Differential equations driven by Hölder continuous functions of order greater than 1/2

Yaozhong Hu and David Nualart

Department of Mathematics, University of Kansas, 405 Snow Hall, Lawrence, Kansas 66045-2142. Y. Hu is supported in part by the National Science Foundation under Grant No. DMS0204613 and DMS0504783.

Summary. We derive estimates for the solutions to differential equations driven by a Hölder continuous function of order $\beta > 1/2$. As an application we deduce the existence of moments for the solutions to stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

1 Introduction

We are interested in the solutions of differential equations on $\mathbb{R}^d$ of the form

$$x_t = x_0 + \int_0^t f(x_r) dy_r, \tag{1}$$

where the driving force $y : [0, \infty) \to \mathbb{R}^m$ is a Hölder continuous function of order $\beta > 1/2$. If the function $f : \mathbb{R}^d \to \mathbb{R}^{md}$ has bounded partial derivatives which are Hölder continuous of order $\lambda > \frac{1}{\beta} - 1$, then there is a unique solution $x : \mathbb{R}^d \to \mathbb{R}$, which has bounded $\frac{1}{\beta}$-variation on any finite interval. These results have been proved by Lyons in [2] using the p-variation norm and the technique introduced by Young in [8]. The integral appearing in (1) is a Riemann-Stieltjes integral.

In [9] Zähle has introduced a generalized Stieltjes integral using the techniques of fractional calculus. This integral is expressed in terms of fractional derivative operators and it coincides with the Riemann-Stieltjes integral $\int_0^T f dg$, when the functions f and g are Hölder continuous of orders λ and μ, respectively and $\lambda + \mu > 1$ (see Proposition 1 below). Using this formula for the Riemann-Stieltjes integral, Nualart and Răşcanu have obtained in [3] the existence of a unique solution for a class of general differential equations that includes (1). Also they have proved that the solution of (1) is bounded on a finite interval $[0, T]$ by $C_1 \exp(C_2 \|y\|_{0,T,\beta}^{\kappa})$, where $\kappa > \frac{1}{\beta}$ if f is bounded and $\kappa > \frac{1}{1-2\beta}$ is f has linear growth. Here $\|y\|_{0,T,\beta}$ denotes the β-Hölder norm of y on the time interval $[0, T]$. These estimates are based on a suitable application of Gronwall's lemma. It turns out that the estimate in the linear growth case is unsatisfactory because κ tends to infinity as β tends to $1/2$.

The main purpose of this paper is to obtain sharper estimates for the solution x_t in the case where f is bounded or has linear growth using a direct approach based on formula (8). In the case where f is bounded we estimate $\sup_{0\leq t\leq T}|x_t|$ by

$$C\left(1+\|y\|_{0,T,\beta}^{\frac{1}{\beta}}\right)$$

and if f has linear growth we obtain the estimate

$$C_1\exp\left(C_2\|y\|_{0,T,\beta}^{\frac{1}{\beta}}\right).$$

In Theorem 2 we provide explicit dependence on f and T for the constants C, C_1 and C_2. We also establish estimates for the solution of a linear equation with rough time dependent coefficient (Theorem 3.2).

Another novelty of this paper is that we establish stability type of results for the solution x_t to (1) on the initial condition x_0, the driving control y and the coefficient f (Theorem 3.2).

As an application we deduce the existence of moments for the solutions to stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H>\frac{1}{2}$. We also discuss the regularity of the solution in the sense of Malliavin Calculus, improving the results of Nualart and Saussereau [4], and we apply the techniques of the Malliavin calculus to establish the smoothness of the density of the solution under suitable non-degeneracy conditions. More precisely, Theorem 3.2 allows us to show that the solution of a stochastic differential equation

2 Fractional integrals and derivatives

Let $a,b\in\mathbb{R}$ with $a<b$. Let $f\in L^1(a,b)$ and $\alpha>0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $x\in(a,b)$ by

$$I_{a+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s)\,ds$$

and

$$I_{b-}^{\alpha}f(t)=\frac{(-1)^{-\alpha}}{\Gamma(\alpha)}\int_t^b(s-t)^{\alpha-1}f(s)\,ds,$$

respectively, where $(-1)^{-\alpha}=e^{-i\pi\alpha}$ and $\Gamma(\alpha)=\int_0^\infty r^{\alpha-1}e^{-r}dr$ is the Euler gamma function. Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p(a,b)$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f\in I_{a+}^{\alpha}(L^p)$ (resp. $f\in I_{b-}^{\alpha}(L^p)$) and $0<\alpha<1$ then the Weyl derivatives are defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}} ds \right) \tag{1}$$

and

$$D_{b-}^{\alpha} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t)-f(s)}{(s-t)^{\alpha+1}} ds \right) \tag{2}$$

where $a \le t \le b$ (the convergence of the integrals at the singularity $s = t$ holds point-wise for almost all $t \in (a,b)$ if $p = 1$ and moreover in L^p-sense if $1 < p < \infty$).

For any $\lambda \in (0,1)$, we denote by $C^{\lambda}(a,b)$ the space of λ-Hölder continuous functions on the interval $[a,b]$. We will make use of the notation

$$\|x\|_{a,b,\beta} = \sup_{a \le \theta < r \le b} \frac{|x_r - x_\theta|}{|r-\theta|^{\beta}},$$

and

$$\|x\|_{a,b,\infty} = \sup_{a \le r \le b} |x_r|,$$

where $x : \mathbb{R}^d \to \mathbb{R}$ is a given continuous function.

