# The space of stochastic differential equations

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## 1 Stochastic differential equations, Hormander representations, and stochastic flows.

### 1.1 Introduction.

One of the main tools arising from Ito's calculus is the theory of stochastic differential equations, now with applications to many areas of science, economics and finance. This article is a remark on some aspects of the geometry and topology of certain spaces of stochastic differential equations, making no claims to relevance to the actual theory or its applications. It is based on work with Yves LeJan & Xue-Mei Li reported in [ELL99],[ELJL04] and in preparation in [ELJL]. It was stimulated by contacts with Steve Rosenberg and his article with Sylvie Paycha, [PR04]. However the topological constructions and remarks, in all except 2.4 (which is taken from [ELJL]), are essentially well known and any novelty arises from their interpretation in terms of stochastic differential equations and flows.

### 1.2 The spaces.

We shall consider Stratonovich equations on a compact, connected, finite dimensional manifold $M$. We shall write them as:

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{1}$$

where A is a vector field on $M$ and for each $x \in M$ we have a continuous linear map

$$X(x) : H \to T_xM$$

of a fixed, real, separable Hilbert space $H$ into the tangent space to $M$ at $x$. Our "noise" $\{B_t : t \geqslant 0\}$ is a standard Brownian motion on $H$, cylindrical if $H$ is infinite dimensional. We are only interested in the case where $X$ and $H$

are sufficiently smooth for a there to be unique solutions for a given initial point and a solution flow of diffeomorphisms, see [Elw82], [IW89], or [Kun90].

The solutions to equation (1) form a diffusion process with generator the diffusion operator $\mathcal{A}$ for

$$\mathcal{A} = 1/2 \sum_j \mathcal{L}_{X^j} \mathcal{L}_{X^j} + \mathcal{L}_A. \tag{2}$$

where $\mathcal{L}_{X^j}$ denotes Lie differentiation by the vector field $X^j$ given by

$$X^j(x) = X(x)(e_j)$$

for $e_j, j = 1, 2, ...$ any orthonormal basis of $H$. Commonly $H$ is finite dimensional, $H = \mathbb{R}^m$, say. However this can be included in the infinite dimensional case by taking $X(x)$ to vanish on some finite codimensional subspace for all $x$, and we know from [Bax84] that to obtain all stochastic flows we need to allow infinite dimensional noise.

We shall fix a diffusion generator $\mathcal{A}$ which is smooth (so has smooth coefficients in local coordinates) and *assume that the principal symbol of $\mathcal{A}$ has constant rank in $TM$*. This latter assumption is equivalent to the existence of a smooth subbundle $E$ of the tangent bundle $TM$ such that for any SDE such as equation (1) the map $X(x)$ maps $H$ onto the fibre $E_x$ of $E$ over $x$. In particular it holds when $\mathcal{A}$ is elliptic, in which case $E = TM$. Note that $X(x)$ or equivalently the symbol of $\mathcal{A}$ determines an inner product, $\langle -, - \rangle_x$ on $E_x$ for each $x \in M$ giving it a Riemannian structure. Without specifying the regularity or giving topologies at this stage let $Hor_{\mathcal{A}}$ denote the set of Hormander form representations, as equation (2) of $\mathcal{A}$, and $SDE_{\mathcal{A}}$ the space of SDE's whose solutions are $\mathcal{A}$- diffusions. Since the natural map from SDE's to Hormander forms depends only on a choice of basis, any such basis determines a bijection

$$Hor_{\mathcal{A}} \cong SDE_{\mathcal{A}}. \tag{3}$$

Moreover, since the choice of the noise coefficient $X$ in an element of $SDE_{\mathcal{A}}$ determines the vector field $A$, both spaces are naturally in one-one correspondence with the space $SDE(E)$ of vector bundle maps $X : M \times H \to E$, of the trivial $H$-bundle onto $E$, which induce the given Riemannian metric on $E$. It is this space which we shall examine in more detail below.

