An application of probability to nonlinear analysis

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Summary. The Martin boundary theory allows to describe all positive solutions of a linear elliptic equation in an arbitrary domain E of a Euclidean space $\mathbb{R}^d$. Our goal is to describe all positive solutions of a semilinear equation $Lu = \psi(u)$. As a result of efforts of probabilists and analysts since early 1990s, now we have a solution of this problem for the equation $\Delta u = u^\alpha$ with $1 < \alpha \leq 2$ in a bounded smooth domain E. The present article contains an exposition of the theory developed to obtain this solution. [1] The central role is played by the boundary trace theory. A survey of this theory is given in Part One. In Part Two we outline the principal steps needed to construct an arbitrary positive solution starting from its trace.

Our main probabilistic tool is (L, ψ)-superdiffusions.

Part One. Trace theory

The trace theory is applicable to a general equation

$$Lu = \psi(u) \quad \text{in } E \tag{I.1}$$

where L is a second order elliptic operator, E is an arbitrary domain in $\mathbb{R}^d$ and ψ is a continuously differentiable convex function on $[0, \infty)$ subject to the conditions

(i) $\psi(u) > 0$ for $u > 0, \psi(0) = 0$.

(ii) There is a constant a such that $\psi(2u) \leq a\psi(u)$ for all u.

(iii) $\int_N^\infty ds \, \left[\int_0^s \psi(u)\, du\right]^{-1/2} < \infty$ for some $N > 0$.

Under these conditions the class $\mathcal{U}$ of all positive solutions of (I.1) is closed under the pointwise convergence.

The trace of a solution u is a pair (Γ, ν) where Γ is a Borel subset of ∂E and ν is a σ-finite measure on $\partial E \setminus \Gamma$. [For a smooth domain E, ∂E is the geometrical boundary of E; in general, this is the Martin boundary.]

A rough version of the trace used in earlier work of Le Gall, Dynkin–Kuznetsov and Marcus–Véron is adequate for small dimensions d: in this

[1] Complete proofs can be found in the books [1] and [2].

case, a solution is uniquely defined by its rough trace. However an example due to Le Gall shows that, in general, infinite many solutions can have the same rough trace. In 1998 Dynkin and Kuznetsov introduced a concept of the fine trace. The solutions in Le Gall's example have distinct fine traces. In [3] all values of the fine trace were described and a 1-1 correspondence was established between them and a class of solutions which we call σ-moderate. [2] Proofs of these results are presented in Chapter 11 of [1]. In the Epilogue to [1], a crucial outstanding question was formulated:

Are all the solutions σ-moderate?

In the case of the equation $\Delta u = u^2$ in a domain of class C^4, a positive answer to this question was given in the thesis of Mselati [6] - a student of J.-F. Le Gall. [3] However his principal tool - the Brownian snake - is not applicable to more general equations. In a series of publications by Dynkin and Kuznetsov, Mselati's result was extended, by using a superdiffusion instead of the snake, to the equation $\Delta u = u^\alpha$ with $1 < \alpha \leq 2$. A systematic presentation of the proofs is contained in the book [2]. In Section 1 we give the definition of the fine trace and formulate its fundamental properties. In Section 2 we explain how these properties can be established by using probabilistic tools: superdiffusions and their relation to conditional diffusions.

Since we consider only the fine trace, we drop the word fine.

1 Definition and properies of trace

1.1 Moderate and σ-moderate solutions

We denote by $\mathcal{U}$ the set of all positive solutions of the equation (I.1) and by $\mathcal{H}$ the set of all positive solutions of the equation

$$Lh = 0 \quad \text{in } E. \tag{1}$$

We call solutions of (1) *harmonic functions*.

If E is smooth [4] and if $k(x, y)$ is the *Poisson kernel* [5] of L in E, then the formula

$$h_\nu(x) = \int_{\partial E} k(x, y)\nu(dy) \tag{2}$$

establishes a 1-1 correspondence between the set $\mathcal{M}(\partial E)$ [6] and the set $\mathcal{H}$.

[2] The definition of this class is given in Section 1.
[3] The dissertation of Mselati was published in 2004 (see [7]).
[4] We use the name smooth for open sets of class $C^{2,\lambda}$ unless another class is indicated explicitely.
[5] For an arbitrary domain, $k(x, y)$ should be replaced by the Martin kernel and ∂E should be replaced by a certain Borel subset E' of the Martin boundary (see Chapter 7 in [1]).
[6] We denote by $\mathcal{M}(S)$ the set of all finite measures on S.

A solution u is called *moderate* if it is dominated by a harmonic function. There exists a 1-1 correspondence between the set $\mathcal{U}_1$ of all moderate solutions and a subset $\mathcal{H}_1$ of $\mathcal{H}$: $h \in \mathcal{H}_1$ is the minimal harmonic function dominating $u \in \mathcal{U}_1$, and u is the maximal solution dominated by h. We put $\nu \in \mathcal{N}_1$ if $h_\nu \in \mathcal{H}_1$. We denote by u_ν the element of $\mathcal{U}_1$ corresponding to h_ν.