Recall from [6] that we have:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$ then
$$I_{a+}^{\alpha}(L^p) = I_{b-}^{\alpha}(L^p) \subset L^q(a,b).$$
- If $\alpha > \frac{1}{p}$ then
$$I_{a+}^{\alpha}(L^p) \cup I_{b-}^{\alpha}(L^p) \subset C^{\alpha - \frac{1}{p}}(a,b).$$

The following inversion formulas hold:

$$I_{a+}^{\alpha}\left(D_{a+}^{\alpha} f\right) = f, \qquad \forall f \in I_{a+}^{\alpha}(L^p) \tag{3}$$

$$I_{a-}^{\alpha}\left(D_{a-}^{\alpha} f\right) = f, \qquad \forall f \in I_{a-}^{\alpha}(L^p) \tag{4}$$

and

$$D_{a+}^{\alpha}\left(I_{a+}^{\alpha} f\right) = f, \quad D_{a-}^{\alpha}\left(I_{a-}^{\alpha} f\right) = f, \quad \forall f \in L^1(a,b)\,. \tag{5}$$

On the other hand, for any $f, g \in L^1(a,b)$ we have

$$\int_a^b I_{a+}^{\alpha} f(t) g(t) dt = (-1)^{\alpha} \int_a^b f(t) I_{b-}^{\alpha} g(t) dt\,, \tag{6}$$

and for $f \in I_{a+}^{\alpha}(L^p)$ and $g \in I_{a-}^{\alpha}(L^p)$ we have

$$\int_a^b D_{a+}^{\alpha} f(t) g(t) dt = (-1)^{-\alpha} \int_a^b f(t) D_{b-}^{\alpha} g(t) dt. \tag{7}$$

Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda+\mu>1$. Then, from the classical paper by Young [8], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives (see [9]).

Proposition 1. *Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda+\mu>1$. Let $\lambda>\alpha$ and $\mu>1-\alpha$. Then the Riemann Stieltjes integral $\int_a^b f dg$ exists and it can be expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f\,(t)\, D_{b-}^{1-\alpha} g_{b-}\,(t)\, dt, \tag{8}$$

where $g_{b-}\,(t) = g\,(t) - g\,(b)$.

3 Estimates for the solutions of differential equations

Suppose that $y:[0,\infty)\to\mathbb{R}^m$ is a Hölder continuous function of order $\beta>1/2$. Fix an initial condition $x_0\in\mathbb{R}^d$ and consider the following differential equation

$$x_t = x_0 + \int_0^t f(x_r) dy_r, \tag{1}$$

where $f:\mathbb{R}^d\to\mathbb{R}^{md}$ is given function. Lyons has proved in [2] that Equation (1) has a unique solution if f is continuously differentiable and it has a derivative f' which is bounded and locally Hölder continuous of order $\lambda>\frac{1}{\beta}-1$.

Our aim is to obtain estimates on x_t which are better than those given by Nualart and Răşcanu in [3].

Theorem 2. *Let f be a continuously differentiable function such that f' is bounded and locally Hölder continuous of order $\lambda>\frac{1}{\beta}-1$.*

(i) Assume $\|f'\|_\infty>0$. There is a constant k depending only on β, such that for all T,

$$\sup_{0\le t\le T}|x_t| \le 2^{1+kT\left[\|f'\|_\infty\vee|f(0)|\right]^{1/\beta}\|y\|_{0,T,\beta}^{1/\beta}}\left(|x_0|+1\right). \tag{2}$$

(ii) Assume that f is bounded. Then, there is a constant k, which depends only on β, such that for all T,

$$\sup_{0\le t\le T}|x_t| \le |x_0| + k\|f\|_\infty\left(T^{\beta}\|y\|_{0,T,\beta}\vee T\|f'\|_\infty^{\frac{1-\beta}{\beta}}\|y\|_{0,T,\beta}^{\frac{1}{\beta}}\right). \tag{3}$$

Proof. Without loss of generality we assume that $d = m = 1$. Set $\|y\|_\beta = \|y\|_{0,T,\beta}$. We can assume that $\|y\|_\beta > 0$, otherwise the inequalities are obvious. Let $\alpha < 1/2$ such that $\alpha > 1-\beta$. Henceforth k will denote a generic constant depending only on β.

Step 1. Assume first that f is bounded. It suffices to assume that $\|f'\|_\infty > 0$. First we use the fractional integration by parts formula given in Proposition 1 to obtain for all $s, t \in [0, T]$,

$$|\int_s^t f(x_r) dy_r| \leq \int_s^t |D_{s+}^\alpha f(x_r)\, D_{t-}^{1-\alpha} y_{t-}(r)| dr.$$

From (1) and (2) it is easy to see

$$|D_{t-}^{1-\alpha} y_{t-}(r)| \leq k\|y\|_{r,t,\beta}(t-r)^{\alpha+\beta-1} \leq k\|y\|_\beta (t-r)^{\alpha+\beta-1} \tag{4}$$

and

$$|D_{s+}^\alpha f(x_r)| \leq k\left[\|f\|_\infty (r-s)^{-\alpha} + \ \|f'\|_\infty \|x\|_{s,t,\beta}(r-s)^{\beta-\alpha}\right]. \tag{5}$$

