Stochastic Integrals and Adjoint Derivatives.

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Summary. In a systematic study form, the present paper concern topics of stochastic calculus with respect to stochastic measures with independent values. We focus on the integration and differentiation with respect to such measures over general space-time products.

Key words: Itô non-anticipating integral, non-anticipating derivative, Malliavin derivative, Skorohod integral, deFinetti-Kolmogorov law.

AMS (2000) Classification: 60H05, 60H07.

Contents

1 Stochastic measures and functions.

1.1 Space-time products.

In this paper we consider some elements of a stochastic calculus for random fields over general space-time products. The various space components which may be specified in the possible applications are here considered altogether and denoted by Θ. We consider Θ to be a general space equiped with some countable σ-algebra. The time-component is an interval $\mathbb{T}$. To simplify notations, we fix $\mathbb{T} = (0, T]$.

In the sequel a basic tool is constituted by the partitions of the involved spaces. As for the interval $\mathbb{T}$, its *partition* with level of refinement n, from now on named n^{th} *-partition*, is represented by the corresponding finite n^{th} *-series* of intervals of type $(s, u]$ such that

$$\mathbb{T} = \sum (s,u] : \qquad \max_{(s,u]} (u - s) \longrightarrow 0 \,, \qquad n \to \infty \tag{1}$$

(note that here and in the sequel we denote the disjoint union of sets by $\sum$). The partitions are such that for $n = 1, 2, ...$, the $(n+1)^{th}$-series is obtained by partitioning the intervals of the previous n^{th}-series. The family of all the sets of all the n^{th}-partitions, $n = 1, 2, ...$, generates the Borel σ-algebra of $\mathbb{T}$.

Hereafter we introduce the n^{th}-partitions ($n = 1, 2, ...$) for the standard product $\Theta \times \mathbb{T}$ of the measurable spaces Θ and $\mathbb{T}$. The *σ-algebra of* $\Theta \times \mathbb{T}$ will be treated as generated by these n^{th}-partitions. Since in the sequel we are dealing with the general σ-finite measure $M = M(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, on the σ-algebra of $\Theta \times \mathbb{T}$, the n^{th} *-partitions of* $\Theta \times \mathbb{T}$ are going to be selected for the increasing sequence (which can be any) of sets

$$\Theta_n \times \mathbb{T} \,, \quad n = 1, 2, ..., \ \text{ such that } \quad \lim_{n \to \infty} \Theta_n \times \mathbb{T} := \bigcup_n \Theta_n \times \mathbb{T} = \Theta \times \mathbb{T} \,,$$

and with

$$M(\Theta_n \times \mathbb{T}) < \infty \,, \qquad n = 1, 2, \dots . \tag{2}$$

The n^{th} *-partition of* $\Theta \times \mathbb{T}$ is then actually a partition of

$$\Theta_n \times \mathbb{T} = \sum \Delta : \quad \Delta \subseteq \Theta \times (s, u] \tag{3}$$

given by the corresponding finite series of sets Δ, related to the n^{th} *-series* of time-intervals $(s, u] \subseteq \mathbb{T}$ in (1). Any set Δ in the n^{th}-series is the (disjoint) union of some elements of the $(n+1)^{th}$-series. Clearly $M(\Delta) < \infty$ for any element Δ of the partitions of $\Theta \times \mathbb{T}$.

We assume that the measure M satisfies

$$M(\Theta \times [t]) \,=\, \lim_{n \to \infty} M(\Theta_n \times [t]) \,=\, 0 \tag{4}$$

whatever the point-set $[t] \subseteq \mathbb{T}$ be. This implies, in particular, that any set $\Delta \subseteq \Theta \times \mathbb{T} : M(\Delta) < \infty$, is *infinitely-divisible* in the sense that Δ admits the n^{th}-partitions.

We will refer to the finite (disjoint) unions of sets belonging to the same n^{th}-series of partitions (3) as the *simple sets* in $\Theta \times \mathbb{T}$. Note that for *any* set $\Delta \subseteq \Theta \times \mathbb{T} \,:\, M(\Delta) < \infty$, we have

$$\Delta = \sum \Delta \cap \Big(\Theta \times (s,u]\Big) : \quad \sum (s,u] = \mathbb{T},$$

where the n^{th}-series of time-intervals (1) have been used. Since $\max(u-s) \to 0$, $n \to \infty$, we have

$$\max_{(s,u]} M\left(\Delta \cap \left(\Theta \times (s,u]\right)\right) \longrightarrow 0, \qquad n \to \infty. \tag{5}$$

We write $M(d\theta\, dt)$, $(\theta,t) \in \Theta \times \mathbb{T}$, for M as integrator.

Any $\Delta \subseteq \Theta \times \mathbb{T} : M(\Delta) < \infty$, can be approximated by simple sets $\Delta^{(n)}$, $n = 1, 2, \ldots$, in the sense that

$$\Delta = \lim_{n\to\infty} \Delta^{(n)}, \text{ i.e. } \quad M\Big((\Delta \setminus \Delta^{(n)}) \cup (\Delta^{(n)} \setminus \Delta)\Big) \longrightarrow 0, \qquad n \to \infty. \tag{6}$$

Note that for any finite number of *disjoint* sets $\Delta_1, \ldots, \Delta_m$: $M(\Delta_j) < \infty$, $j = 1, \ldots, m$, the approximation above can be given by the corresponding sequences of *disjoint* simple sets $\Delta_1^{(n)}, \ldots, \Delta_m^{(n)}$ $(n = 1, 2, \ldots)$.

1.2 Stochastic measures with independent values.

For the complete probability space $(\Omega, \mathfrak{A}, P)$, let $L_2(\Omega)$ be the standard (complex) space of random variables $\xi = \xi(\omega)$, $\omega \in \Omega$, with finite norm

$$\| \xi \| = \Big(E\, |\xi|^2\Big)^{1/2}. \tag{7}$$

We write $\mu = \mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, for the *additive* set-function with the *real* values $\mu(\Delta) \in L_2(\Omega)$ and such that $E\mu(\Delta) = 0$, $E\mu(\Delta)^2 = M(\Delta)$. Here the variance $M = M(\Delta)$, $\Delta \in \Theta \times \mathbb{T}$, is a measure which satisfies the conditions (2)-(4). The additive set-function μ is considered on all the sets $\Delta : M(\Delta) < \infty$. The values of μ on *disjoint* sets are *independent* random variables.