An element u of $\mathcal{U}$ is called *σ-moderate solutions* if there exist $u_n \in \mathcal{U}_1$ such that $u_n(x) \uparrow u(x)$ for all x. The labeling of moderate solutions by measures $\nu \in \mathcal{N}_1$ can be extended to σ-moderate solutions by the convention: if $\nu_n \in \mathcal{N}_1$, $\nu_n \uparrow \nu$ and if $u_{\nu_n} \uparrow u$, then put $\nu \in \mathcal{N}_0$ and $u = u_\nu$.

1.2 Lattice structure in $\mathcal{U}$

We write $u \le v$ if $u(x) \le v(x)$ for all $x \in E$. This determines a partial order in $\mathcal{U}$. For every $\tilde{\mathcal{U}} \subset \mathcal{U}$, there exists a unique element u of $\mathcal{U}$ with the properties: (a) $u \ge v$ for every $v \in \tilde{\mathcal{U}}$; (b) if $\tilde{u} \in \mathcal{U}$ satisfies (a), then $u \le \tilde{u}$. [7] We denote this element Sup $\tilde{\mathcal{U}}$.

For every $u, v \in \mathcal{U}$, we put $u \vee v = \mathrm{Sup}\{u, v\}$ and we put $u \oplus v = \mathrm{Sup} W$ where W is the set of all $w \in \mathcal{U}$ such that $w \le u + v$. Note that $u \oplus v$ and $u \vee v$ are moderate if u and v are moderate and they are σ-moderate if so are u and v.

In general, Sup $\tilde{\mathcal{U}}$ does not coincide with the pointwise supremum (the latter does not belong to $\mathcal{U}$). However, both are equal if $u \vee v \in \tilde{\mathcal{U}}$ for every $u, v \in \tilde{\mathcal{U}}$. Moreover, in this case there exist $u_n \in \tilde{\mathcal{U}}$ such that $u_n(x) \uparrow u(x) =$ Sup $\tilde{\mathcal{U}}$ for all $x \in E$. Therefore, if $\tilde{\mathcal{U}}$ is closed under $\vee$ and if it consists of moderate solutions, then Sup $\tilde{\mathcal{U}}$ is σ-moderate. Since $u \vee v$ is moderate for all moderate u and v, to every Borel subset Γ of ∂E there corresponds a σ-moderate solution

$$u_\Gamma = \mathrm{Sup}\{u_\nu : \nu \in \mathcal{N}_1, \nu \text{ is concentrated on } \Gamma\}. \tag{3}$$

We also associate with Γ another solution w_Γ. First, we define w_K for closed K by the formula

$$w_K = \mathrm{Sup}\{u \in \mathcal{U} : u = 0 \quad \text{on } \partial E \setminus K\}. \tag{4}$$

For every Borel subset Γ of ∂E, we put

$$w_\Gamma = \mathrm{Sup}\{w_K : \text{closed } K \subset \Gamma\}. \tag{5}$$

Proving that $u_\Gamma = w_\Gamma$ was a key part of the program outlined in [1]. A sketch of the proof will be presented in Section 4.

[7] The existence is proved in Section 8, Chapter 5 in [1].

1.3 Singular points of a solution u

We consider classical solutions of (I.1) which are twice continuously differentiable in E. However they can tend to infinity as $x \to y \in \partial E$. We say that y is a *singular point of* u if it is a point of rapid growth of $\psi'(u)$. [A special role of $\psi'(u)$ is due to the fact that the tangent space to $\mathcal{U}$ at point u is described by the equation $Lv = \psi'(u)v$.] An analytic definition of rapid growth involves the Poisson kernel (or Martin kernel) $k_\ell(x, y)$ of the operator $Lu - \ell u$. Namely, $y \in \partial E$ is a point of rapid growth for a positive continuous function ℓ if $k_\ell(x, y) = 0$ for all $x \in E$.

A transparent probabilistic definition of singular points is given in Section 2.5.

We say that a Borel subset Γ of ∂E is *f-closed* if Γ contains all singular points of the solution u_Γ defined by (3).

1.4 Definition and properties of trace

The trace of $u \in \mathcal{U}$ (which we denote $\mathrm{Tr}(u)$) is defined as a pair (Γ, ν) where Γ is the set of all singular points of u and ν is a measure on $\partial E \setminus \Gamma$ given by the formula

$$\nu(B) = \sup\{\mu(B) : \mu \in \mathcal{N}_1, \mu(\Gamma) = 0, u_\mu \leq u\}. \tag{6}$$

We have

$$u_\nu = \mathrm{Sup}\{ \text{ moderate } u_\mu \leq u \text{ with } \mu(\Gamma) = 0\}$$

and therefore u_ν is σ-moderate.

The trace of every solution u has the following properties: [8]

Assumption 1.1

1.1.A *Γ is a Borel f-closed set;* [9] *ν is a σ-finite measure of class $\mathcal{N}_0$ such that $\nu(\Gamma) = 0$ and all singular points of u_ν belong to Γ.*

1.1.B *If $\mathrm{Tr}(u) = (\Gamma, \nu)$, then*

$$u \geq u_\Gamma \oplus u_\nu. \tag{7}$$

Moreover, $u_\Gamma \oplus u_\nu$ is the maximal σ-moderate solution dominated by u.