Therefore

$$\begin{aligned}|\int_s^t f(x_r) dy_r| &\leq k\|y\|_\beta \int_s^t \left[\|f\|_\infty (r-s)^{-\alpha}(t-r)^{\alpha+\beta-1}\right.\\ &\qquad \left. + \|f'\|_\infty \|x\|_{s,t,\beta}(r-s)^{\beta-\alpha}(t-r)^{\alpha+\beta-1}\right] dr\\ &\leq k\|y\|_\beta \left[\|f\|_\infty (t-s)^\beta + \|f'\|_\infty \|x\|_{s,t,\beta}(t-s)^{2\beta}\right].\end{aligned}$$

Consequently, we have

$$\|x\|_{s,t,\beta} \leq k\|y\|_\beta \left[\|f\|_\infty + \|f'\|_\infty \|x\|_{s,t,\beta}(t-s)^\beta\right].$$

Choose Δ such that

$$\Delta = \left(\frac{1}{2k\,\|f'\|_\infty \|y\|_\beta}\right)^{\frac{1}{\beta}}.$$

Then, for all s and t such that $t - s \leq \Delta$ we have

$$\|x\|_{s,t,\beta} \leq 2k\|y\|_\beta \|f\|_\infty. \tag{6}$$

Therefore,

$$\|x\|_{s,t,\infty} \leq |x_s| + \|x\|_{s,t,\beta}(t-s)^\beta \leq |x_s| + 2k\|y\|_\beta \|f\|_\infty \Delta^\beta. \tag{7}$$

If $\Delta \geq T$ we obtain the estimate

$$\|x\|_{0,T,\infty} \leq |x_0| + 2k\|y\|_\beta \|f\|_\infty T^\beta. \tag{8}$$

Assume $\Delta < T$. Then, from (7) we get

$$\|x\|_{s,t,\infty} \leq |x_s| + \|f\|_\infty \|f'\|_\infty^{-1}. \tag{9}$$

Divide the interval $[0,T]$ into $n=[T/\Delta]+1$ subintervals (where $[a]$ denotes the largest integer bounded by a). Applying the inequality (9) for $s=0$ and $t=\Delta$ we obtain

$$\sup_{0\le t\le \Delta} |x_t| \le |x_0| + \|f\|_\infty \|f'\|_\infty^{-1}.$$

Then, applying the inequality (9) on the intervals $[\Delta, 2\Delta]$, ..., $[(n-1)\Delta, n\Delta]$ recursively, we obtain

$$\begin{aligned}\sup_{0\le t\le T} |x_t| &\le |x_0| + n\|f\|_\infty \|f'\|_\infty^{-1} \le |x_0| + \Delta^{-1}(T+\Delta)\|f\|_\infty \|f'\|_\infty^{-1} \\ &\le |x_0| + Tk\|f\|_\infty \|f'\|_\infty^{\frac{1-\beta}{\beta}} \|y\|_\beta^{\frac{1}{\beta}}. \end{aligned} \tag{10}$$

The inequality (3) follows from (8) and (10).

Step 2. In the general case, assuming $\|f'\|_\infty > 0$, instead of (5) we have

$$|D^\alpha_{s+} f(x_r)| \le k\left[\left(|f(0)| + \|f'\|_\infty |x_r|\right)(r-s)^{-\alpha} + \|f'\|_\infty \|x\|_{s,t,\beta} (r-s)^{\beta-\alpha} \right].$$

As a consequence,

$$\|x\|_{s,t,\beta} \le k\|y\|_\beta \left[|f(0)| + \|f'\|_\infty \|x\|_{s,t,\infty} + \|f'\|_\infty \|x\|_{s,t,\beta}(t-s)^\beta \right].$$

Suppose that Δ satisfies

$$\Delta \le \left(\frac{1}{3k\|f'\|_\infty \|y\|_\beta} \right)^{\frac{1}{\beta}}. \tag{11}$$

Then, for all s and t such that $t-s\le \Delta$ we have

$$\|x\|_{s,t,\beta} \le \frac{3}{2} k\|y\|_\beta \left(|f(0)| + \|f'\|_\infty \|x\|_{s,t,\infty} \right).$$

Therefore,

$$|x_t| \le |x_s| + \frac{3}{2} k\|y\|_\beta \left(|f(0)| + \|f'\|_\infty \|x\|_{s,t,\infty} \right)\Delta^\beta,$$

and

$$\|x\|_{s,t,\infty} \le |x_s| + \frac{3}{2} k\|y\|_\beta \left(|f(0)| + \|f'\|_\infty \|x\|_{s,t,\infty} \right)\Delta^\beta.$$

Using again (11) we get

$$\|x\|_{s,t,\infty} \le 2|x_s| + 2k\|y\|_\beta |f(0)|\Delta^\beta.$$

Assume also that

$$\Delta \le \left(\frac{1}{k\,|f(0)|\,\|y\|_\beta} \right)^{\frac{1}{\beta}}. \tag{12}$$

Then

$$\|x\|_{s,t,\infty} \leq 2\left(|x_s|+1\right).$$

Hence,

$$\sup_{0\leq r\leq t}|x_r| \leq 2\left(\sup_{0\leq r\leq s}|x_r|+1\right). \tag{13}$$

As before, divide the interval $[0,T]$ into $n=[T/\Delta]+1$ subintervals, and use the estimate (13) in every interval to obtain

$$\sup_{0\leq t\leq T}|x_t| \leq 2^n\left(|x_0|+1\right). \tag{14}$$

Choose

$$\Delta = \left(k\|y\|_\beta\left(\|f'\|_\infty \vee |f(0)|\right)\right)^{-\frac{1}{\beta}},$$

in such a way that (11) and (12) hold. Then, (14) implies

$$\sup_{0\leq t\leq T}|x_t| \leq 2^{1+kT\left[\|f'\|_\infty\vee|f(0)|\right]^{1/\beta}\|y\|_\beta^{1/\beta}}\left(|x_0|+1\right).$$

The proof of the theorem is now complete.