Note that μ, initially considered just on the simple sets in $\Theta \times \mathbb{T}$ (related to some partitions), can be extended on all $\Delta \subseteq \Theta \times \mathbb{T} : M(\Delta) < \infty$, via the limits

$$\mu(\Delta) = \lim_{n\to\infty} \mu(\Delta^{(n)}), \text{ i.e. } \quad \big\|\mu(\Delta) - \mu(\Delta^{(n)})\big\| \longrightarrow 0, \qquad n \to \infty, \tag{8}$$

where the simple sets $\Delta^{(n)}$, $n = 1, 2, \ldots$, approximate Δ, i.e.

$$M\left(\left(\Delta \setminus \Delta^{(n)}\right) \cup \left(\Delta^{(n)} \setminus \Delta\right)\right) = \Big\|\mu(\Delta) - \mu(\Delta^{(n)})\Big\|^2 \longrightarrow 0, \quad n \to \infty.$$

Cf. (6). We refer to μ as *the stochastic measure with indepenent values of the type*

$$E\,\mu = 0, \qquad E\,\mu^2 = M. \tag{9}$$

And we write $\mu(d\theta\, dt)$, $(\theta, t) \in \Theta \times \mathbb{T}$, for μ as the integrator.

Let μ_k, $k = 1, ..., K$ $(K \leq \infty)$, be a number of the *independent* stochastic measures of type $E\mu_k = 0$, $E\mu_k^2 = M_k$ on the corresponding space-time products $\Theta_k \times \mathbb{T}$. Let

$$\Theta \times \mathbb{T} := \sum_k (\Theta_k \times \mathbb{T}).$$

The *mixture* of μ_k, $k = 1, ..., K$, is a stochastic measure μ on the space-time product $\Theta \times \mathbb{T}$ formally introduced above defined as

$$\mu(\Delta) := \sum_k \mu_k \Big(\Delta \cap (\Theta_k \times \mathbb{T}) \Big), \qquad \Delta \subseteq \Theta \times \mathbb{T}. \tag{10}$$

This stochastic measure is of the type $E\mu = 0$, $E\mu^2 = M$, where

$$M(\Delta) = \sum_k M_k \Big(\Delta \cap (\Theta_k \times \mathbb{T}) \Big), \qquad \Delta \subseteq \Theta \times \mathbb{T}.$$

Cf. (9). Naturally in the expression above the sets $\Theta_k \times \mathbb{T}\,(k = 1, ..., K)$ formally represent some partition sets of $\Theta \times \mathbb{T}$. To illustrate, let μ_k, $k = 1, ..., K$ $(K \leq \infty)$, be stochastic measures on the time interval $\mathbb{T}$. Then the space-time product $\Theta \times \mathbb{T}$ that can be applied has space component $\Theta = \{1, ..., K\}$.

1.3 The events generated.

Let $\mu = \mu(\Delta)$, $\Delta \in \Theta \times \mathbb{T}$, be a general stochastic measure with independent values of the type (9). In particular, it can be the *mixture* of a *number* of independent components - cf. (10). We write

$$\mathfrak{A}_\Delta, \qquad \Delta \subseteq \Theta \times \mathbb{T}, \tag{11}$$

for the σ-algebras generated by μ over the subsets of Δ and augmented by all the events of zero probability. To be more precise, $\mathfrak{A}_\Delta$is the minimal augmented σ-algebra containing all the standard events $\{\mu(\Delta') \in B\}$ for all $B \subseteq \mathbb{R}$ and the subsets $\Delta' \subseteq \Delta$. To simplify notations and terminology, we assume that the σ-algebra

$$\mathfrak{A} = \mathfrak{A}_{\Theta \times \mathbb{T}} \tag{12}$$

represents all the events $\mathcal{A} \subseteq \Omega$.

We remark that, in some sense, the σ-algebras $\mathfrak{A}_\Delta$ are *continuous* with respect to the sets $\Delta \subseteq \Theta \times \mathbb{T}$. To explain, on one hand we have

$$\lim_{n\to\infty} \mathfrak{A}_{\Delta^{(n)}} := \bigvee_n \mathfrak{A}_{\Delta^{(n)}} = \mathfrak{A}_\Delta, \tag{13}$$

for any sequence of *increasing* sets $\Delta^{(n)}$, $n = 1, 2, ...$, such that $\lim_{n\to\infty} \Delta^{(n)} = \Delta$, i.e. $M\left((\Delta^{(n)} \setminus \Delta) \cup (\Delta \setminus \Delta^{(n)})\right) \to 0$, $n \to \infty$ (here, the sign $\bigvee$ defines the minimal σ-algebra containing the involved components)-cf. (8). On the other hand, we have the following result. See e.g. [13].

Theorem 1. *Let $\Delta^{(n)}$, $n = 1, 2, ...$, be a sequence of decreasing sets and let $\Delta = \bigcap_n \Delta^{(n)}$, then we have*

$$\lim_{n\to\infty} \mathfrak{A}_{\Delta^{(n)}} := \bigcap_n \mathfrak{A}_{\Delta^{(n)}} = \mathfrak{A}_{\Delta} . \tag{14}$$

Proof. Note that

$$\mathfrak{A}_{\Delta^{(1)}} = \mathfrak{A}_{\Delta} \bigvee \mathfrak{A}_{\Delta^{(1)}\setminus\Delta} \text{ where } \mathfrak{A}_{\Delta^{(1)}\setminus\Delta} = \bigvee_n \mathfrak{A}_{\Delta^{(1)}\setminus\Delta^{(n)}} .$$

Cf. (13). Accordingly, we have

$$H_{\Delta^{(1)}} = H_{\Delta} \bigvee H_{\Delta^{(1)}\setminus\Delta} \text{ where } H_{\Delta^{(1)}\setminus\Delta} = \bigvee_n H_{\Delta^{(1)}\setminus\Delta^{(n)}} .$$

for the subspaces in $L_2(\Omega)$ of random variables measurable with respect to the corresponding σ-algebras (here above, the sign $\bigvee$ defines the linear closure of the involved components). The products $\xi\cdot\xi' : \xi \in H_\Delta$, $\xi' \in H_{\Delta^{(1)}\setminus\Delta^{(n)}}$, $n > 1$, constitute a complete system in $H_{\Delta^{(1)}}$. Hence, the orthogonal projections

$$\xi\cdot\xi' - E\,(\xi\cdot\xi'|\mathfrak{A}_\Delta) = \xi(\xi' - E\,\xi'),$$

on the orthogonal complement $H_{\Delta^{(1)}} \ominus H_\Delta$ to the subspace $H_\Delta \subseteq H_{\Delta^{(1)}}$, constitute a complete system in $H_{\Delta^{(1)}} \ominus H_\Delta$. For the subspace H^+_Δ of the random variables in $H_{\Delta^{(1)}}$ measurable with respect to the σ-algebra $\mathfrak{A}^+_\Delta := \bigcap_n \mathfrak{A}_{\Delta^{(n)}}$, any $\xi^+ \in H^+_\Delta$ is independent from all $\xi' \in H_{\Delta^{(1)}\setminus\Delta^{(n)}}$, $n > 1$, and this implies that

$$E\left(\xi^+\cdot\xi(\xi' - E\xi')\right) = E\,(\xi^+\cdot\xi)\cdot E\,(\xi' - E\xi') = 0 .$$