Closely related to these spaces is the space $Flow_{\mathcal{A}}$ of stochastic flows of diffeomorphisms of M whose one point motions are $\mathcal{A}$-diffusions. Following Baxendale, [Bax84], these can be considered as Wiener processes on the diffeomorphism group, $DiffM$ of $M$, and determine and are determined by a Hilbert space $\mathcal{H}_\gamma$ of sections of $E$ with the property that the evaluation map $ev_x : \mathcal{H}_\gamma \to E_x$ is surjective and induces the given inner product, for each $x \in M$. In turn this is determined by a suitable *reproducing kernel* $k_\gamma(x, y) : E_x^* \to E_y$, [Bax76],[ELL99], defined by

$$k_\gamma(x, -) = (ev_x)^* : E_x^* \to H_\gamma. \tag{4}$$

Let $RKH(E)$ denote the space of such Hilbert subspaces and $RK(E)$ the, isomorphic, space of their reproducing kernels. Using the inner product on $E_x$ to identify it with its dual space, the latter can be identified with the space of those sections $k^\sharp$ of the bundle of linear maps $\mathbb{L}(E;E)$ over $M \times M$ such that $k^\sharp(x,y) : E_x \to E_y$ satisfies

(i) $k^\sharp(x,y) = k^\sharp(y,x)^*$;
(ii) $k^\sharp(x,x) = identity : E_x \to E_x$;
(iii) for any finite set $x_1, ..., x_q$ of elements of $M$ we have

$$\sum_{i,j=1}^{q} \langle k^\sharp(x_i,x_j)u_i, u_j\rangle_{x_j} \geqslant 0$$

for all $\{u_j\}_{j=1}^q$ with $u_j \in E_{x_j}$.

It is easy to see that these form a convex subset of the space of all sections. It is natural to identify the space of smooth flows $Flow_{\mathcal{A}}^\infty$ in $Flow_{\mathcal{A}}$ with the space of smooth elements of $RK(E)$ with topology induced from the $C^\infty$ topology on the sections of the bundle $\mathbb{L}(E;E)$ over $M \times M$. This topology is a reasonable topology for the space of flows: for example if $K$ is a smooth compact manifold with a map $f : K \to Flow_{\mathcal{A}}^\infty$ which is smooth in the sense that it is smooth when identified with a map into every Sobolev space of sections of $\mathbb{L}(E;E)$, then there is a smooth stochastic flow on $K \times M$ which restricts to $f(k)$ on each of the leaves $\{k\} \times M, k \in M$. Convexity tells us that given any two flows in $Flow_{\mathcal{A}}^\infty$ there is a (canonical) smooth flow on $[0,1] \times M$ which restricts to a flow on each $\{k\} \times M$ in $Flow_{\mathcal{A}}^\infty$ agreeing with the given ones at $k = 0, 1$. In this sense:

- *The space of smooth stochastic flows on $M$ whose one point motions have $\mathcal{A}$ as generator, is contractible.*

There is the natural map taking an SDE to its flow. It corresponds to the map

$$\mathcal{H} : SDE(E) \to RKH(E) \tag{5}$$

given by $\mathcal{H}(X) = \{X(-)(e) : e \in H\}$ with inner product induced from $H$. When $H$ is infinite dimensional this is surjective. Note that given some $\mathcal{H}_\gamma$ in $RKH(E)$ we can obtain an SDE in $SDE(E)$ which maps to $\mathcal{H}_\gamma$ by choosing a linear map $U : \mathcal{H}_\gamma \to H$ which is an isometry into $H$ and defining $X(x)e = U^*(e)(x)$, for $e \in H, x \in M$. Thus $\mathcal{H}^{-1}(\mathcal{H}_\gamma)$ is not connected in general. See also the end of section 2.2 below.

Our main interest is in $C^\infty$ equations and flows. To do differential calculus on the various manifolds of $C^\infty$ mappings which will arise it would be natural to use the Froelicher-Kriegl calculus, see [KM97], as Michor in [Mic91]. However Banach manifolds are more familiar and we will generally consider manifolds of Sobolev spaces of mappings of sufficiently high differentiability class. Taking $s$ very large compared to the dimension of $M$ let $SDE(E)^s$,

$RK(E)^s$, etc., denote the relevant subsets of Sobolev spaces of mappings of class $H^s$, (i.e. those whose weak derivatives of order $s$ lie in $L^2$, see [Pal68]). In particular let $\mathcal{D}^s$ denote the Hilbert manifold of all diffeomorphisms of class $H^s$ . By standard approximation techniques the homotopy class of these spaces does not depend on $s$ given that $s$ is large enough.