Consider now the following system of equations

$$x_t = x_0 + \int_0^t f(x_r)dy_r,$$
$$z_t = z_0 + \int_0^t g(x_r)z_r dy_r,$$

where $y:[0,\infty)\to\mathbb{R}^m$ is a Hölder continuous function of order $\beta>1/2$. $f:\mathbb{R}^d\to\mathbb{R}^{md}$ and $g:\mathbb{R}^d\to\mathbb{R}^{M^2d}$ are given functions and $x_0\in\mathbb{R}^m$, $z_0\in\mathbb{R}^M$. We make the following assumptions:

H1) f is bounded with a bounded derivative f' which is locally Hölder continuous of order $\lambda > \frac{1}{\beta}-1$.
H2) g is bounded with bounded derivative.

Theorem 3. *Assume conditions H1) and H2). Then, there is a constant k depending only on β, such that for all T,*

$$\sup_{0\leq t\leq T}|z_t| \leq 2^{1+kT\left[\|f'\|_\infty\vee\left(\|g\|_\infty+\sqrt{\|g'\|_\infty\|f\|_\infty}\right)\right]^{1/\beta}\|y\|_{0,T,\beta}^{1/\beta}}|z_0|. \tag{15}$$

Proof. Without loss of generality we assume that $d=m=M=1$. Set $\|y\|_\beta=\|y\|_{0,T,\beta}$. We can assume that $\|y\|_\beta>0$, otherwise the inequality is obvious. Let $\alpha<1/2$ such that $\alpha>1-\beta$.

If we choose Δ such that

$$\Delta^{\beta} \leq \frac{1}{2k\, \|f'\|_{\infty}\, \|y\|_{\beta}},$$

by (6) for all s and t such that $t-s \leq \Delta$ we have

$$\|x\|_{s,t,\beta} \leq 2k \|y\|_{\beta}\, \|f\|_{\infty}\ . \tag{16}$$

On the other hand, using the fractional integration by parts formula we obtain for all $s,t \in [0,T]$,

$$\left|\int_s^t g(x_r) z_r dy_r\right| \leq \int_s^t \left|D_{s+}^{\alpha}\ \left(g(x_r)z_r\right)\ D_{t-}^{1-\alpha} y_{t-}(r)\right| dr. \tag{17}$$

From (1) we get

$$\left|D_{s+}^{\alpha}\left(g(x_r)z_r\right)\right| \leq k\ \left(\|g\|_{\infty}\, \|z\|_{s,r,\infty} (r-s)^{-\alpha} + \int_s^r \frac{|g(x_r)z_r - g(x_\theta)z_\theta|}{|r-\theta|^{\alpha+1}} d\theta \right).$$

Now if $0 \leq s \leq r \leq t \leq T$, then

$$\begin{aligned}
\int_s^r \frac{|g(x_r)z_r - g(x_\theta)z_\theta|}{|r-\theta|^{\alpha+1}} d\theta &\leq k\, \|g\|_{\infty} \int_s^r \|z\|_{s,r,\beta} |r-\theta|^{\beta-\alpha-1} d\theta \\
&\quad + k\, \|g'\|_{\infty} \int_s^r \|z\|_{s,r,\infty} \|x\|_{s,r,\beta} |r-\theta|^{\beta-\alpha-1} d\theta \\
&\leq k \left(\|g\|_{\infty}\, \|z\|_{s,t,\beta} + \|g'\|_{\infty}\, \|z\|_{s,r,\infty} \|x\|_{s,r,\beta} \right) |r-s|^{\beta-\alpha}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|D_{s+}^{\alpha}\left(g(x_r)z_r\right)\right| &\leq k(\|g\|_{\infty}\, \|z\|_{s,r,\infty} (r-s)^{-\alpha} \\
&+ \left(\|g\|_{\infty}\, \|z\|_{s,t,\beta} + \|g'\|_{\infty}\, \|z\|_{s,r,\infty} \|x\|_{s,r,\beta} \right) |r-s|^{\beta-\alpha}).
\end{aligned} \tag{18}$$

Substituting (18) and (4) into (17) yields

$$\begin{aligned}
\left|\int_s^t g(x_r) z_r dy_r\right| &\leq k \|y\|_{\beta} \Bigg(\|g\|_{\infty}\, \|z\|_{s,t,\infty} (t-s)^{\beta} \\
&\quad + \left(\|g\|_{\infty}\, \|z\|_{s,t,\beta} + \|g'\|_{\infty}\, \|z\|_{s,t,\infty} \|x\|_{s,t,\beta} \right) (t-s)^{2\beta} \Bigg).
\end{aligned}$$