Thus, ξ^+ is orthogonal to all the elements $\xi\cdot(\xi' - E\xi')$ of the complete system in $H_{\Delta^{(1)}} \ominus H_\Delta$. Accordingly, $\xi^+ \in H_\Delta$. This justifies that $H^+_\Delta = H_\Delta$, $\mathfrak{A}^+_\Delta = \mathfrak{A}_\Delta$. □

The σ-algebras

$$\mathfrak{A}_t := \mathfrak{A}_{\Theta\times(0,t]} , \qquad t \in \mathbb{T} \tag{15}$$

- cf. (11), represent the flow of events in the course of time on $\mathbb{T} = (0, T]$. Thanks to the condition (4), *the σ-algebras $\mathfrak{A}_t$ are continuous with respect to $t \in \mathbb{T}$*:

$$\begin{aligned} \lim_{s\to t-0} \mathfrak{A}_s &:= \bigvee_{s<t} \mathfrak{A}_s = \mathfrak{A}_t \quad (0 < t \le T) , \\ \lim_{u\to t+0} \mathfrak{A}_u &:= \bigcap_{u>t} \mathfrak{A}_u = \mathfrak{A}_t \quad (0 \le t < T) . \end{aligned} \tag{16}$$

Cf. (13)-(14). Note that, here above, for $t = 0$ we have the *trivial* σ-algebra $\mathfrak{A}_0$. We remark that, for any t, the values $\mu(\Delta)$, $\Delta \in \Theta \times (0, t]$, are $\mathfrak{A}_t$-measurable and the values $\mu(\Delta)$, $\Delta \in \Theta \times (t, T]$, are independent of $\mathfrak{A}_t$.

1.4 The de Finetti-Kolmogorov infinitely-divisible law.

Let us set

$$\mathbb{R} \setminus [0] := (-\infty, 0) \cup (0, \infty) .$$

Similar to the stochastic processes with independent increments (cf. e.g. [51], see also e.g. [49], [3]) the stochastic measure $\mu = \mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, of the type $E\mu = 0$, $E\mu^2 = M$ - cf. (9), can be characterized as follows. We can refer to [18] for the following result. We also refer for example to [30] for some results in this direction with respect to random measures.

Theorem 2. *The values* $\mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, *obey the infinitely-divisible law*

$$\begin{aligned} log E\, e^{i\lambda\mu(\Delta)} = \iint_{\Delta} \Big[& -\frac{\lambda^2}{2}\sigma^2(\theta,t) \\ & + \int_{\mathbb{R}\setminus[0]} \big(e^{i\lambda x} - 1 - i\lambda x\big)\, L(dx,\theta,t)\Big] M(d\theta\, dt), \quad \lambda \in \mathbb{R} : \quad (17) \end{aligned}$$

$$\sigma^2(\theta,t) + \int_{\mathbb{R}\setminus[0]} x^2\, L(dx,\theta,t) \equiv 1 .$$

Proof. The values $\mu(\Delta)$ are infinitely-divisible random variables - cf. (5). Hence, according to the deFinetti [8] and Kolmogorov [34] law (see [36], [33]), we have

$$log E\, e^{i\lambda\mu(\Delta)} = -\frac{\lambda^2}{2}\sigma^2_\Delta + \int_{\mathbb{R}\setminus[0]} \big(e^{i\lambda x} - 1 - i\lambda x\big)\, L_\Delta(dx) , \quad \lambda \in \mathbb{R} \qquad (18)$$

with

$$\sigma^2_\Delta + \int_{\mathbb{R}\setminus[0]} x^2\, L_\Delta(dx) \equiv M(\Delta) , \qquad \Delta \subseteq \Theta \times \mathbb{T} , \qquad (19)$$

where the constant σ^2_Δ and the Borel measure $L_\Delta = L_\Delta(B)$, $B \subseteq \mathbb{R} \setminus [0]$, depend on $\Delta \subseteq \Theta \times \mathbb{T}$ as additive set-functions. Taking the relationship (19) with the variance measure $M = M(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, into account we can see that σ^2_Δ, L_Δ admit the integral representations

$$\sigma^2_\Delta = \iint_{\Delta} \sigma^2(\theta,t)\, M(d\theta\, dt) , \qquad L_\Delta(B) = \iint_{\Delta} L(B,\theta,t)\, M(d\theta\, dt) .$$

The integrands $\sigma^2(\theta,t)$ and $L(B,\theta,t)$, $(\theta,t) \in \Theta \times \mathbb{T}$, are elements of the standard L_1-space (with respect to the measure M). Moreover they are *additive* in their dependence on the Borel sets $B \subseteq \mathbb{R} \setminus [0]$. The above stochastic function $L(B,\theta,t)$, $B \subseteq \mathbb{R} \setminus [0]$, $(\theta,t) \in \Theta \times \mathbb{T}$, can be modified on a set of *zero* M-measure in a way that yields a lifting to a *new* equivalent integrand such that, whatever $(\theta,t) \in \Theta \times \mathbb{T}$ be, the set-function $L(B,\theta,t)$, $B \subseteq \mathbb{R} \setminus [0]$, is a *measure* on $\mathbb{R} \setminus [0]$ (see e.g. [24]). So, the probability law (18) admits a representation in the form (17). □

Example 1.1. The *Gaussian stochastic measure* μ, having Gaussian random variables as values $\mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, corresponds to the probability law (17) with $\sigma^2 \equiv 1$ and $L \equiv 0$.