1.1.C *Suppose that (Γ, ν) is an arbitrary pair subject to the condition 1.1. If $\mathrm{Tr}(u_\Gamma \oplus u_\nu) = (\Gamma', \nu)$, then the symmetric difference between Γ and Γ' is not charged by any measure $\mu \in \mathcal{N}_1$. Moreover, $u_\Gamma \oplus u_\nu$ is the minimal solution with this property and the only one which is σ-moderate.*

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[8] See Theorems 7.1-7.2 in Chapter 11 of [1].

[9] This part will be also proved in Section 2.5 below.

2 Diffusions and superdiffusions

2.1 *L*-diffusion and its transformations

A diffusion describes a random motion of a particle. An example is the Brownian motion in $\mathbb{R}^d$. This is a Markov process with continuous paths and with the transition density

$$p_t(x,y) = (2\pi t)^{-d/2} e^{-|x-y|^2/2t}$$

which is the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u.$$

A Brownian motion in a domain E can be obtained by killing the path at the first exit time from E. By replacing $\frac{1}{2}\Delta$ by an elliptic operator L, we define a Markov process (ξ_t, Π_x) called *L-diffusion*.

Suppose that (ξ_t, Π_x) is an L-diffusion in E with the transition density $p_t(x,y)$. To every $h \in \mathcal{H}$ there corresponds a finite measure Π_x^h such that, for all $0 < t_1 < \dots < t_n$ and every Borel subsets $B_1, \dots, B_n$ of E,

$$\begin{aligned} &\Pi_x^h\{\xi_{t_1} \in B_1, \dots, \xi_{t_n} \in B_n\} \\ &= \int_{B_1} dz_1 \dots \int_{B_n} dz_n \; p_{t_1}(x, z_1) p_{t_2 - t_1}(z_1, z_2) \dots p_{t_n - t_{n-1}}(z_{n-1}, z_n) h(z_n). \end{aligned} \tag{8}$$

Note that $\Pi_x^h(\Omega) = h(x)$ and therefore $\hat{\Pi}_x^h = \Pi_x^h / h(x)$ is a probability measure. $(\xi_t, \hat{\Pi}_x^h)$ is a Markov process with continuous paths and with the transition density

$$p_t^h(x,y) = \frac{1}{h(x)} p_t(x,y) h(y).$$

For every $y \in \partial E$, we put $\Pi_x^y = \Pi_x^h$ with $h(x) = k(x,y)$. The process $(\xi_t, \hat{\Pi}_x^y)$ can be interpreted as an L-diffusion conditioned to exit from E at point y:

$$\hat{\Pi}_x^y\{C\} = \Pi_x\{C | \xi_{\tau_E} = y\}$$

where τ_E is the first exit time of ξ_t from E.

2.2 (L,ψ)-superdiffusion

An (L,ψ)-superdiffusion is a model of random evolution of a cloud of particles. Each particle performs an L-diffusion. It dies at a random time leaving a random offspring of size controlled by the function ψ. All children move independently of each other (and of the family history) with the same transition and procreation mechanism as the parent.

Superdiffusions appeared, first, (under the name“continuous state branching processes”) in a pioneering paper of S. Watanabe [8]. Important contributions to the theory of these processes were made by Dawson and Perkins.

We consider a superdiffusion as a family of the exit measures (X_D, P_μ) from open sets $D \subset E$. An intuitive picture of (X_D, P_μ) is explained on Figure 1 (borrowed from [1]).

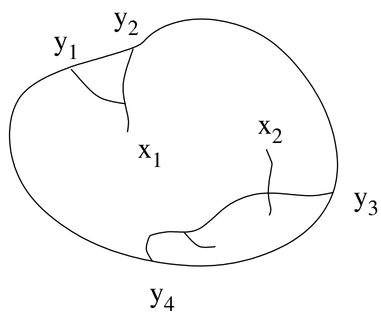

Fig. 1.

Here we have a scheme of a process started by two particles located at points x_1, x_2 in D. The first particle produces at its death time two children that survive until they reach ∂D at points y_1, y_2. The second particle has three children. One reaches the boundary at point y_3, the second one dies childless and the third one has two children. Only one of them hits ∂D at point y_4. The initial and exit measure are described by the formulae

$$\mu = \sum \delta_{x_i}, \quad X_D = \sum \delta_{y_i}.$$

To get an (L, ψ)-superdiffusion, we pass to the limit as the mass of each particle and its expected life time tend to 0 and an initial number of particles tends to infinity. We refer for details to [1].

2.3 Superdiffusions as a special class of branching exit Markov systems

The concept of a *branching exit Markov [BEM] system* (in a more general setting) is introduced in [1], Chapter 3. Suppose that to every $D \subset E$ and to every $\mu \in \mathcal{M}(E)$ there corresponds a random measure (X_D, P_μ). We say that this family is a BEM system if $X_D \in \mathcal{M}(E)$ for all D and if:

Assumption 2.1

2.1.A [Continuous branching property] For all positive Borel functions $f_1, \dots,$ f_n, all subdomains $D_1, \dots, D_n$ of E and every $\mu \in \mathcal{M}(E)$,

$$\log P_\mu e^{-Z} = \int \log P_y e^{-Z} \mu(dy) \tag{9}$$

where

$$Z = \sum_1^n \langle f_i, X_{D_i} \rangle \tag{10}$$

and $P_y = P_{\delta_y}$.

2.1.B [Markov property.] The σ*-algebra* $\mathcal{F}_{\subset D}$ *generated by* $X_{D'}, D' \subset D$ *and the* σ*-algebra* $\mathcal{F}_{\supset D}$ *generated by* $X_{D''}, D'' \supset D$ *are conditionally independent given* X_D.