Consequently, is $t-s \leq \Delta$, applying (16) yields

$$\begin{aligned}
\|z\|_{s,t,\beta} &\leq k \|y\|_{\beta} \Big\{ \|g\|_{\infty}\, \|z\|_{s,t,\infty} \\
&\quad + \left(\|g\|_{\infty}\, \|z\|_{s,t,\beta} + \|g'\|_{\infty}\, \|z\|_{s,t,\infty} \|x\|_{s,t,\beta} \right) \Delta^{\beta} \Big\} \\
&\leq k \|y\|_{\beta} \Big\{ \|g\|_{\infty}\, \|z\|_{s,t,\infty} \\
&\quad + \left(\|g\|_{\infty}\, \|z\|_{s,t,\beta} + \|g'\|_{\infty}\, \|f\|_{\infty}\, \|y\|_{\beta} \|z\|_{s,t,\infty} \right) \Delta^{\beta} \Big\}.
\end{aligned}$$

Supposse that Δ is sufficiently small such that

$$\Delta \leq \left(\frac{1}{2k\|g\|_\infty \|y\|_\beta} \right)^{\frac{1}{\beta}} . \tag{19}$$

Then we have

$$\|z\|_{s,t,\beta} \leq 2k\|y\|_\beta \|z\|_{s,t,\infty} \left(\|g\|_\infty + \|g'\|_\infty \|f\|_\infty \|y\|_\beta \Delta^\beta \right) .$$

This implies that

$$\|z\|_{s,t,\infty} \leq |z_s| + k\|y\|_\beta \Delta^\beta \|z\|_{s,t,\infty} \left(\|g\|_\infty + \|g'\|_\infty \|f\|_\infty \|y\|_\beta \Delta^\beta \right) .$$

If Δ satisfies

$$\|g\|_\infty \Delta^\beta + \|g'\|_\infty \|f\|_\infty \|y\|_\beta \Delta^{2\beta} \leq \frac{1}{2k\|y\|_\beta} \tag{20}$$

then we have

$$\|z\|_{s,t,\infty} \leq 2|z_s|$$

Hence,

$$\sup_{0\leq r\leq t} |z_r| \leq 2 \sup_{0\leq r\leq s} |z_r|. \tag{21}$$

As before, divide the interval $[0,T]$ into $n = [T/\Delta] + 1$ subintervals, and use the estimate (21) in every interval to obtain

$$\|z\|_{0,T,\infty} \leq 2^n |z_0| . \tag{22}$$

Notice that for (20) to hold it suffices that

$$\begin{aligned} \Delta^\beta \|y\|_\beta &\leq \frac{\sqrt{\|g\|_\infty^2 + \frac{2}{k}\|g'\|_\infty \|f\|_\infty} - \|g\|_\infty}{2\|g'\|_\infty \|f\|_\infty} \\ &= \frac{1}{k\left(\sqrt{\|g\|_\infty^2 + \frac{2}{k}\|g'\|_\infty \|f\|_\infty} + \|g\|_\infty \right)}. \end{aligned}$$

If we choose

$$\Delta = \left[k\|y\|_\beta \max\left(\|f'\|_\infty , \|g\|_\infty + \sqrt{\|g'\|_\infty \|f\|_\infty} \right) \right]^{-\frac{1}{\beta}} ,$$

then (22) yields

$$\|z\|_{0,T,\infty} \leq 2^{1+kT\left[\|f'\|_\infty \vee \left(\|g\|_\infty + \sqrt{\|g'\|_\infty \|f\|_\infty} \right) \right]^{1/\beta} \|y\|_{0,T,\beta}^{1/\beta}} |z_0| .$$

The proof is now complete.

Suppose now that we have two differential equations of the form

$$x_t = x_0 + \int_0^t f(x_s) dy_s,$$

and

$$\tilde{x}_t = \tilde{x}_0 + \int_0^t \tilde{f}(\tilde{x}_s) \tilde{y}_s\,,$$

where y and $\widetilde{y}$ are Hölder continuous functions of order $\beta > 1/2$, and f and $\widetilde{f}$ are two functions which are continuously differentiable with locally Hölder continuous derivatives of order $\lambda > \frac{1}{\beta} - 1$. Then, we have the following estimate.

Theorem 4. *Suppose in addition that f is twice continuously differentiable and f'' is bounded. Then there is a constant k such that*

$$\begin{aligned}\sup_{0\le r\le T} |x_r - \tilde{x}_r| \le k 2^{kD^{1/\beta}\|y\|_{0,T,\beta}^{1/\beta} T} & \\ \times\Bigg\{ |x_0 - \tilde{x}_0| + \|y\|_{0,T,\beta} & \left[\|f - \tilde{f}\|_\infty + \|x\|_{0,T,\beta} \|f' - \tilde{f}'\|_\infty \right] \\ & + \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|x\|_{0,T,\infty} \right] \|y - \tilde{y}\|_{0,T,\beta} \Bigg\},\end{aligned}$$

where

$$D = \|f'\|_\infty \vee \left(\|f'\|_\infty + \|f''\|_\infty (\|x\|_{0,T,\beta} + \|\tilde{x}\|_{0,T,\beta}) T^\beta \right).$$

Remark 5. The above inequality is valid only when each term appeared on the right hand side is finite.