Example 1.2. The *Poisson (centred) stochastic measure* μ, having values

$$\mu(\Delta) \,=\, \nu(\Delta) - E\nu(\Delta)\,, \qquad \Delta \subseteq \Theta \times \mathbb{T}\,,$$

where $\nu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, are Poisson random variables, corresponds to the probability law (17) with $\sigma^2 \equiv 0$ and L concentrated the point $x = 1$ in $\mathbb{R} \setminus [0]$ with unit mass, i.e.

$$logE\, e^{i\lambda\mu(\Delta)} \,=\, (e^{i\lambda} - 1 - i\lambda)\, M(\Delta)\,, \qquad \lambda \in \mathbb{R}.$$

We recall that the non-negative additive set-function $\nu = \nu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, has values $\nu(\Delta) \in L_2(\Omega)$ which are integer random variables $\nu(\Delta) = \nu(\Delta, \omega)$, $\omega \in \Omega$. In the case Θ is a *complete separable metric space* equipped with the σ-algebra of its Borel sets, we have that $\nu = \nu(\Delta)$, $\Delta \in \Theta \times \mathbb{T}$, *admits an equivalent modification*, which is referred to as

$$\nu \,=\, \nu(\cdot, \omega),\; \omega \in \Omega \,:\, E\nu \,=\, M\,,$$

with values $\nu(\Delta) = \nu(\Delta, \omega)$, $\omega \in \Omega$, *representing the measures* $\nu(\cdot, \omega) = \nu(\Delta, \omega)$, $\Delta \subseteq \Theta \times \mathbb{T}$, *depending on* $\omega \in \Omega$ *as parameter*. For some (which can be any) sequence of increasing sets $\Theta_n \times \mathbb{T}$, such that $M(\Theta_n \times \mathbb{T}) < \infty$, $n = 1, 2, ...$, and $\lim_{n\to\infty} \Theta_n \times \mathbb{T} = \Theta \times \mathbb{T}$, the measures $\nu(\cdot, \omega)$ can be defined in a way that $\nu(\Theta_n \times \mathbb{T}, \omega) < \infty$, $n = 1, 2, ...$, and all the finite values $\nu(\Delta, \omega)$ are integers. So, the measures $\nu(\cdot, \omega)$ are *purely discrete*, concentrated on the corresponding atoms $(\theta_\omega, t_\omega) \in \Theta \times \mathbb{T}$. In particular for $\Delta \subseteq \Theta \times \mathbb{T}$ with $M(\Delta) < \infty$, the possibility of having one atom $(\theta_\omega, t_\omega) \subseteq \Delta$ with $\nu(\theta_\omega, t_\omega, \omega) > 1$ or of having a couple of atoms in Δ with the *same* time components occur with zero probability. To explain, the limit

$$\lim_{n\to\infty} P\Big\{ \max_{(s,u]} \mu\big(\Delta \cap (\Theta \times (s, u])\big) > 1 \Big\} \,=\, 0$$

holds true for the n^{th} -series of partitions of $\mathbb{T}$ - cf. (1), $\mathbb{T} = \sum (s, u] \,:\, \max(s - u) \to 0$, $n \to \infty$, and the corresponding partitions of $\Theta \times \mathbb{T}$:

$$\Delta \,=\, \sum \Delta \cap \big(\Theta \times (s, u]\big)\,, \qquad n = 1, 2, ...,$$

- cf. (2)-(5). Hence, *all the atoms* $(\theta_\omega, t_\omega)$ *are in one-to-one correspondence*

$$\Theta \times \mathbb{T} \ni (\theta_\omega, t_\omega) \Longleftrightarrow t_\omega \in \mathbb{T}$$

with their time components and we have

$$\nu(\theta_\omega, t_\omega, \omega) \,\equiv\, 1.$$

1.5 Non-anticipating and predictable stochastic functions.

We write $L_2(\Theta \times \mathbb{T} \times \Omega)$ for the standard (complex) space of the stochastic functions $\varphi = \varphi(\theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$, with values $\varphi(\theta, t) = \varphi(\theta, t, \omega)$, $\omega \in \Omega$, in $L_2(\Omega)$:

$$\|\varphi\|_{L_2} = \Big(\iiint_{\Theta\times\mathbb{T}\times\Omega} |\varphi|^2 M(d\theta\, dt) \times P(d\omega) \Big)^{1/2}$$

$$= \Big(\iint_{\Theta\times\mathbb{T}} \|\varphi\|^2 M(d\theta\, dt) \Big)^{1/2} . \quad (20)$$

Cf. (7). Here, $P = P(\mathcal{A})$, $\mathcal{A} \in \mathfrak{A}$, is the probability on the σ-algebra $\mathfrak{A} = \mathfrak{A}_{\Theta\times\mathbb{T}}$ of all events $\mathcal{A} \subseteq \Omega$ and the product-measure $M \times P$ on $\Theta \times \mathbb{T} \times \Omega$ is considered on the σ-algebra generated by the product-sets

$$\Delta \times \mathcal{A} : \quad \Delta \subseteq \Theta \times (s, u] , \quad \mathcal{A} \in \mathfrak{A} . \quad (21)$$

The component $M = M(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, satisfies (2)-(4). For the product-sets (21) we have

$$\iiint_{\Delta\times\mathcal{A}} M(d\theta\, dt) \times P(d\omega) = M(\Delta) \cdot P(\mathcal{A}) .$$

We say that φ is a *simple function* if it admits the representation

$$\varphi = \sum \varphi \cdot 1_\Delta$$

where the sum is taken on some finite series of disjoint sets $\Delta \subseteq \Theta \times \mathbb{T}$: $M(\Delta) < \infty$, and the indicated element $\varphi \in L_2(\Omega)$ in each component $\varphi \cdot 1_\Delta$ is the value of the simple function on Δ. Note that the simple functions represented by the indicators

$$1_{\Delta\times\mathcal{A}} = 1_\mathcal{A} \cdot 1_\Delta : \quad \Delta \subseteq \Theta \times (s, u] , \quad \mathcal{A} \subseteq \Omega ,$$

with Δ belonging to the partitions of $\Theta \times \mathbb{T}$ - see (3), constitute a complete system in $L_2(\Theta \times \mathbb{T} \times \Omega)$. Cf. (3)-(6) and (21).

Let us turn to the σ-algebras

$$\mathfrak{A}_t, \quad t \in \mathbb{T},$$

characterized in (15) which represent the flow of events in the course of time. The *non-anticipating simple function* φ is characterized by the representation

$$\varphi = \sum \varphi \cdot 1_\Delta \quad (22)$$

where each component $\varphi \cdot 1_\Delta$ has $\Delta \subseteq \Theta \times (s, u]$ and the indicated value $\varphi \in L_2(\Omega)$ on Δ is an $\mathfrak{A}_s$-measurable random variable.

In general, we refer to $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$ as a *non-anticipating function* if its values $\varphi(\theta, t) \in L_2(\Omega)$ in the course of time are determined by the "past" events. To be more precise, for any $t \in \mathbb{T}$, the random variable $\varphi(\theta, t)$ is measurable with respect to the σ-algebra $\mathfrak{A}_t$.