## 2 Induced connections and the action of the gauge group.

### 2.1 The gauge group and its universal bundle.

For $\mathcal{A}$ and $E$ as above let $q$ be the fibre dimension of $E$. Suppose that $H$ is infinite dimensional. Consider the Grassmanian $G(q, H)$ of all $q$-dimensional linear subspaces of $H$, the space $V(q, H)$ of all $q$-frames in $H$, and the natural projection $p : V(q, H) \to G(q, H)$. Identify $\mathbb{R}^q$ with a subspace of $H$ and let $H^{\infty-q}$ be its orthogonal complement. Let $O(H)$, $O(\infty - q)$, and $O(q)$ be the orthogonal groups of $H$, $H^{\infty-q}$, and $\mathbb{R}^q$, respectively. Then $V(q, H)$, which is naturally the space of all isometries of $\mathbb{R}^q$ into $H$, can be identified with the homogeneous space $O(H)/O(\infty-q)$ with the natural right action of $O(q)$ making $p$ a smooth, even real analytic, principal $O(q)$-bundle, [KM97]. Here we can furnish $G(q, H)$ with the manifold structure it inherits as a homogeneous space or, equivalently, as a manifold modelled on the Hilbert space of continuous linear maps $\mathbb{L}(\mathbb{R}^q; H^{\infty-q})$. Thus both $G(q, H)$ and the total space $V(q, H)$ are modelled on Hilbert spaces.

By Kuiper's theorem $O(H)$ and $O(\infty-q)$ are contractible, and so therefore is $V(q, H)$, making $p$ a universal $O(q)$-bundle, as is frequently used. This means that if $p' : B \to M$ is any smooth principal $O(q)$-bundle over $M$ there is a smooth map $\chi : M \to G(q, H)$ classifying $p'$ in the sense that the the pull back by $\chi$ of $V(q, H)$ is equivalent to $B$; in other words there is a diagram of smooth maps:

$$\begin{tikzcd} B \arrow[r, "\bar{\chi}"] \arrow[d, "p'"'] & V(q,H) \arrow[d, "p"] \\ M \arrow[r, "\chi"'] & G(q,H) \end{tikzcd}$$

where $\bar{\chi}$ is a diffeomorphism on the fibres and is equivariant with respect to the right actions of $O(q)$. Such a lift $\bar{\chi}$ exists over any smooth map homotopic to $\chi$, e.g see [Ste51]. It is not uniquely determined by $\chi$; the space of all such lifts is $\{\chi \circ \alpha : \alpha \in \mathcal{G}\}$ where $\mathcal{G}$ is the *gauge group* of $B$, i.e. the group of all smooth $O(q)$ equivariant diffeomorphisms $\alpha : B \to B$ over the identity map of $M$. Following Atiyah &Bott, [AB83], let $\mathbb{H}^s_B(M; G(q, H))$ be the space of $H^s$ maps classifying $B$ and $\mathbb{H}^s_{O(q)}(B; V(q, H))$ the space of equivariant maps of $B$ into $V(q, H)$ of class $H^s$. There is the natural projection

$$p^{\mathcal{G}} : \mathbb{H}^s_{O(q)}(B; V(q;H)) \to \mathbb{H}^s_B(M; G(q,H))$$

say, which coincides with the quotient map by the right action of $\mathcal{G}^s$, the $H^s$ version of $\mathcal{G}$. Note that $\mathbb{H}^s_B(M; G(q,H))$ is a smooth manifold with Hilbert model since it is a connected component of the space of all $H^s$ maps of $M$ into $G(q,H)$; that $\mathcal{G}^s$ is, and is a Lie group, is shown in [MV81]; while $\mathbb{H}^s_{O(q)}(B; V(q,H))$ is the fixed point set of the natural action of the compact group $O(q)$ on the Hilbert manifold $\mathbb{H}^s(B; V(q,H))$, and so a smooth submanifold of $\mathbb{H}^s(B; V(q,H))$ by [Pal79].

