\delta \theta^a}, \tag{148}$$

where

$$W^a = \eta^a - \theta^a_\mu X^\mu. \tag{149}$$

Choose a vector field of the form

$$\Lambda = T \frac{\partial}{\partial t} + Q^i \frac{\partial}{\partial q^i} + U \left(\theta^* \frac{\partial}{\partial \theta^*} - \theta \frac{\partial}{\partial \theta} \right). \tag{150}$$

These vector fields are just another representation of the first order PDOs $T\frac{\partial}{\partial t} + Q^i \frac{\partial}{\partial q^i} + U$ employed above. The Lie algebras are isomorphic. – As usual, T, Q^i and U only depend on t and q. In this general case, Noether's theorem states that $\Lambda^{(2)}(\mathsf{L}) + \mathsf{L} D_\mu X^\mu = D_\mu B^\mu$ for some vector field B if and only if $\mathsf{L} X^0 + W^a \partial \mathsf{L} / \partial \dot{\theta}^a$ is the density of a conservation law. The latter quantity is

$$I = I_\Lambda = T \cdot \frac{1}{2}(\dot{\theta}\theta^* - \theta\dot{\theta}^*) + Q^i \cdot \frac{1}{2}(\theta_i \theta^* - \theta\theta^*_i) + U\theta\theta^*. \tag{151}$$

$I/\theta\theta^*$ is just a symmetric version of the function $\widehat{E}T + \widehat{p}_i Q^i + U$ in (118) above. The Noether theorem gives the same densities for conservation laws as we encountered in Sect. 6.1.

6.5 Connecting the classical and the quantum algebras. Examples

The total (i.e., space-time) differential $\sigma := \overline{d}A$ of the action density $A = \log\theta$ defines a section of T^*M. If $F = F_\Lambda$, then

$$\widehat{E}T + \widehat{p}_i Q^i + U = \sigma^* F = F \circ \sigma =: \widehat{F}. \tag{152}$$

Write $F_j = F_{\Lambda_j}$. Theorem 28 implies, with obvious notation,

$$\widehat{\{F_1, F_2\}} = \widehat{F}_{[1,2]}. \tag{153}$$

One readily shows the following commutator formula:

$$\widehat{F}_{[1,2]} = X_1(\widehat{F}_2) - X_2(\widehat{F}_1). \tag{154}$$

We now characterise the classical Lie algebra for a class of quadratic potentials. The proof is left out.

Theorem 31. *Suppose $N = \mathbb{R}^n$. All potentials on the form*

$$V(t,q) = \frac{1}{2}a(t)|q|^2 + b(t)\cdot q + c(t), \tag{155}$$

where a, b and c are smooth functions of t, with $a > 0$, yield isomorphic classical Lie algebras for the Hamiltonian $H = -\frac{1}{2}|p|^2 + V$. They can be represented as

$$\mathfrak{h} + \mathfrak{b}_1 + \mathfrak{b}_2, \tag{156}$$

where $\mathfrak{h} = \langle 1, \xi_i, \eta_j\rangle_{i,j=0}^n$ is a representation of the Heisenberg algebra: $\{\xi_i, \eta_j\} = \delta_{ij}$ and all other brackets vanish. The centre is generated by 1, $\mathfrak{h}$ is an ideal and $\mathfrak{b}_1$ and $\mathfrak{b}_2$ commute. $\mathfrak{b}_1 = \langle |\xi|^2, \xi\cdot\eta, |\eta|^2\rangle$ is a representation of sl_2, and $\mathfrak{b}_2$ is a representation of so_n. The dimension is $2n+4+n(n-1)/2$.

Finally, we shall compare the two Lie algebras in this case, so $V(t,q) = \frac{1}{2}a(t)|q|^2 + b(t)\cdot q + c(t)$. If we integrate (C3)–(C4), omitting the case when $\nabla\omega$ is antisymmetric, i.e., the case of infinitesimal rotations, we first find $Q = \frac{1}{2}\dot{T}q + \alpha$, where α is a function of t, and then

$$U = -\frac{1}{4}\ddot{T}|q|^2 - \dot{a}\cdot q + \beta, \tag{157}$$

where $\beta = \beta(t)$. Equation (C2) leads to

$$-\frac{1}{4}\dddot{T} + 2a\dot{T} + \dot{a}T = 0, \quad \ddot{\alpha} - a\alpha = \frac{3}{2}\dot{T}b + T\dot{b}, \quad \dot{\beta} = -(Tc)^{\cdot} + \alpha b. \tag{158}$$

Now $\Delta U = -\frac{n}{2}\ddot{T}$, so the only change caused by the heat equation is that β must fulfill

$$\dot{\beta} = -(Tc)^{\cdot} + \alpha b - \frac{n}{4}\ddot{T}. \tag{159}$$

This is the Itô correction. It only concerns one of the elements of the sl_2-part. Changing the representation for this part, in case of the heat equation, one finds that the classical Lie algebra and the heat Lie algebra are isomorphic.