Let us consider also the functions $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$ measurable with respect to the σ-algebra generated by the product-sets

$$\Delta \times \mathcal{A} : \quad \Delta \subseteq \Theta \times (s, u] , \quad \mathcal{A} \in \mathfrak{A}_s \tag{23}$$

- cf. (21). Following the common terminology (see e.g. [11]), we refer to the above functions φ as the *predictable functions* and the σ-algebra generated by the sets (23) as the *predictable σ-algebra*. Note that all the *non-anticipating simple* functions are predictable. We remark that all the predictable functions are non-anticipating. The following result details the study of the converse relationship. Note that this coming result holds thanks to the left-continuity of the flow of σ-algebras $\mathfrak{A}_t$, $t \in \mathbb{T}$ - cf. (16).

Theorem 3. *Any non-anticipating function $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$ can be identified with the corresponding predictable function given by the limit*

$$\varphi = \lim_{n\to\infty} \varphi^{(n)}, \ \textit{i.e.} \quad \left\|\varphi - \varphi^{(n)}\right\|_{L_2} \longrightarrow 0 , \qquad n \to \infty ,$$

of the non-anticipating simple functions $\varphi^{(n)}$, $n = 1, 2, \ldots$, defined along the n^{th}-series of sets $\Delta \subseteq \Theta \times (s, u]$ of the partitions of $\Theta \times \mathbb{T}$ - cf. (3), as

$$\varphi^{(n)} = \sum \varphi^{(n)} \cdot 1_\Delta, \ \textit{with} \ \varphi^{(n)} = \frac{1}{M(\Delta)} E\Big(\iint_\Delta \varphi \, M(d\theta\, dt) \,\big|\mathfrak{A}_s \Big). \tag{24}$$

Proof. At first, let us show that *any* function $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$ is the limit $\varphi = \lim_{n\to\infty} \varphi^{(n)}$ of simple approximations of the form

$$\varphi^{(n)} = \sum \varphi^{(n)} \cdot 1_\Delta \text{ with } \varphi^{(n)} = \frac{1}{M(\Delta)} \iint_\Delta \varphi \, M(d\theta\, dt), \tag{25}$$

where the sum is taken on the sets $\Delta \subseteq \Theta \times (s, u]$ of the n^{th}-series of the partitions of $\Theta \times \mathbb{T}$. For $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$, there are some simple functions

$$\psi^{(n)} = \sum \psi^{(n)} 1_\Delta \text{ with } \Delta \subseteq \Theta \times (s, u] ,$$

such that

$$\varphi = \lim_{n\to\infty} \psi^{(n)}, \text{ i.e.} \quad \left\|\varphi - \psi^{(n)}\right\|_{L_2} \longrightarrow 0 , \qquad n \to \infty .$$

For the indicated values $\varphi^{(n)}$, $\psi^{(n)}$ on the n^{th}-series sets Δ, we have

$$\begin{aligned} \left\|\varphi^{(n)} - \psi^{(n)}\right\|^2 &= \left\| \frac{1}{M(\Delta)} \iint_\Delta (\varphi - \psi^{(n)}) \, M(d\theta\, dt) \right\|^2 \\ &\leq \frac{1}{M(\Delta)} \iint_\Delta \|\varphi - \psi^{(n)}\|^2 \, M(d\theta\, dt) . \end{aligned} \tag{26}$$

So, we also have

$$\left\|\varphi^{(n)} - \psi^{(n)}\right\|_{L_2}^2 \le \sum \iint_{\Delta} \|\varphi - \psi^{(n)}\|^2 \, M(d\theta\, dt) \;\le\; \left\|\varphi - \psi^{(n)}\right\|_{L_2}^2$$

which implies that

$$\left\|\varphi - \varphi^{(n)}\right\|_{L_2} \le 2 \left\|\varphi - \psi^{(n)}\right\|_{L_2} \longrightarrow 0\,, \qquad n \to \infty\,.$$

Next, let us turn to the non-anticipating functions φ such that, on some (which can be any) n^{th}-series sets $\Delta \subseteq \Theta \times (s,u]$ of the considered (3)-partitions, the values $\varphi(\theta,t)$ for $(\theta,t) \in \Delta$ are measurable with respect to the σ-algebras $\mathfrak{A}_s$. For these functions, when n is large enough ($n \to \infty$), the approximations (24) are identical to the approximations (25). Any non-anticipating function φ admits its approximations in $L_2(\Theta \times \mathbb{T} \times \Omega)$ by the above type functions. To explain, for any $(\theta,t) \in \Theta \times \mathbb{T}$ and any set $\Delta \subseteq \Theta \times (s,u]$ f the n^{th}-series of partitions of $\Theta \times \mathbb{T}$ such that $(\theta,t) \in \Delta$, the corresponding increasing σ-algebras $\mathfrak{A}_s$ have limit $\lim_{n\to\infty} \mathfrak{A}_s = \mathfrak{A}_t$. Thus

$$\varphi(\theta,t) = E\Big(\varphi(\theta,t)|\mathfrak{A}_t\Big) = \lim_{n\to\infty} E\Big(\varphi(\theta,t)\,\big|\,\mathfrak{A}_s\Big)$$

in $L_2(\Omega)$ and

$$\varphi = \lim_{n\to\infty} \sum E\Big(\varphi(\theta,t)|\mathfrak{A}_s\Big) \cdot 1_\Delta$$

in $L_2(\Theta \times \mathbb{T} \times \Omega)$. Cf. (3)-(5) and (16). So, we can see that φ is the limit $\varphi = \lim_{n\to\infty} \psi^{(n)}$ in $L_2(\Theta \times \mathbb{T} \times \Omega)$ of *some* non-anticipating simple functions $\psi^{(n)}$, $n = 1, 2, ...$, of the form

$$\psi^{(n)} \;=\; \sum \psi^{(n)} \cdot 1_\Delta$$

related to the sets $\Delta \subseteq \Theta \times (s,u]$ of the n^{th}-series of the partitions of $\Theta \times \mathbb{T}$ - see (3). Hence, by the same arguments applied for the approximations (25), we can conclude that φ is the limit $\varphi = \lim_{n\to\infty} \varphi^{(n)}$ of the non-anticipating simple functions (24). □

Now, let us consider the simple functions $\varphi = \sum \varphi \cdot 1_\Delta$ where for each component $\varphi \cdot 1_\Delta$ the indicated value φ on $\Delta \subseteq \Theta \times \mathbb{T}$ is measurable with respect to the corresponding σ-algebra

$$\mathfrak{A}_{]\Delta[} \;:\quad]\Delta[= \;\Theta \times \mathbb{T} \setminus \Delta\,, \tag{27}$$

generated by the stochastic measure μ over the complement set $]\Delta[$ to Δ - cf. (11). We have the following result - see [18].