2.1.C For all μ *and* D,

$$P_\mu\{X_D(D) = 0\} = 1.$$

2.1.D If $\mu(D) = 0$, *then*

$$P_\mu\{X_D = \mu\} = 1.$$

Condition 2.1.A implies that

$$P_\mu e^{-Z} = \prod P_{\mu_n} e^{-Z}$$

if $\mu = \sum \mu_n$.

A BEM system is an (L, ψ)-*superdiffusion* if

$$u(x) = -\log P_x e^{-\langle f, X_D \rangle}$$

satisfies the equation

$$u(x) + \Pi_x \int_0^{\tau_D} \psi[u(\xi_t)]dt = \Pi_x f(\xi_{\tau_D}) \tag{11}$$

where (ξ_t, Π_x) is an L-diffusion. If D is smooth and bounded and f is continuous and bounded, then (11) is equivalent to the conditions

$$\begin{aligned} Lu &= \psi(u) \quad \text{in } D, \\ u &= f \quad \text{on } \partial D. \end{aligned} \tag{12}$$

[The problem (12) has a unique solution.]

The existence of an (L, ψ)-superdiffusion is proved, in particular, for

$$\psi(u) = bu^2 + \int_0^\infty (e^{-tu} - 1 + tu) N(dt) \tag{13}$$

under the conditions

$$b \geq 0, \int_1^\infty tN(dt) < \infty, \int_0^1 t^2 N(dt) < \infty. \tag{14}$$

An important special case is the function

$$\psi(u) = u^\alpha, 1 < \alpha \le 2 \tag{15}$$

corresponding to $b = 0$ and

$$N(dt) = \ell t^{-1-\alpha} dt$$

where

$$\ell = [\int_0^\infty (e^{-\lambda} - 1 + \lambda)\lambda^{-1-\alpha} d\lambda]^{-1}.$$

The class $\mathcal{U}$ under investigation can be characterized probabilistically by the following *mean value property*: $u \in \mathcal{U}$ if and only if

$$P_x e^{-\langle u, X_D \rangle} = e^{-u(x)} \tag{16}$$

for all D such that $\bar{D} \subset E$. [10]

2.4 Stochastic boundary values

Denote by $\mathcal{M}_c(E)$ the set of all finite measures on E concentrated on compact subsets of E. Suppose that, for all $\mu \in \mathcal{M}_c(E)$

$$\langle u, X_{D_n} \rangle \to Z \quad P_\mu\text{-a.s.} \tag{17}$$

for every sequence of domains D_n such that $\bar{D}_n \subset D_{n+1}$ and E is the union of D_n. Then we say that Z is the *stochastic boundary value of* u and we write $Z = \mathrm{SBV}(u)$. The stochastic boundary values exist for all $u \in \mathcal{U}$ and for all $h \in \mathcal{H}$. Put

$$Z_u = \mathrm{SBV}(u), \quad Z_\nu = \mathrm{SBV}(u_\nu).$$

It follows from (17) and the mean value property (16) that

$$P_x e^{-Z_u} = e^{-u(x)} \quad \text{for every } u \in \mathcal{U}. \tag{18}$$

In particular,

$$P_x e^{-Z_\nu} = e^{-u_\nu(x)} \quad \text{for every } \nu \in \mathcal{N}_1. \tag{19}$$

We have

$$Z_{u \oplus v} = Z_u + Z_v, \tag{20}$$

$$Z_{cu} = cZ_u \quad \text{for any constant } c \ge 0, \tag{21}$$

$$Z_{u_n} \uparrow Z_u \quad \text{if } u_n \uparrow u. \tag{22}$$

[See Section 1.3, Chapter 9 in [1].]

[10] See [1], Chapter 8, 2.1.D.

2.5 Relation between superdiffusions and conditional diffusions

We start with the promised probabilistic definition of singular points of a solution u. Put

$$\Phi_u = \int_0^{\tau_E} \psi'[u(\xi_t)]dt. \tag{23}$$

A point $y \in \partial E$ is singular for u if $\Phi_u = \infty$ Π_x^y-a.s. for every $x \in E$.

The following relation plays a fundamental role for the developing the trace theory. For every $u \in \mathcal{U}$ and every $\nu \in \mathcal{N}_1$,

$$P_x Z_\nu e^{-Z_u} = e^{-u(x)} \Pi_x^\nu e^{-\Phi_u} \tag{24}$$

where

$$\Pi_x^\nu = \int \nu(dy) \Pi_x^y.$$

[See Theorem 3.1 in Chapter 9 of [1].] The formula (24) is a key tool for proving the properties 1.1.A–1.1.C of the trace. To illustrate how it is applied, we prove that the set Γ of all singular points of u is f-closed. If ν is concentrated on Γ, then $\Phi_u = \infty$ $\mathcal{P}_x^\nu$-a.s. By (24), P_x-a.s.,$P_x Z_\nu e^{-Z_u} = 0$. Hence, P_x-a.s., either $Z_\nu = 0$ or $Z_u = \infty$. In both cases $P_x\{Z_\nu \leq Z_u\} = 1$. By (18) and (19), this implies $u_\nu \leq u$ and, by (3), $u_\Gamma \leq u$. Hence, every singular point of u_Γ is a singular point of u that is it belongs to Γ.