Proof. Fix $s, t \in [0, T]$. Set

$$x_t - \tilde{x}_t - (x_s - \tilde{x}_s) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_s^t [f(x_r) - f(\tilde{x}_r)]\, dy_r \\ I_2 &= \int_s^t \left[f(\tilde{x}_r) - \tilde{f}(\tilde{x}_r) \right] dy_r \\ I_3 &= \int_s^t \tilde{f}(\widetilde{x}_r) d\, [y_r - \tilde{y}_r]\,. \end{aligned}$$

The terms I_2 and I_3 can be estimated easily. In fact, we have

$$|I_2| \le k \|y\|_\beta \left[\|f - \tilde{f}\|_\infty (t-s)^\beta + \|f' - \tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta} (t-s)^{2\beta} \right]$$

and

$$|I_3| \le k\|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty (t-s)^\beta + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}(t-s)^{2\beta}\right] ,$$

where $\|y\|_\beta = \|y\|_{0,T,\beta}$ and $\|y-\tilde{y}\|_\beta = \|y-\tilde{y}\|_{0,T,\beta}$. The term I_1 is a little more complicated.

$$\begin{aligned}
|I_1| &\le \int_s^t |D^\alpha_{s+}\left[f(x_r)-f(\tilde{x}_r)\right]||D^{1-\alpha}_{t-}y_{t-}(r)|dr \\
&\le k\int_s^t \|y\|_{s,t,\beta}(t-r)^{\alpha+\beta-1}\left[|f(x_r)-f(\tilde{x}_r)|(r-s)^{-\alpha}\right. \\
&\quad +\|f'\|_\infty \|x-\tilde{x}\|_{s,r,\beta}(r-s)^{\beta-\alpha} \\
&\quad \left.+\|f''\|_\infty \|x-\tilde{x}\|_{s,r,\infty}\left[\|x\|_{s,r,\beta}+\|\tilde{x}\|_{s,r,\beta}\right](r-s)^{\beta-\alpha}\right] dr \\
&\le k\|y\|_\beta \left\{\|f'\|_\infty \|x-\tilde{x}\|_{s,t,\infty}(t-s)^\beta + \|f'\|_\infty \|x-\tilde{x}\|_{s,t,\beta}(t-s)^{2\beta}\right. \\
&\quad \left.+\|f''\|_\infty \|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right](t-s)^{2\beta}\right\} .
\end{aligned}$$

Therefore

$$\begin{aligned}
\|x-\tilde{x}\|_{s,t,\beta} \le k\|y\|_\beta &\left\{\|f'\|_\infty \|x-\tilde{x}\|_{s,t,\infty} + \|f'\|_\infty \|x-\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right. \\
&\quad +\|f''\|_\infty \|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right](t-s)^\beta \\
&\left.+\|f-\tilde{f}\|_\infty + \|f'-\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right\} \\
&+k\|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right].
\end{aligned}$$

Rearrange it to obtain

$$\begin{aligned}
\|x-\tilde{x}\|_{s,t,\beta} \le k(1-k\|f'\|_\infty \|y\|_\beta (t-s)^\beta)^{-1} &\Bigg\{ \|y\|_\beta \Bigg[\|f'\|_\infty \|x-\tilde{x}\|_{s,t,\infty} \\
&+\|f''\|_\infty \|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right](t-s)^\beta \\
&+\|f-\tilde{f}\|_\infty + \|f'-\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}(t-s)^\beta \Bigg] \\
&+k\|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right] \Bigg\}.
\end{aligned}$$

Set $\Delta = t-s$, and $A = k\|f'\|_\infty \|y\|_\beta$. Then

$$
\begin{aligned}
\|x-\tilde{x}\|_{s,t,\infty} &\le |x_s-\tilde{x}_s| + \|x-\tilde{x}\|_{s,t,\beta}(t-s)^\beta \\
&\le |x_s-\tilde{x}_s| + k(1-A\Delta^\beta)^{-1}\Delta^\beta \Bigg\{ \|y\|_\beta \Bigg[\|f'\|_\infty \|x-\tilde{x}\|_{s,t,\infty} \\
&\quad + \|f''\|_\infty \|x-\tilde{x}\|_{s,t,\infty} \left[\|x\|_{s,t,\beta} + \|\tilde{x}\|_{s,t,\beta}\right]\Delta^\beta \\
&\quad + \|f-\tilde{f}\|_\infty + \|f'-\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \Bigg] \\
&\quad + k\|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \Bigg\}.
\end{aligned}
$$

Denote

$$
B = k\|y\|_\beta \left(\|f'\|_\infty + \|f''\|_\infty (\|x\|_{0,T,\beta} + \|\tilde{x}\|_{0,T,\beta}) T^\beta \right) .
$$

Then

$$
\begin{aligned}
\|x-\tilde{x}\|_{s,t,\infty} &\le \left(1-(1-A\Delta^\beta)^{-1}\Delta^\beta B\right)^{-1} \\
&\quad \times \Bigg\{ |x_s-\tilde{x}_s| + k(1-A\Delta^\beta)^{-1}\Delta^\beta \\
&\quad \times \Bigg[\|y\|_\beta \left[\|f-\tilde{f}\|_\infty + \|f'-\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \\
&\quad + \|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \Bigg] \Bigg\}.
\end{aligned}
$$