Theorem 4. *Any function $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$ is the limit*

$$\varphi \;=\; \lim_{n\to\infty} \varphi^{(n)}, \;\; i.e. \quad \left\|\varphi - \varphi^{(n)}\right\|_{L_2} \longrightarrow 0\,, \quad n \to \infty,$$

of the simple functions $\varphi^{(n)}$, $n = 1, 2, \dots,$ *defined along the sets of the* n^{th} *-series of the partitions* (3) *as*

$$\varphi^{(n)} = \sum \varphi^{(n)} \cdot 1_{\Delta} \quad with \quad \varphi^{(n)} = \frac{1}{M(\Delta)} E\Big(\iint_{\Delta} \varphi\, M(d\theta\, dt) \,\Big|\, \mathfrak{A}_{]\Delta[} \Big). \quad (28)$$

Proof. The proof uses the same arguments as in the proof of Theorem 1.3. Here, to explain, we just note that for any $(\theta, t) \in \Theta \times \mathbb{T}$ and any set Δ of the n^{th}-series of the partitions of $\Theta \times \mathbb{T}$ such that $(\theta, t) \in \Delta$, we have

$$\varphi(\theta, t) = \lim_{n\to\infty} E\big(\varphi(\theta, t) | \mathfrak{A}_{]\Delta[}\big).$$

In fact the increasing σ-algebras $\mathfrak{A}_{]\Delta[}$ have limit $\lim_{n\to\infty} \mathfrak{A}_{]\Delta[} = \mathfrak{A}$, where $\mathfrak{A} = \mathfrak{A}_{\Theta\times\mathbb{T}}$ represents *all* the events in Ω. Cf. (3)-(5) and (13). □

2 The Itô non-anticipating integral.

2.1 A general definition and related properties.

The Itô integration scheme [26] (see also e.g. [39]) can be applied to the non-anticipating integration on the general space-time product $\Theta \times \mathbb{T}$ with respect to the stochastic measure $\mu = \mu(d\theta\, dt)$, $(\theta, t) \in \Theta \times \mathbb{T}$, of type (9): $E\mu = 0$, $E\mu^2 = M$. In particular, it can be applied in the modeling of stochastic processes of the form

$$\xi(t) = \iint_{\Theta\times(0,t]} \varphi\, \mu(d\theta\, ds)\,, \qquad t \in \mathbb{T}\,. \quad (1)$$

The term *non-anticipating* is referred to the family of σ-algebras

$$\mathfrak{A}_t, \quad t \in \mathbb{T},$$

which represent the flow of events in time - cf. (15). The integrands φ in (1) are the *non-anticipating* stochastic functions treated as elements of the functional space $L_2(\Theta \times \mathbb{T} \times \Omega)$ - cf. (20). The non-anticipating functions $\varphi = \varphi(\theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$, with $\varphi(\theta, t) \in L_2(\Omega)$ $\mathfrak{A}_t$-measurable, for any (θ, t), constitute the subspace

$$L_2^I(\Theta \times \mathbb{T} \times \Omega) \subseteq L_2(\Theta \times \mathbb{T} \times \Omega) \quad (2)$$

of all the integrands. To be more precise, this subspace is the closure of all the non-anticipating simple functions (24). Cf. Theorem 1.3.

Let us consider non-anticipating *simple* functions

$$\varphi = \sum \varphi \cdot 1_{\Delta}$$

where the sum is taken on a finite series of disjoint sets $\Delta \subseteq \Theta \times (s,u]$ $M(\Delta) < \infty$, and, for each component $\varphi \cdot 1_\Delta$, the $\mathfrak{A}_s$-measurable random variable $\varphi \in L_2(\Omega)$ is the value of μ on the indicated set Δ. The integration results in the random variable

$$I\varphi = \sum \varphi \cdot \mu(\Delta) \tag{3}$$

belonging to $L_2(\Omega)$. And here we have

$$\|I\varphi\| = \|\varphi\|_{L_2}.$$

Cf. (7) and (20). So, the integration formula (3) defines the *isometric* linear operator I:

$$L_2^I(\Theta \times \mathbb{T} \times \Omega) \ni \varphi \Longrightarrow I\varphi \in L_2(\Omega)$$

on the domain of all the non-anticipating simple functions, dense in $L_2^I(\Theta \times \mathbb{T} \times \Omega)$. The standard extension of this linear operator on $L_2^I(\Theta \times \mathbb{T} \times \Omega)$ is the *non-anticipating integral*

$$I\varphi = \iint_{\Theta \times \mathbb{T}} \varphi\, \mu(d\theta\, dt)\,.$$

Namely, for any $\varphi \in L_2^I(\Theta \times \mathbb{T} \times \Omega)$, i.e. the limit

$$\varphi = \lim_{n\to\infty} \varphi^{(n)} \text{ i.e. } \quad \|\varphi - \varphi^{(n)}\|_{L_2} \longrightarrow 0\,, \quad n \to \infty,$$

of the non-anticipating simple functions $\varphi^{(n)}$, $n = 1, 2, \dots$, we have

$$I\varphi = \lim_{n\to\infty} I\varphi^{(n)}, \text{ i.e. } \quad \|I\varphi - I\varphi^{(n)}\| \longrightarrow 0\ n \to \infty. \tag{4}$$

In particular, the integration can be carried through via the standard non-anticipating simple approximations of type (24).

For all the integrands, the integral

$$\iint_\Delta \varphi\, \mu(d\theta\, dt) := \iint_{\Theta \times \mathbb{T}} (\varphi \cdot 1_\Delta)\, \mu(d\theta\, dt)\,, \quad \Delta \subseteq \Theta \times \mathbb{T}, \tag{5}$$

is well-defined. Cf. (3)-(4).

In this line, all the functions φ of form

$$\varphi = \sum_k \varphi_k = \sum_k \varphi 1_{\Theta_k \times \mathbb{T}} \text{ where } \sum_k \Theta_k \times \mathbb{T} = \Theta \times \mathbb{T},$$

with the components $\varphi_k = \varphi 1_{\Theta_k \times \mathbb{T}}$, $k = 1, ..., K$ $(K \leq \infty)$, represent the integrands with respect to the measure μ as integrator on $\Theta_k \times \mathbb{T}$ - cf. (10).

We remark that

$$E\left(\iint_\Delta \varphi\, \mu(d\theta\, dt) \Big| \mathfrak{A}_s \right) = 0 \tag{6}$$

and

$$E\left(\iint_{\Delta}\varphi\,\mu(d\theta\,dt)\cdot\iint_{\Delta'}\varphi'\,\mu(d\theta\,dt)\Big|\mathfrak{A}_s\right) = \iint_{\Delta\bigcap\Delta'} E\left(\varphi\cdot\varphi'\big|\mathfrak{A}_s\right) M(d\theta\,dt) \tag{7}$$

for the integrands φ, φ' and Δ, $\Delta'\subseteq\Theta\times(s,T]$, $0\le s<T$.