This is related to Gaussian diffusions, oscillator-like systems and the semi-classical limit. See Brandão [B1], Brandão-Kolsrud [BK1], [BK2], M. Kolsrud [K0] and Kolsrud-Zambrini [KZ2].

Comments and acknowledgements

The major part of this article is included in the unpublished article [K1], most of which was written during a stay in Lisbon 1995. I thank my friends there once more, in particular the late Augusto Brandão Correia, 1961–2004.

It was a pleasure to participate in the Abel Symposium 2005. Many thanks to my Norwegian friends for kindly inviting me to the university where, almost twenty years ago, I first learned from Raphael Høegh-Krohn the beauty of the relations between the classical and the quantum. Thanks also to my uncle Marius K., with whom I have discussed this and other areas of mathematical physics many late evenings.

References

[AI] Anderson, R.L., Ibragimov, N.H.: Lie-Bäcklund Transformations in Applications. SIAM, Philadelphia (1979)

[Arn] Arnold, V.I.: Mathematical methods in classical mechanics. Springer, Berlin Heidelberg New York (1978)

[AG] Arnold, V.I., Givental, A.B.: Symplectic Geometry. In: Arnold, V.I. and Novikov, S.P. (eds) Dynamical Systems IV, Encyclopedia of Mathematical Sciences, Vol 4. Springer, Berlin Heidelberg New York (1990)

[AKN] Arnold, V.I., Kozlov, V.V., Neishtadt, A.I.: Mathematical aspects of classical and celestial mechanics. In: Arnold, V.I. (ed) Dynamical Systems III, Encyclopedia of Mathematical Sciences, Vol 3. Springer, Berlin Heidelberg New York (1988)

[B1] Brandão, A.: Symplectic structure for Gaussian diffusions. J. Math. Phys., **39**, 4257–4283 (1998)

[BK1] Brandão, A., Kolsrud T.: Phase space transformations of Gaussian diffusions. Potential Anal., **10**, 119–132 (1999)

[BK2] Brandão, A., Kolsrud T.: Time-dependent conservation laws and symmetries for classical mechanics and heat equations. In: Harmonic morphisms, harmonic maps, and related topics (Brest, 1997). Chapman & Hall/CRC Res. Notes Math., 413, Chapman & Hall/CRC, Boca Raton FL (2000)

[CZ] Cruzeiro, A.B., Zambrini, J.-C.: Malliavin calculus and Euclidean quantum mechanics. I. Functional calculus. J. Funct. Anal., **96**, 62–95 (1991)

[D] Darling, R.W.R.: Martingales in manifolds—Definition, examples and behaviour under maps. In: Azéma, J., Yor, M. (eds) Séminaire de Probabilités XVI, Lect. Notes Math. 921, Springer-Verlag, Berlin Heidelberg New York (1982)

[DK] Djehiche, B., Kolsrud, T.: Canonical transformations for diffusions. C. R. Acad. Sci. Paris, **3**21, I, 339–44 (1995)

[DKN] Dubrovin, B.A., Krichever, I.M., Novikov, S.P.: Integrable Systems I. In: Arnold, V.I., Novikov, S.P. (eds) Dynamical Systems IV, Encyclopedia of Mathematical Sciences, Vol 4., Springer, Berlin Heidelberg New York (1990)

[DNF] Doubrovine, B., Novikov, S., Fomenko, A.: Géométrie contemporaine. Méthodes et applications. 1re partie. Traduction française. Éditions Mir, Moscou (1982)

[EL] Eells, J., Lemaire, L.: Selected Topics in Harmonic Maps. CBMS Regional Conf. Ser. in Math 50. Amer. Math. Soc., Providence RI (1983)

[Em] Emery, M.: Stochastic Calculus in Manifolds. Springer, Berlin Heidelberg New York (1989)

[FH] Feynman, R.P., Hibbs, A.R.: Quantum mechanics and path integrals. McGraw-Hill, New York (1965)

[F1] Fuglede, B.: Harmonic morphisms between Riemannian manifolds. Ann. Inst. Fourier, **28**, 107–44 (1978)

[F2] Fuglede, B., Harmonic morphisms. In: Springer Lect. Notes in Math. 747. Springer, Berlin Heidelberg New York (1979)