Remark 1. To apply (24) we need to assume the existence of (L, ψ)-superdiffusion. The original version of the trace theory was developed under this assumption. Later the theory was extended to more general ψ [11] by using an inequality which follows from (24) but can be proved without assuming the existence of (L, ψ)-superdiffusion. The price is less transparent and more lengthy arguments.

Part Two. Representation of solutions in terms of their traces

Suppose that $\mathrm{Tr}(u) = (\Gamma, \nu)$. We claim that u can be represented by the formula

$$u = u_\Gamma \oplus u_\nu \tag{II.1}$$

where u_ν is defined in Section 1 and u_Γ is defined in Section 1.2. A probabilistic version of this formula is given in Section 3.2 (see (29)).

By 1.1.B,

$$u \geq u_\Gamma \oplus u_\nu. \tag{II.2}$$

Since u_Γ and u_ν are σ-moderate, (II.1) implies that u is σ-moderate.

Formula (II.1) will follow if we prove that

[11] See Chapter 11 in [1].

$$w_\Gamma = u_\Gamma, \tag{II.3}$$

and

$$u \le w_\Gamma \oplus u_\nu. \tag{II.4}$$

We establish (II.3) for a bounded smooth domain E and $\psi(u) = u^\alpha$ where $1 < \alpha \le 2$. [12] The bound (II.4) is proved under an additional assumption that $L = \frac{1}{2}\Delta$ (that is a superdiffusion is the super-Brownian motion).

In Section 3 we prepare tools for proving (II.3): $\mathbb{N}$-measures, range of a superdiffusion and Poisson capacities. A special role is played by an inequality (32) relating superdiffusions in two domains $D \subset E$. We call it (D, E)-*inequality*.

Section 4 is devoted to proof of (II.3) and Section 5 to proof of (II.4).

3 Tools

3.1 $\mathbb{N}$-measures

An introduction of **measures** $\mathbb{N}_x$ in parallel to measures P_x is a recent enhancement of the superdiffusion theory. First, $\mathbb{N}$-measures appeared as excursion measures of the Brownian snake introduced by Le Gall. These measures were used by him and his school for investigating the equation $\Delta u = u^2$. In particular, they played a key role in Mselati's dissertation. In Le Gall's theory, measures $\mathbb{N}_x$ are defined on the space of continuous paths. We define their analog on the same space Ω as measures P_μ.

The measures $\mathbb{N}_x$ are constructed by using the integral representation of infinitely divisible random measures (X_D, P_x). They are related to P_x by the formula

$$\mathbb{N}_x(1 - e^{-Z}) = -\log P_x e^{-Z} \tag{25}$$

for every Z of form (10) and for $Z = Z_u$ where $u \in \mathcal{U}$. In particular, for every bounded smooth domain E and every continuous function f,

$$u(x) = \mathbb{N}_x(1 - e^{-\langle f, X_E\rangle})$$

is a solution of (I.1) with the boundary value f.

In contrast to probability measures P_x, measures $\mathbb{N}_x$ are infinite (but $\mathbb{N}_x Z_\nu < \infty$ for all $\nu \in \mathcal{N}_1$).

For every $\nu \in \mathcal{N}_1$,

$$\mathbb{N}_x\{Z_\nu \neq 0\} = \lim_{n\to\infty} u_{n\nu}(x). \tag{26}$$

Indeed, $\mathbb{N}_x\{1 - e^{-nZ_\nu}\} \to \mathbb{N}_x\{Z_\nu \neq 0\}$ as $n \to \infty$ and therefore (26) follows from (25), (18) and (21).

[12] By using purely analytic method, Marcus and Véron proved in [5] that in the case $L = \Delta$ the equality (II.3) holds for all $\alpha > 1$.

An increasing sequence $n\nu$ tends to a measure $\infty \cdot \nu$ equal to 0 on sets of ν-measure 0 and equal to ∞ on the rest of Borel sets. Note that $\infty \cdot \nu \in \mathcal{N}_0$ and, by (26),

$$\mathbb{N}_x\{Z_\nu \neq 0\} = u_{\infty\cdot\nu}(x). \tag{27}$$

3.2 Range

The **range of a superdiffusion** X is the area hit by X. More precisely, the range is a closed set $\mathcal{R} = \mathcal{R}(\omega)$ with the properties:

(i) For every $D \subset E$ and every $\mu \in \mathcal{M}(E)$, X_D is concentrated, P_μ-a.s. on $\mathcal{R}$.

(ii) If $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\omega)$ is a closed set such that, for some $\mu \in \mathcal{M}(E)$ and for all $D \subset E$, X_D are concentrated, P_μ-a.s., on $\tilde{\mathcal{R}}$, then, P_μ-a.s., $\tilde{\mathcal{R}} \supset \mathcal{R}$.

(iii) For every $D \subset E$ and every $x \in D$, X_D is concentrated, $\mathbb{N}_x$-a.s. on $\mathcal{R}$.

(iv) If $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\omega)$ is a closed set such that, for some $x \in E$ and for all $D \subset E$ which contain x, X_D are concentrated, $\mathbb{N}_x$-a.s., on $\tilde{\mathcal{R}}$, then, $\mathbb{N}_x$-a.s., $\tilde{\mathcal{R}} \supset \mathcal{R}$.