The non-anticipating integration on general product spaces with the time component $\mathbb{T}$ was considered in [15]. With respect to a particular generalization of the Itô stochastic integral on the product space of the form $\mathbb{T}\times\mathbb{T}$ we can refer for example to [5].

Example 2.1. Let us consider the *optional* (*stopping*) time τ:

$$\{\tau\le t\}\in\mathfrak{A}_t,\quad t\in\mathbb{T},$$

and the *optional* σ-algebra $\mathfrak{A}_\tau$ of the events $\mathcal{A}\subseteq\Omega$ such that

$$\mathcal{A}\cap\{\tau\le t\}\in\mathfrak{A}_t,\quad t\in\mathbb{T}.$$

Thanks to the *right-continuity* of $\mathfrak{A}_t$, $t\in\mathbb{T}$, we have that the stochastic functions

$$\xi\cdot 1_{(\tau,T]}\cdot\varphi\in L_2(\Theta\times\mathbb{T}\times\Omega)$$

are integrands whatever $\mathfrak{A}_t$-measurable random variables ξ and integrands φ be applied. Moreover we have

$$\iint_{\Theta\times\mathbb{T}}\xi\cdot 1_{(\tau,T]}\varphi\,\mu(d\theta dt)=\xi\cdot\iint_{\Theta\times\mathbb{T}}1_{(\tau,T]}\varphi\,\mu(d\theta dt).$$

2.2 The stochastic Poisson integral.

As continuation of Example 1.2, we specify the Itô non-anticipating integral with respect to the *Poisson (centred) stochastic measure* $\mu:=\nu-E\nu$ treated through its Poisson components

$$\nu\;=\;\nu(\cdot,\omega),\;\;\omega\in\Omega:\quad E\nu\;=\;M\,.$$

Here, the pure discrete measures

$$\nu(\cdot,\omega)\;=\;\nu(\Delta,\omega)\,,\qquad\Delta\subseteq\Theta\times\mathbb{T}\,, \tag{8}$$

which depend on $\omega\in\Omega$ as parameter, are concentrated on the atoms

$$(\theta_\omega,t_\omega):\quad\nu(\theta_\omega,t_\omega,\omega)\equiv 1.$$

All these atoms are in one-to-one correspondence

$$\Theta\times\mathbb{T}\ni(\theta_\omega,t_\omega)\Longleftrightarrow t_\omega\in\mathbb{T}$$

with their time components.

The integrands φ are *predictable* functions in $L_2(\Theta \times \mathbb{T} \times \Omega)$. We assume that they satisfy

$$\iint_{\Theta\times\mathbb{T}} \big(E\,|\varphi|\big)\, M(d\theta\, dt) < \infty. \tag{9}$$

Note that (9) holds for all the integrands in the case M is a finite measure - cf. (20). Now let us consider the Poisson stochastic measure

$$\nu \;=\; \nu(\Delta, \omega)\,, \qquad \Delta \subseteq \Theta\times\mathbb{T} \qquad (\omega\in\Omega),$$

and the product-measure $\nu\times P$ on $\Theta\times\mathbb{T}\times\Omega$ with values

$$(\nu\times P)(\Delta\times\mathcal{A}) = \iiint_{\Delta\times\mathcal{A}} \nu(d\theta\, dt, \omega)\times P(d\omega)$$

on the product-sets $\Delta\times\mathcal{A}$: $\Delta\subseteq\Theta\times\mathbb{T}$, $\mathcal{A}\subseteq\Omega$ - cf. (21). In particular we can see that

$$\nu\times P \;\equiv\; M\times P$$

on the predictable σ-algebra, i.e. the σ-algebra generated by the product-sets $\Delta\times\mathcal{A}$: $\Delta\subseteq\Theta\times(s,u]$, $\mathcal{A}\subseteq\mathfrak{A}_s$ - cf. (23). To explain, we have

$$\iiint_{\Delta\times\mathcal{A}} \nu(d\theta\, dt, \omega)\times P(d\omega)$$
$$= E\,\big(1_{\mathcal{A}}\times\nu(\Delta)\big) = E\,1_{\mathcal{A}}\times E\,\nu(\Delta) = \iiint_{\Delta\times\mathcal{A}} M(d\theta\, dt)\times P(d\omega)$$

since the values $\nu(\Delta)\,:\,\Delta\subseteq\Theta\times(s,u]$, are *independent* from the events $\mathcal{A}\in\mathfrak{A}_s$. For the predictable function φ which, we recall, is a function measurable with respect to the predictable σ-algebra, the condition (9) says that

$$\iiint_{\Theta\times\mathbb{T}\times\Omega} |\varphi|\,\nu(d\theta\, dt,\omega)\times P(d\omega) \;=\; \iint_{\Theta\times\mathbb{T}} \big(E|\varphi|\big)\, M(d\theta\, dt) < \infty\,.$$

Accordingly, the *stochastic Poisson integral*

$$\iint_{\Delta} \varphi\,\nu(d\theta\, dt) \;:=\; \iint_{\Delta} \varphi(\cdot,\omega)\,\nu(d\theta\, dt,\omega)\,, \qquad \omega\in\Omega\,,$$

is well-defined via the realizations (trajectories) $\varphi(\cdot,\omega) = \varphi(\theta,t,\omega)$, $(\theta,t)\in\Theta\times\mathbb{T}$, integrable with respect to the measures $\nu(\cdot,\omega)$ for almost all $\omega\in\Omega$. In this scheme, we can see that the Itô non-anticipating integral with respect to the integrator $\mu=\nu-M$ is related to the stochastic Poisson integral in the following way:

$$\iint_{\Delta} \varphi\,\mu(d\theta\, dt) \;=\; \iint_{\Delta} \varphi\,\nu(d\theta\, dt) \;-\; \iint_{\Delta} \varphi\, M(d\theta\, dt)\,, \qquad \Delta\subseteq\Theta\times\mathbb{T}\,. \tag{10}$$

This is obvious for the non-anticipating simple functions φ and it is true in general via the limit

$$\varphi = \lim_{n\to\infty} \varphi^{(n)}, \text{ i.e. } \quad \left\|\varphi - \varphi^{(n)}\right\|_{L_2} \longrightarrow 0\,, \qquad n \to \infty\,,$$

of the non-anticipating simple functions $\varphi^{(n)}$, $n = 1, 2, ...$, thanks to the identity

$$\iiint_{\Theta\times\mathbb{T}\times\Omega} \left|\varphi - \varphi^{(n)}\right|^2 M(d\theta\, dt) \times P(d\omega) \\ = \iiint_{\Theta\times\mathbb{T}\times\Omega} \left|\varphi - \varphi^{(n)}\right|^2 \nu(d\theta\, dt, \omega) \times P(d\omega).$$

With respect to the representation (10), we remark that the stochastic Poisson integral is actually

$$\iint_{\Delta} \varphi\, \nu(d\theta\, dt) = \sum_{(\theta_\omega, t_\omega)\in\Delta} \varphi(\theta_\omega, t_\omega, \omega)\,, \qquad \omega \in \Omega, \tag{11}$$

and the above stochastic series converges absolutely with

$$\int_{\Omega} \Big[\sum_{(\theta_\omega, t_\omega)\in\Delta} \left|\varphi(\theta_\omega, t_\omega, \omega)\right| \Big] P(d\omega) = \iint_{\Delta} \big(E|\varphi|\big)\, M(d\theta\, dt) < \infty\,.$$

Cf. (8)-(9).