Let Δ satisfy

$$
A\Delta^\beta \le 1/3\,, \quad B\Delta^\beta \le 1/3
$$

Namely, we take

$$
\Delta = \left(\frac{1}{3\,(A \vee B)} \right)^{1/\beta} .
$$

Then

$$
\|x-\tilde{x}\|_{s,t,\infty} \le 2\left[|x_s-\tilde{x}_s| + C\Delta^\beta \right],
$$

where

$$
\begin{aligned}
C = \frac{3}{2}k \Bigg[& \|y\|_\beta \left[\|f-\tilde{f}\|_\infty + \|f'-\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \\
& + \|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \Bigg].
\end{aligned}
$$

Applying the above estimate recursively we obtain

$$\sup_{0\le r\le T}|x_r-\tilde{x}_r|\le 2^n\left[|x_0-\tilde{x}_0|+C\Delta^\beta\right],$$

where $n=[T/\Delta]+1$. Or we have

$$\sup_{0\le r\le T}|x_r-\tilde{x}_r|\le k2^{k(\|f'\|_\infty\vee(\|f'\|_\infty+\|f''\|_\infty(\|x\|_{0,T,\beta}+\|\tilde{x}\|_{0,T,\beta})T^\beta))^{1/\beta}\|y\|_{0,T,\beta}^{1/\beta}T}$$
$$\times\left\{|x_0-\tilde{x}_0|+\|y\|_{0,T,\beta}\left[\|f-\tilde{f}\|_\infty+\|\tilde{x}\|_{0,T,\beta}\|f'-\tilde{f}'\|_\infty\right]\right.$$
$$\left.+\left[\|\tilde{f}\|_\infty+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{0,T,\infty}\right]\|y-\tilde{y}\|_{0,T,\beta}\right\}.$$

4 Stochastic differential equations driven by a fBm

Let $B=\{B_t,t\ge 0\}$ be an m-dimensional fractional Brownian motion (fBm) with Hurst parameter $H>1/2$. That is, B is a Gaussian centered process with the covariance function $E(B_t^iB_s^j)=R_H(t,s)\delta_{ij}$, where

$$R_H(t,s)=\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

Consider the stochastic differential equation on $\mathbb{R}^d$

$$X_t=X_0+\int_0^t\sigma(X_s)dB_s, \tag{1}$$

where X_0 is a fixed d-dimensional random variable and the stochastic integral is is a path-wise Riemann-Stieltjes integral. ([1]). This equation has a unique solution (see [2] and [3]) provided σ is continuously differentiable, and σ' is bounded and Hölder continuous of order $\lambda>\frac{1}{H}-1$.

Then, using the estimate (2) in Theorem 2 we obtain the following estimate for the solution of Equation (1), if we choose $\beta\in\left(\frac{1}{2},H\right)$. Notice that $\frac{1}{\beta}<2$.

$$\sup_{0\le t\le T}|X_t|\le 2^{1+kT\left(\|\sigma'\|_\infty\vee|\sigma(0)|\right)\|B\|_{0,T,\beta}^{1/\beta}}\left(|X_0|+1\right). \tag{2}$$

If σ is bounded and $\|\sigma'\|\neq 0$ we can make use of the estimate (3) and we obtain

$$\sup_{0\le t\le T}|X_t|\le|X_0|+k\|\sigma\|_\infty\left(T^\beta\|B\|_{0,T,\beta}^{\frac{1}{\beta}}\vee T\|\sigma'\|_\infty^{\frac{1-\beta}{\beta}}\|B\|_{0,T,\beta}^{\frac{1}{\beta}}\right). \tag{3}$$

These estimates improve those obtained by Nualart and Răşcanu in [3] based on a suitable version of Gronwall's lemma. The estimates (2) and (3) allow us to establish the following integrability properties for the solution of Equation (1).

Theorem 6. *Consider the stochastic differential equation (1), and assume that* $E(|X_0|^p) < \infty$ *for all* $p \geq 2$*. If* σ' *is bounded and Hölder continuous of order* $\lambda > \frac{1}{H} - 1$*, then*

$$E\left(\sup_{0\leq t\leq T} |X_t|^p\right) < \infty \tag{4}$$

for all $p \geq 2$*. If furthermore* σ *is bounded and* $E\left(\exp(\lambda|X_0|^\gamma)\right) < \infty$ *for any* $\lambda > 0$ *and* $\gamma < 2H$*, then*

$$E\left(\exp\lambda\left(\sup_{0\leq t\leq T} |X_t|^\gamma\right)\right) < \infty \tag{5}$$

for any $\lambda > 0$ *and* $\gamma < 2H$*.*

In [4] Nualart and Saussereau have proved that the random variable X_t belongs locally to the space $\mathbb{D}^\infty$ if the function σ is infinitely differentiable and bounded together with all its partial derivatives. As a consequence, they have derived the absolute continuity of the law of X_t for any $t > 0$ assuming that the initial condition is constant and the vector space spanned by $\{(\sigma^{ij}(x_0))_{1\leq i\leq d}, 1 \leq j \leq m\}$ is $\mathbb{R}^d$.