The existence of $\mathcal{R}$ is proved for all superdiffusions. It is also proved that

$$w_\Gamma(x) = -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset\} = \mathbb{N}_x\{\mathcal{R} \cap \Gamma \neq \emptyset\}. \tag{28}$$

Moreover, for every $\nu \in \mathcal{N}_0$,

$$\begin{aligned} w_\Gamma \oplus u_\nu &= -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-Z_\nu}\} \\ &= \mathbb{N}_x\{\mathcal{R} \cap \Gamma \neq \emptyset\} + \mathbb{N}_x\{\mathcal{R} \cap \Gamma = \emptyset, 1 - e^{-Z_\nu}\}.^{13} \end{aligned} \tag{29}$$

In combination with (II.1) and (II.3) this formula provides a probabilistic representation of a solution with the trace (Γ, ν).

3.3 Poisson capacities

To every constant $\alpha > 1$ there corresponds the Poisson capacity [14] defined by the formula

$$\mathrm{Cap}(\Gamma) = \sup\{\mathrm{e}(\nu)^{-1} : \nu \in \mathcal{P}(\Gamma)\}$$

where $\mathcal{P}(\Gamma)$ is the set of all probability measures on Γ and

$$\mathrm{e}(\nu) = \int_E d(y, \partial E) dy [h_\nu(y)]^\alpha.$$

($d(x, K)$ stands for the distance from x to K. Function h_ν is given by (2).) We also use the capacities

[13] See Theorem 3.4 in Section 4, [2]. Writing $P\{A, X\}$ means $\int_A X dP$.

[14] Analysts work with the Bessel capacity $\mathrm{Cap}_{2/\alpha,\alpha'}$. Bounds for the Poisson capacity in terms of the Bessel capacity proved in [2] in the Appendix written by I. E. Verbitsky imply that the results of both approaches are equivalent.

$$\mathrm{Cap}_x(\Gamma) = \sup\{\mathrm{e}_x(\nu)^{-1} : \nu \in \mathcal{P}(\Gamma)\}$$

where

$$\mathrm{e}_x(\nu) = \int_E g(x,y)dy[h_\nu(y)]^\alpha. \tag{30}$$

and g is the Green function in E for L.

We establish the following relation between $\mathrm{Cap}(K)$ and $\mathrm{Cap}_x(K)$. Put

$$E_K = \{x \in E : d(x,K) \geq \frac{1}{4}\mathrm{diam}(K), \quad \varphi(x,K) = d(x,\partial E)d(x,K)^{-d}$$

where $\mathrm{diam}(K)$ means the diameter of K. There exists a constant C such that

$$\mathrm{Cap}(K) \leq C\varphi(x,K)\mathrm{Cap}_x(K) \tag{31}$$

for all K and all $x \in E_K$.

3.4 (D,E)-inequality

The (D,E)**-inequality** involves $\mathbb{N}$-measures, the range, the stochastic boundary values Z_ν of u_ν and the integrals (30).

Suppose that $D \subset E$ are bounded smooth domains. Put

$$D^* = \{x \in \bar{D} : d(x, E \setminus D) > 0\} = D \cup L$$

where $L = \{x \in \partial E : d(x, E \setminus D) > 0\}$. For every $\nu \in \mathcal{N}_1$ and every $x \in E$,

$$\mathbb{N}_x\{\mathcal{R} \subset D^*, Z_\nu \neq 0\} \geq \mathrm{const.}\mathbb{N}_x\{\mathcal{R} \subset D^*, Z_\nu\}^{\alpha/(\alpha-1)}\mathrm{e}_x(\nu)^{-1/(\alpha-1)}. \tag{32}$$

[This is Theorem 1.1 in Chapter 7 of [2].]

4 Proof of equation (II.3)

4.1 Reduction to $\mathbb{N}$-inequality

First, we prove that (II.3) can be deduced from the following proposition which we call the $\mathbb{N}$-*inequality*:

($\mathbb{N}$) *For every K, there exists a constant C with the property: for every x, there exists a measure $\nu \in \mathcal{M}(K)$ such that*

$$\mathbb{N}_x\{\mathcal{R} \cap K \neq \emptyset\} \leq C\mathbb{N}_x\{Z_\nu \neq 0\}. \tag{33}$$

By (28) and (27), this inequality is equivalent to

$$w_K \leq C u_{\infty\cdot\nu} \tag{34}$$

If $\nu \in \mathcal{M}(\partial E)$ and $\mathrm{e}_x(\nu) < \infty$, then $\nu \in \mathcal{N}_1$. [15] Denote by $\mathcal{N}_1(K)$ the class of all $\nu \in \mathcal{N}_1$ concentrated on K.

It follows easily from the definitions of u_Γ and w_Γ [(3)–(5)] that:

(i) If (II.3) is true for compact subsets of ∂E, then it is true for all Borel $\Gamma \subset \partial E$.

(ii) $w_K \geq u_K$ for all compact K, and so it is sufficient to prove that $w_K \leq u_K$.