Atiyah & Bott observe that $\mathbb{H}^s_{O(q)}(B; V(q;H))$ is contractible and hence $p^{\mathcal{G}}$ is a universal $\mathcal{G}^s$-bundle, so that $\mathbb{H}^s_B(M; G(q,H))$ is a classifying space for $\mathcal{G}^s$- bundles. To see this contractibility it suffices, by a theorem of J.H.C.Whitehead, to prove that any two continuous maps $f_j, j = 1, 2$ of a finite dimensional complex $K$, say, into $\mathbb{H}^s_{O(q)}(B; V(q,H))$ are homotopic. However such maps determine a bundle map to $V(q,H)$ of the restriction to $\{0,1\} \times K \times M$ of the $O(q)$-bundle $\mathbf{I} \times \mathbf{I} \times B$ over $[0,1] \times K \times M$. By the universal property of $p : V(q,H) \to G(q,H)$ this extends over the whole bundle projecting down to give the required homotopy, c.f. the proof of Theorem 19.3 in [Ste51]. There is also a proof in [Hus94].

### 2.2 Stochastic differential equations, their filtrations, and gauge equivalence.

Now take $B$ to be the orthonormal frame bundle, $O(E)$, of our subbundle $E$ of $TM$. Note that an element $X$ in $SDE(E)^s$ is equally determined by the $H$-valued one-form $Y$ on $E$ given by the its adjoint map: $Y_x = X(x)^* : E_x \to H$. From this we obtain the diagram, [ELL99],

$$\begin{tikzcd} O(E) \arrow[r, "\Phi"] \arrow[d, "p'"'] & V(q,H) \arrow[d, "p"] \\ M \arrow[r, "\Phi_0"'] & G(q,H) \end{tikzcd}$$

defined by: $\Phi_0(x) = Image Y_x, x \in M$ and $\Phi(u) = (Y_x u(e_1), ..., Y_x u(e_q))$, for $u \in O(E)$ where $e_1, ..., e_q$ is an orthonormal base for $\mathbb{R}^q$. In particular we obtain $\Phi$ belonging to $\mathbb{H}^s_{O(q)}(B; V(q,H))$ and so a smooth map $\kappa^s : SDE(E)^s \to \mathbb{H}^s_{O(q)}(B; V(q,H))$. Elements of $\mathcal{G}$ can be considered as automorphisms of the Riemannian bundle $E$ and so act on the right on $SDE(E)^s$ by $(X, \alpha) \mapsto \alpha^{-1} \circ X(\cdot)$ so that multiplication by $\alpha$ maps $Y$ to $Y \circ \alpha$. This action is free and we see that $\kappa^s$ is an equivariant diffeomorphism which descends to give an isomorphism of $\mathcal{G}^s$-bundles:

$$\begin{tikzcd} SDE(E)^s \arrow[r, "\kappa^s"] \arrow[d, "proj."'] & \mathbb{H}^s_{O(q)}(B; V(q,H)) \arrow[d, "p"] \\ SDE(E)^s/\mathcal{G}^s \arrow[r, "\kappa_0"'] & \mathbb{H}^s_{O(E)}(M; G(q,H)) \end{tikzcd}$$

We will say that two stochastic differential equations determined by $X$ and $X'$ in $SDE(E)^s$ are *gauge equivalent* if they are in the same orbit of $\mathcal{G}^s$ i.e. if there exists some $\alpha : E \to E$ in $\mathcal{G}^s$ such that $X'(x) = \alpha X(x)$ for all $x$ in $M$ .