Applying Theorem 3.2 we can show that the derivatives of X_t possess moments of all orders, and we can then derive the C^∞ property of the density. Define the matrix

$$\alpha(x) = \left(\sum_{l=1}^{m} \sigma^{il}(x)\sigma^{jl}(x)\right)_{1\leq i,j\leq d}.$$

Theorem 7. *Consider the stochastic differential equation (1), with constant initial condition* x_0*. Suppose that* $\sigma(x)$ *is bounded infinitely differentiable with bounded derivatives of all orders, and* $\alpha(x)$ *is uniformly elliptic. Then, for any* $t > 0$ *the probability law of* X_t *has an* C^∞ *density.*

Proof. Let us first show that X_t belongs to the space $\mathbb{D}^\infty$. From Equation (34) of [4] we have

$$D_r^j X_t^i = \sigma^{ij}(X_r) + \sum_{k=1}^{d} \int_0^t \sum_{l=1}^{m} \partial_k \sigma^{il}(X_u) D_r^j X_u^k dB_u^l. \tag{6}$$

As a consequence, (15) applied to the system formed by the equations (1) and (6) yields

$$\left|D_r^j X_t^i\right| \leq 2^{1+kT\left[\|\sigma\|_\infty \vee \left(\|\sigma'\|_\infty + \sqrt{\|\sigma''\|_\infty \|\sigma\|_\infty}\right)\right]^{\frac{1}{\beta}} \|B\|_{0,T,\beta}^{1/\beta}} \|\sigma\|_\infty.$$

This implies that for all $p \geq 2$

$$E\left(\left|\sum_{j=1}^{m} \int_0^t \int_0^t D_s^j X_t^i D_r^j X_t^i |r-s|^{2H-2} ds dr\right|^p\right) < \infty,$$

and the random variable X_t^i belongs to the Sobolev space $\mathbb{D}^{1,p}$ for all $p \geq 2$. In a similar way, writing down the linear equations satisfied by the iterated derivatives, one can show that X_t^i belongs to the Sobolev space $\mathbb{D}^{k,p}$ for all $p \geq 2$ and $k \geq 2$.

In order to show the nongeneracy of the density we use the notation of [4] and follow the idea of [7]. By Itô's formula we have

$$D_r^j X_t^i D_{r'}^j X_t^{i'} = \sigma^{ij}(X_r)\sigma^{i'j}(X_{r'}) + \sum_{k=1}^d \sum_{l=1}^m \int_0^t \partial_k \sigma^{i'l}(X_u) D_r^j X_u^i D_{r'}^j X_u^k dB_u^l$$
$$+ \sum_{k=1}^d \sum_{l=1}^m \int_0^t \partial_k \sigma^{il}(X_u) D_{r'}^j X_u^{i'} D_r^j X_u^k dB_u^l\,.$$

Denote

$$\beta_l(X_u) = \left(\partial_k \sigma^{i'l}(X_u)\right)_{1\leq i',k\leq d}$$
$$\Gamma_t = \left(\sum_{j=1}^m \int_0^t \int_0^t |r - r'^{2H-2} D_r^j X_t^i D_{r'}^j X_t^{i'} drdr'\right)_{1\leq i,i'\leq d}.$$

Then $H(2H-1)\Gamma_t$ is the Malliavin covariance matrix of the random vector X_t, and we need to show that Γ_t^{-1} is in L_p for any $p \geq 1$ and for all $t > 0$. We have

$$\Gamma_t = \alpha_0 + \sum_{l=0}^m \int_0^t \left(\beta_l(X_u)\Gamma_u + \Gamma_u \beta_l^T(X_u)\right) dB_u^l,$$

where

$$\alpha_0 = \sum_{j=1}^m \int_0^t \int_0^t |r - r'|^{2H-2} \sigma_{ij}(X_r)\sigma_{i'j}(X_{r'}) drdr'.$$

By using Itô formula again we have

$$\Gamma_t^{-1} = \alpha_0^{-1} - \sum_{l=0}^m \int_0^t \left(\Gamma_u^{-1}\beta_l(X_u) + \beta_l^T(X_u)\Gamma_u^{-1}\right) dB_u^l. \tag{7}$$

By the estimate (15) applied to the equations (1) and (7) , we see that Γ_t^{-1} is in L_p for any $p \geq 1$. This proves the theorem (see [5]).

References

1. Hu, Y. Integral transformations and anticipative calculus for fractional Brownian motions. Mem. Amer. Math. Soc. 175 (2005), no. 825.
2. Lyons, T. Differential equations driven by rough signals (I): An extension of an inequality of L. C. Young. *Mathematical Research Letters* **1** (1994) 451-464.

3. Nualart, D., Răşcanu, A. Differential equations driven by fractional Brownian motion. *Collect. Math.* **53** (2002) 55-81.
4. Nualart, D., Saussereau, B. Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. Preprint.
5. Nualart, D. The Malliavin calculus and related topics. Probability and its Applications (New York). Springer-Verlag, New York, 2006.
6. Samko S. G., Kilbas A. A. and Marichev O. I. *Fractional Integrals and Derivatives. Theory and Applications*. Gordon and Breach, 1993.
7. Stroock, D. Some applications of stochastic calculus to partial differential equations. Eleventh Saint Flour probability summer school—1981 (Saint Flour, 1981), 267–382, Lecture Notes in Math., 976, Springer, Berlin, 1983.
8. Young, L. C. An inequality of the Hölder type connected with Stieltjes integration. *Acta Math.* **67** (1936) 251-282.
9. Zähle, M. Integration with respect to fractal functions and stochastic calculus. I. *Prob. Theory Relat. Fields* **111** (1998) 333-374.