[Ga] Gallavotti, G.: The Elements of Mechanics (transl. of *Meccanica Elementare*. Ed. Boringhieri, Torino (1980)). Springer, Berlin Heidelberg New York (1983)

[Go] Goldberg, S.I.: Curvature and homology. Academic Press, New York (1962)

[Gol] Goldstein, H.: Classical mechanics, 2nd ed. Addison-Wesley, New York (1980)

[Gorb] Gorbatsevich, V.V., Onishchik, A.L., Vinberg, E.B.: Foundations of Lie Theory and Lie Transformation Groups. Springer, Berlin Heidelberg New York (1993)

[Ibr2] Ibragimov, N.H.: Transformation Groups Applied to Mathematical Physics. Nauka, Moscow (1983) (English translation by D. Reidel, Dordrecht (1985))

[Ibr3] Ibragimov, N.H. (ed) CRC handbook of Lie group analysis of differential equations, Vol 1, Symmetries, exact solutions and conservation laws. CRC Press, Boca Raton FL (1993)

[IK] Ibragimov, N.H., Kolsrud T.: Lagrangian approach to evolution equations: symmetries and conservation laws. Nonlinear Dynam., **3**6, 29–40 (2004)

[IW] Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes, 2nd ed. North-Holland, Kodansha Amsterdam Tokyo (1989)

[Ish] Ishihara, T.: A mapping of Riemannian manifolds which preserves harmonic functions. J. Math. Kyoto Univ., **19**, 215–229 (1979)

[I] Itô, K.: The Brownian motion and tensor fields on a Riemannian manifold. In: Proc. Int. Congr. Math., Stockholm 1962. Inst. Mittag-Leffler, Djursholm (1963)

[Ko] Kobayashi, S.: Transformation groups in differential geometry. Springer. Berlin Heidelberg New York (1972)

[K0] Kolsrud, M.: Exact quantum dynamical solutions for oscillator-like systems. Phys. Mat. Univ. Osloensis, **28** (1965)

[K1] Kolsrud, T.: Quantum constants of motion and the heat Lie algebra in a Riemannian manifold. Preprint TRITA-MAT, Royal Institute of Technology, Stockholm (1996)

[K2] Kolsrud, T.: Symmetries for the Euclidean Non-Linear Schrödinger Equation and Related Free Equations. In: Proc. of MOGRAN X (Cyprus 2004). Also preprint TRITA-MAT, Royal Institute of Technology, Stockholm (2005)

[KL] Kolsrud, T., Loubeau, E.: Foliated manifolds and conformal heat morphisms. Ann. Global Anal. Geom., **2**1, 241–67 (2002)

[KZ1] Kolsrud, T., Zambrini, J. C.: The general mathematical framework of Euclidean quantum mechanics. In: Stochastic analysis and applications (Lisbon 1989), Birkhäuser, Basel (1991)

[KZ2] Kolsrud, T., Zambrini, J.C.: An introduction to the semiclassical limit of Euclidean quantum mechanics. J. Math. Phys., **33**, 1301–1334 (1992)

[LL] Landau, L.D., Lifshitz, E.M., Course of Theoretical Physics, Vol 1, Mechanics, Vol 3, Quantum mechanics, 3rd ed. Pergamon Press, Oxford (1977)

[Lie] Lie, S.: Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen. Arch. Math., **6**, 328–368 (1881)

[Lob] Loubeau, E.: Morphisms of the heat equation. Ann. Global Anal. Geom., **15**, 487–496 (1997)

[M] Malliavin, P.: Géométrie differéntielle stochastique. Séminaire de Mathèmatiques Supérieures 64. Presses de l'Université de Montréal, Montreal (1978)

[MF] Morse, P.M., Feshbach, H.: Methods of theoretical physics, Vol. I-II. McGraw-Hill, New York (1953)

[N] Nelson, E.: Quantum Fluctuations. Princeton University Press (1985)

[Or1] Olver, P. J.: Applications of Lie groups to differential equations, 2nd ed. Springer, Berlin Heidelberg New York (1993)

[Ok] Øksendal, B.: When is a stochastic integral a time change of a diffusion? J. Th. Prob., **3**, 207–226 (1990)

[TZ1] Thieullen, M., Zambrini, J.C.: Probability and quantum symmetries I. The theorem of Noether in Schrödinger's euclidean quantum mechanics. Ann. Inst. Henri Poincaré, Phys. Théorique, **67**, 297–338 (1997)

[TZ2] Thieullen, M., Zambrini, J.C.: Symmetries in the stochastic calculus of variations. Probab. Theory Relat. Fields, **107**, 401–427 (1997)