This leads to one of our main observations:

- *Let $\mathcal{A}$ be a smooth diffusion generator on a compact manifold $M$ whose symbol has constant rank. Then for all sufficiently large $s$ the space of stochastic differential equations, $SDE^s_{\mathcal{A}}$, whose solutions are $\mathcal{A}$-diffusions is contractible. Moreover the natural right action of the group of $H^s$-automorphisms, $\mathcal{G}^s$, on $SDE(E)^s$ makes the latter into the total space of a universal bundle for $\mathcal{G}^s$. In particular the space of equivalence classes of elements in $SDE(E)^s$ under gauge equivalence has a natural topology which makes it a classifying space for $\mathcal{G}^s$-bundles. The corresponding results hold for smooth stochastic differential equations.*

In fact each gauge equivalence class corresponds to a map from $M$ to $G(q,H)$, namely that given by the map $\Phi_0$ above. Intuitively it tells us which part of the cylindrical noise is acting infinitesimally at a given point of $M$. It may be illuminating to consider the following, rather artificial, problem: suppose we are given a smooth map $\Theta$ of the product $K \times M$ of $M$ with a compact connected manifold $K$, into the Grassmanian $G(q,H)$, and wish to construct a smooth family of stochastic differential equations in $SDE^s_{\mathcal{A}}$ parametrised by $K$ so that at each point $(k,x)$ the SDE is driven by the noise in the subspace $\Theta(k,x)$; what conditions on $\Theta$ are needed? From above we know that for each $k$ in $K$ we must have $x \mapsto \Theta(k,x)$ in the correct homotopy class of maps, $\mathbb{H}^s_{O(E)}(M; G(q,H))$, to classify $E$. To get a family of SDE's continuous in $K$ we also need the resulting map $\theta : K \to \mathbb{H}^s_O(E)(M; G(q,H))$ to lift to a continuous map of $K$ into $\mathbb{H}^s_{O(q)}(O(E); V(q,H))$. This holds if and only if $\theta$ is homotopic to a constant. The fibre over a point $k \in K$ of the pull back by $\theta$ of $\mathbb{H}^s_{O(q)}(O(E); V(q,H))$ can be identified with the space of all those stochastic differential equations which use the noise in the subspaces determined by $\Theta(k,-)$, and a section of the pull back bundle will give us the required family.

For another equivalence relation with more standard probabilistic significance it will be convenient to fix a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ on which our cylindrical noise $B$ is defined. For each equation in $SDE^s_{\mathcal{A}}$ we obtain the (completed) filtration, $\mathcal{F}^X_t : 0 \leqslant t < \infty$, say, determined by its solution flow, where $X$ is the corresponding element in $SDE(E)^s$. Clearly gauge equivalent equations give the same filtration. On the other hand the filtration is the same as the filtration of the, possibly cylindrical, Brownian motion of $\mathcal{H}(X)$, and so using the martingale representation theorem for cylindrical Brownian motions, as in [AH04], we see

- *Two stochastic differential equations $X$ and $X'$ in $SDE^s_{\mathcal{A}}$ give the same filtrations if and only if the kernels of their induced maps $H \to \mathcal{H}(X)$ and $H \to \mathcal{H}(X')$ are the same.*

Thus the space of all possible such filtrations can be identified with the set $\{(q, F) : q \in \mathbb{Z} \cup \{\infty\}, q \geqslant r(E) \& F \text{is a } q\text{-dimensional subspace of } H\ \}$, where $r(E)$ is the minimal fibre dimension of a trivial bundle over $M$ which contains a copy of $E$. In other words it can be identified with $\bigcup_{\infty \geqslant q \geqslant r(E)} G(q, H)$, the space of all closed linear subspaces of $H$. If we give this space the topology corresponding to strong convergence of the corresponding orthogonal projections, the Wijsman topology, [Tsi], it will agree with the usual topology on the finite dimensional Grassmanians. Also, any such filtration is immersed in that of our underlying cylindrical Brownian motion $\{B_t : t \geqslant 0\}$ in the sense of Tsirelson, [Tsi], and so is determined by the $\sigma$-algebra $\mathcal{F}^X_\infty$. This shows that this description fits in with the much more general discussion of filtrations in [Tsi].