The relation $u_\mu \vee u_\nu = u_{\mu \vee \nu}$ implies that $\mathcal{N}_1(K)$ is closed under $\vee$ and therefore, according to (3), for every $x \in E$, $u_K(x)$ is equal to $\sup u_\nu(x)$ over $\nu \in \mathcal{N}_1(K)$. For every $\nu \in \mathcal{N}_1(K)$, $u_{n\nu} \leq u_K$ and therefore $u_{\infty \cdot \nu}(x) \leq u_K(x)$. To prove that $w_K \leq u_K$ it is sufficient to demonstrate that, for every x, there exists $\nu \in \mathcal{N}_1(K)$ such that $w_K(x) \leq u_{\infty \cdot \nu}(x)$. Put $\eta = \infty \cdot \nu$. It follows from (21) that $Z_\eta = Z_{C\eta} = CZ_\eta$. Therefore the bound $w_K(x) \leq u_{\infty \cdot \nu}(x)$ will follow from (18) if we prove that (34) holds with C independent of x.

4.2 Proof of the $\mathbb{N}$-inequality

We establish a number of estimates in terms of $\mathrm{Cap}_x(K)$.

A. An upper bound for $w_K(x) = \mathbb{N}_x\{\mathcal{R} \cap K \neq \emptyset\}$.

B. A lower bound (for sufficiently large n) for

$$\mathbb{N}_x\{\mathcal{R} \subset B_n(x, K), Z_\nu\}$$

where

$$B_n(x, K) = \{z : |x - z| < nd(x, K)\}$$

C. A lower bound (for sufficiently big n) for

$$\mathbb{N}_x\{\mathcal{R} \subset B_n(x, K), Z_\nu \neq 0\}.$$

Part A is based on an estimate

$$w_K(x) \leq C\varphi(x, K)\mathrm{Cap}(K)^{1/(\alpha-1)}, \tag{35}$$

where the constant C does not depend on K and x. [16] It follows from (35) and (31) that

$$w_K(x) \leq C[\varphi(x, K)^\alpha \mathrm{Cap}_x(K)]^{1/(\alpha-1)}. \tag{36}$$

In part B we use the relations between superdiffusions and conditional diffusions and bounds for conditional diffusions involving first exit times from E and from a ball of radius r centered at x. As a result, we prove the existence of C and n such that, for all K, all $x \in E_K$ and all $\nu \in \mathcal{P}(K)$ such that $\mathrm{e}_x(\nu) < \infty$,

$$\mathbb{N}_x\{\mathcal{R} \subset B_n(x, K), Z_\nu\} > C\varphi(x, K). \tag{37}$$

[15] This follows from 2.1.A, Chapter 12 in [1].

[16] The bound (35) was proved for $\alpha = 2$ by Mselati [7] and for $1 < \alpha < 2$ by Kuznetsov [4].

Part C is deduced from the definition of $\mathrm{Cap}_x(K)$ and from (37) by (D,E)-inequality (32) applied to $D = E \cap B_n(x,K)$. We prove this way the existence of C and n with the property: for every K and every $x \in E_K$,

$$\mathbb{N}_x\{\mathcal{R} \subset B_n(x,K), Z_\nu \neq 0\} \geq C[\varphi(x,K)^\alpha \mathrm{Cap}_x(K)]^{1/(\alpha-1)} \tag{38}$$

for some $\nu \in \mathcal{P}(K)$ such that $\mathrm{e}_x(\nu) < \infty$.

It follows from (28), (36) and (38) that:

($\mathbb{M}$) There exist constants C and n such that, for every K and every $x \in E_K$, there is a $\nu \in \mathcal{N}_1(K)$ with the property

$$\mathbb{N}_x\{\mathcal{R} \cap K \neq \emptyset\} \leq C\mathbb{N}_x\{\mathcal{R} \subset B_n(x,K), Z_\nu \neq 0\}. \tag{39}$$

It remains to deduce ($\mathbb{N}$) from ($\mathbb{M}$). In both propositions we have upper estimates for $\mathbb{N}_x\{\mathcal{R} \cap K \neq \emptyset\}$. However (39) holds only for $x \in E_K$ and (33) holds for all $x \in E$. On the other hand, C in (33) depends on K and in (39) it is independent of K. Following Mselati [6] and [7], [17] we cover K by closed sets K_m to which we can apply $\mathbb{M}$. We get this way measures $\nu_m \in \mathcal{P}(K_m)$ with $\mathrm{e}_x(\nu_m) < \infty$. Their sum ν satisfies (33).

To realize this plan, we fix $x \in E$ and $K \subset \partial E$ and we put

$$K_m = \begin{cases} \{z \in K : |x-z| \leq 2\delta\} & \text{for } m = 1, \\ \{z \in K : 2^{m-1} \leq \delta \leq 2^m\delta\} & \text{for } m > 1 \end{cases}$$

where $\delta = d(x,K)$. The set M of m such that K_m is not empty is finite and $x \in E_{K_m}$ for every $m \in M$. By ($\mathbb{M}$), there exist constants C, n and measures $\nu_m \in \mathcal{N}_1(K_m)$ such that $\mathrm{e}_x(\nu_m) < \infty$ and

$$\mathbb{N}_x\{\mathcal{R} \cap K_m \neq \emptyset\} \leq C\mathbb{N}_x\{\mathcal{R} \subset B_n(x,K_m), Z_{\nu_m} \neq 0\}. \tag{40}$$