From this we can also return to equation (5) and observe that a stochastic differential equation in $SDE_{\mathcal{A}}$ is determined, up to a right action of $O(q) \times I_{\infty - q}$, by its flow and its filtration, where the filtration is determined by $(q, F)$ and the group is considered as the subgroup of $O(H)$ which acts as the identity on the orthogonal complement of $F$.

### 2.3 The connection induced on *E*.

Narasimhan &Ramanan showed in [NR61] that there is a "universal connection", $\varpi$, say, on any universal $O(q)$-bundle and given a metric connection on $E$, or equivalently any connection $\varpi_E$ on $O(E)$, there is a classifying map which pulls $\varpi$ back to $\varpi_E$. In fact they show this holds for the finite dimensional Stiefel bundles, where $H$ is replaced by a sufficiently high dimensional Euclidean space. The universal connection in this situation is described in an Appendix in [ELL99]. In particular we can use any $X$ in $SDE(E)^s$ to obtain a connection $(\kappa(X))^*(\varpi)$ on $O(E)$ and any metric connection on $E$ is obtained that way. The covariant derivative operator $\breve{\nabla}$ on sections of $E$ corresponding to $(\kappa(X))^*(\varpi)$ has the very simple expression

$$\breve{\nabla}_v(U) = X(x)d[y \mapsto Y_y(U(y))](v) \tag{6}$$

and in [ELL99] this connection was called the *LeJan-Watanabe connection* of the flow since a special case had been noted in the context of stochastic flows in [LW84], see also [AMV96]. A direct proof that all metric connections on $E$ can be obtained by a suitable $X$ with $H$ finite dimensional is in [Qui88].

The right action of $\mathcal{G}$ on $RK(E)$ given by $(k, \alpha) \mapsto k^\alpha$ with $(k^\alpha)^\sharp(x, y) = \alpha(y)^{-1} k^\sharp(x, y)\alpha(y)$ determines a right action of $\mathcal{G}$ on the space of smooth flows $Flow_{\mathcal{A}}^{\infty}$, though it seems far from clear if it has any significance for the behaviour of the flows. The map from SDE to flows is equivariant with respect to this action since the reproducing kernel $k^X$ say of $\mathcal{H}(X)$ is given by $(k^X)^\sharp(x,y) = X(y)Y_x$. We see we have a factorisation by equivariant maps:

$$SDE_{\mathcal{A}} \to Flow_{\mathcal{A}}^{\infty} \to \mathcal{C}_E$$

of the map $X \mapsto (\kappa(X))^*(\varpi)$ into the space of smooth metric connections $\mathcal{C}_E$ on $E$. (Note that by its contractibility, observed in 1.2 we can also consider the quotient of $Flow_{\mathcal{A}}^{\infty}$ by the action of $\mathcal{G}$ as a classifying space for $\mathcal{G}$). Each part of this factorisation is surjective. In the final section, next, we lift results from [ELJL], see also [ELJL04], which give information about the fibres of the second map.

### 2.4 The induced semi-connection on the diffeomorphism bundle.

Fix some point $x_0$ of $M$ and let $\pi : \mathcal{D}^s \to M$ be the evaluation map $\pi(\theta) = \theta(x_0)$. We shall think of this as a principal bundle with group $\mathcal{D}^s_{x_0}$, those $H^s$-diffeomorphisms which fix $x_0$, acting on the right by composition. Since the action is not smooth we need to be careful; alternatively we can consider smooth diffeomorphisms using the approach in [KM97] and [Mic91].

Consider a smooth stochastic flow with corresponding element $k \in RK(E)$. From it we obtain a smooth *horizontal lift map*:

$$\Xi_\theta : E_{\pi(\theta)} \to T_\theta \mathcal{D}^s$$

given by

$$\Xi_\theta(u)(y) = k^\sharp(\theta(x_0), \theta(y))(u) \in E_{\theta(y)}$$

for $u \in E_{\theta(x_0)}, y \in M, \theta \in \mathcal{D}^s$, where we identify the tangent space $T_\theta\mathcal{D}^s$ at $\theta$ to the diffeomorphism group with the space of $H^s$-maps of $M$ into $TM$ which lie over $\theta$. This is invariant under the action of $\mathcal{D}^s_{x_0}$ on $\mathcal{D}^s$. We call such an object a *semi-connection on* $\mathcal{D}^s$ *over* $E$ and let $SC_E(\mathcal{D}^s)$ denote the set of all of these objects. They are also called " partial connections" or "connections over E", see [Gro96]. In the elliptic case, $E = TM$, they are the usual connections. They give a procedure for obtaining horizontal lifts $\tilde{\sigma} : [0,T] \to \mathcal{D}^s$ of those smooth curves $\sigma : [0,T] \to M$ with the property that $\dot{\sigma}(t) \in E_{\sigma(t)}$ for all $t$. For the semi-connection determined by our kernel $k$ this lift, starting from a given diffeomorphism $\theta$ with $\theta(x_0) = \sigma(0)$, is the composition $\widetilde{\sigma}(t) = \Psi(t) \circ \theta$ where $\Psi$ is the flow of the time dependent dynamical system on $M$,

$$\dot{z}(t) = k^\sharp(\sigma(t), z(t))\dot{\sigma}.$$

Our diffeomorphism bundle can be considered as a universal natural bundle on $M$, and each element of $SC_E(\mathcal{D}^s)$ determines a semi-connection over $E$ on each natural bundle over $M$, (see [KMS93]). In particular it gives an element of $SC_E(GL(M))$ the space of semi-connections on the full linear frame bundle of $E$: for this the lift of our curve $\sigma$ to $GL(M)$ starting at a frame $u$ is just $T\widetilde{\sigma} \circ u$, the composition of the derivative of our lift $\widetilde{\sigma}$ with the frame. This determines a partial covariant derivative operator $\nabla'$, say, which allows us to differentiate arbitrary smooth vector fields but only in $E$-directions, i.e a "semi-connection over $E$ on $TM$" as defined in [ELL99]. There is a map between connections on $E$ itself and such semi-connections : to $\nabla$ the covariant derivative of a connection on E there corresponds the semi-connection with covariant derivative $\nabla'$ given by

$$\nabla'_u(V) = \nabla_v(U) - [V,U](x)$$

for $U$ a smooth section of $E$, $V$ a smooth vector field, $x \in M$ and $U(x) = u, V(x) = v$. Following Driver for the case $E = TM$, we say the semi-connection and connection are "adjoints", [ELL99]. From [ELJL04] we have:

- *The semi-connection on $GL(M)$ induced by a stochastic flow is the adjoint of the metric connection on $E$ determined by the flow.*

In [ELJL] there is the following:

- *The map described from smooth flows with $\mathcal{A}$ as generator of their one point motions to smooth semi-connections over $E$ on the diffeomorphism bundle is injective.*

From this the induced semi-connection must contain all information about the flow. We can rephrase some of these statements to :

- *The adjoint semi-connection of a metric connection $\varpi_E$ on $E$ has many "prolongations" to a semi-connection on the diffeomorphism bundle and so to a coherent system of semi-connections on all natural bundles over $M$. Some of these are induced by a stochastic flow, (necessarily unique), and from then by the choice of a classifying map for the bundle $E$ into the infinite dimensional Grassmanian. The latter will pull back the universal connection to the given connection $\varpi_E$.*

We can summarise some of these observations in the following diagram:

$$\begin{tikzcd}
C^\infty_{O(q)}(O(E);V(q,H)) \arrow[r, "(\kappa^\infty)^{-1}"] \arrow[d, "NR"'] & SDE(E) \arrow[r, "\mathcal{H}"] & Flow_{\mathcal{A}} \cong RK(E) \arrow[d, "\Xi"] \\
\mathbf{C}_E \arrow[r, "adjoint"] & SC_E(TM) & SC_E(DiffM) \arrow[l]
\end{tikzcd}$$

The maps on the top row are $\mathcal{G}$-equivariant and surjective, with $(\kappa^\infty)^{-1}$ bijective; the map $NR$ refers to the pull-back of Narasimhan & Ramanan's universal connection and so is surjective and $\mathcal{G}$-equivariant.

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