If $2^p > n$, then, for every positive m, $B_n(x,K_m) \subset B_{2^{p+m}}(x,K)$ and, by (40),

$$\mathbb{N}_x\{\mathcal{R} \cap \mathcal{K}_m \neq \emptyset\} \leq C\mathbb{N}_x(Q_m)$$

where

$$Q_m = \{\mathcal{R} \subset B_{2^{p+m}}(x,K)\}.$$

The sum ν of ν_m is a finite measure and $\mathrm{e}_x(\nu) < \infty$.

$$\nu = \sum_M \nu_m \in \mathcal{N}_1(K)$$

and

$$\mathbb{N}_x\{\mathcal{R} \cap K \neq \emptyset\} \leq \sum_M \mathbb{N}_x\{\mathcal{R} \cap K_m \neq \emptyset\} \leq C\sum_1^\infty \mathbb{N}_x(Q_m).$$

Now we need to bound the right side from above. First, we prove that

[17] See also [2].

$$\mathbb{N}_x\{Q_m \cap Q_{m'}\} = 0 \quad \text{if } m' \geq m + p + 1. \tag{41}$$

Indeed,

$$Q_m \cap Q_{m'} \subset \{\mathcal{R} \cap K_{m'} = \emptyset, Z_{\nu_{m'}} \neq 0\}.$$

Since $\nu_{m'}$ is concentrated on $K_{m'}$,

$$\mathbb{N}_x\{\mathcal{R} \cap K_{m'} = \emptyset, Z_{\nu_{m'}} \neq 0\} = 0$$

which implies (41).

Every integer $m \geq 1$ has a unique representation $m = n(p+1) + j$ where $j = 1, \dots, p+1$ and therefore

$$\mathbb{N}_x\{\mathcal{R} \cap K \neq \emptyset\} \leq C_\kappa \sum_{j=1}^{p+1} \sum_{n=0}^{\infty} \mathbb{N}_x(Q_{n(p+1)+j}). \tag{42}$$

It follows from (41) that $\mathbb{N}_x\{Q_{n(p+1)+j} \cap Q_{n'(p+1)+j}\} = 0$ for $n' > n$. Therefore, for every j,

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{N}_x\{Q_{n(p+1)+j}\} &= \mathbb{N}_x\left\{\bigcup_{n=0}^{\infty} Q_{n(p+1)+j}\right\} \\ &\leq \mathbb{N}_x\left\{\sum_{n=0}^{\infty} Z_{\nu_{n(p+1)+j}} \neq 0\right\} \leq \mathbb{N}_x\{Z_\nu \neq 0\}\end{aligned} \tag{43}$$

because

$$\sum_{n=0}^{\infty} Z_{\nu_{n(p+1)+j}} \leq \sum_{m=1}^{\infty} Z_{\nu_m} = Z_\nu.$$

The bound (33) follows from (42) and (43).

5 Proof of bound (II.4)

In this section we assume that u is a positive solution of the equation

$$\Delta u = u^\alpha \quad \text{in } E \tag{44}$$

with the trace (Γ, ν) and that $1 < \alpha \leq 2$ and we investigate the class $\mathfrak{E}$ of all domains E for which the bound (II.4) is true. The final result is: *all bounded domains of class C^4 belong to $\mathfrak{E}$.*

The main steps in the proof are:

A.There is a class $\mathfrak{E}_1 \subset \mathfrak{E}$ with the property: $E \in \mathfrak{E}_1$ if, for every $y \in \partial E$, there exists a domain $D \in \mathfrak{E}_1$ such that $D \subset E$ and $\partial D \cap \partial E$ contains a neighborhood of y in ∂E.

B. $\mathfrak{E}_1$ contains all star domains. [18]

C. If E is a C^4 domain, then, for every $y \in \partial E$, there exists a star domain $D \subset E$ such that $\partial D \cap \partial E$ contains a neighborhood of y in ∂E.

Here is the definition of class $\mathfrak{E}_1$: $E \in \mathfrak{E}_1$ if, for every $v \in \mathcal{U}(E)$ and every $\Gamma \subset \partial E$, the conditions $\mathrm{Tr}(v) = (\Lambda, \mu)$, $\Lambda \subset \Gamma$ and $\mu(\Gamma \setminus \Lambda) = 0$ imply that $v \leq w_\Gamma$.

In part A we use connections between $\mathrm{Tr}(v)$ and $\mathrm{Tr}(v')$ where $v \in \mathcal{U}(E)$ and v' is the restriction of v to $D \subset E$.

Step B is based on a self-similarity property of the equation $\Delta u = u^\alpha$: if E is a star domain relative to 0, then, for every $0 < r \leq 1$,

$$u_r(x) = r^{2/(\alpha-1)} u(rx)$$

also belongs to $\mathcal{U}(E)$. A crucial role is played by the following absolute continuity result which is also of independent interest: if $A \in \mathcal{F}_{\supset D}$, then either $P_x(A) = 0$ for all $x \in D$ or $P_x(A) > 0$ for all $x \in D$. In other words, on the σ-algebra $\mathcal{F}_{\supset D}$, P_{x_1} is absolutely continuous with respect to P_{x_2} for all $x_1, x_2 \in D$.

Step C is based on elementary arguments of differential geometry.

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[18] A domain E is called a star domain relative to a point c if, for every $x \in E$, the line segment $[c, x]$ connecting c and x is contained in E.