2.3 The jumping stochastic processes.

To continue the scheme (8)-(11), we apply it to the cadlag stochastic processes

$$\xi(t) = \iint_{\Theta\times(0,t]} \varphi\mu(d\theta\, ds)\,, \qquad t \in \mathbb{T}\,,$$

- cf. (1), where $\varphi = \varphi(\theta, t, \omega)$, $(\theta, t, \omega) \in \Theta \times \mathbb{T} \times \Omega$, are the real predictable integrands with respect to the Poisson (centred) stochastic measure μ of the type $E\mu = 0$, $E\mu^2 = M$, treated as $\mu = \nu - E\nu$ through its Poisson component $\nu = \nu(\cdot, \omega)$, $\omega \in \Omega \,:\, E\nu = M$. Recall that the involved pure discrete measures $\nu(\cdot, \omega) = \nu(\Delta, \omega)$, $\Delta \subseteq \Theta \times \mathbb{T}$, have atoms $(\theta_\omega, t_\omega) \in \Theta \times \mathbb{T}$ which are in one-to-one correspondence $(\theta_\omega, t_\omega) \Leftrightarrow t_\omega$ with the times $t_\omega \in \mathbb{T}$. Hence we can see that *all* the jumps of the realizations (trajectories) of the above process $\xi(t)$, $t \in \mathbb{T}$:

$$\xi(t, \omega) = \sum_{0<t_\omega\le t} \varphi(\theta_\omega, t_\omega, \omega) \\ - \iint_{\Theta\times(0,t]} \varphi(\cdot, \omega)\, M(d\theta\, dt)\,, \qquad t \in \mathbb{T} \quad (\omega \in \Omega)\,, \tag{12}$$

are

$$\rho_\omega := \xi(t_\omega,\omega) - \xi(t_\omega - 0,\omega) \equiv \varphi(\theta_\omega, t_\omega, \omega)\,, \qquad (\theta_\omega, t_\omega) \in \Theta \times \mathbb{T}\,.$$

Cf. (10)-(11). Accordingly, whatever real function

$$F(x,\cdot) = F(x,\theta,t,\omega)\,, \qquad (x,\theta,t,\omega) \in \mathbb{R} \times \Theta \times \mathbb{T} \times \Omega\,,$$

be considered such that $F(\varphi,\cdot)$ is a predictable integrand with

$$\iint_{\Theta\times\mathbb{T}} \Big(E\big|F(\varphi,\cdot)\big|\Big)\, M(d\theta\, dt) < \infty$$

- cf. (9), we obtain that, for *all* the jumps $\rho_\omega := \xi(t_\omega,\omega) - \xi(t_\omega - 0,\omega)$ of the trajectories (12), the corresponding trajectories

$$\eta(t,\omega) = \sum_{0<t_\omega\le t} F(\rho_\omega, \theta_\omega, t_\omega, \omega) - \iint_{\Theta\times(0,t]} F(\varphi,\cdot)\, M(d\theta\, dt), \quad t\in\mathbb{T} \quad (\omega\in\Omega), \quad (13)$$

represent the stochastic process

$$\eta(t) = \iint_{\Theta\times(0,t]} F(\varphi,\cdot)\, \mu(d\theta\, ds), \quad t\in\mathbb{T}.$$

Now let φ and $F(\varphi,\cdot) = F(\varphi(\theta,t),\theta,t)$, $(\theta,t) \in \Theta\times\mathbb{T}$, be *deterministic* real functions. Then $\eta(t)$, $t\in\mathbb{T}$, here above is the process with independent increments characterized by the infinitely-divisible probability law

$$\log e^{i\lambda\eta(t)} = \iint_{\Theta\times(0,t]} \big(e^{i\lambda F(\varphi,\cdot)} - 1 - i\lambda F(\varphi,\cdot)\big)\, M(d\theta\, ds), \quad \lambda\in\mathbb{R} \quad (t\in\mathbb{T}).$$

Example 2.2. In relation to the probability law (17), let us turn to the Poisson (centred) stochastic measure μ on the space-time product $(\mathbb{R}\setminus[0])\times \Theta\times\mathbb{T}$:

$$E\mu = 0\,, \qquad E\mu^2 = L\times M\,,$$

with the variance represented by the standard product-measure $L\times M$ on $(\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}$ with the component $L = L(B,\theta,t)$, $B\subseteq \mathbb{R}\setminus[0]$ such that

$$\int_{\mathbb{R}\setminus[0]} x^2 L(dx,\theta,t) \le 1\,.$$

Assuming that Θ is a complete separable metric space, we can apply the scheme generally described in (8)-(13) and consider the stochastic measure $\mu = \nu - E\nu$, with the Poisson component

$$\nu = \nu(\cdot,\omega), \qquad \omega \in \Omega : E\nu = L \times M$$

on $(\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T}$ represented by the pure discrete measures $\nu(\cdot,\omega)$ having atoms

$$(x_\omega, \theta_\omega, \omega) \in (\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T}.$$

For the sets

$$\Delta \subseteq \Theta \times (s,u] : \quad M(\Delta) < \infty$$

in $\Theta \times \mathbb{T}$ and

$$B \subseteq \{|x| > r\} : \quad r > \epsilon > 0$$

in $\mathbb{R} \setminus [0]$, let us consider the function

$$\varphi = 1_\Delta(\theta,t) x \, 1_{\{|x|>\epsilon\}} , \qquad (x,\theta,t) \in (\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T} ,$$

and the integrand

$$F(\varphi,\cdot) := 1_B(\varphi) \equiv 1_{B\times\Delta}$$

on $(\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T}$. Let

$$\xi(t) = \iiint_{(\mathbb{R}\setminus[0])\times\Theta\times(0,t]} 1_\Delta x \, 1_{\{|x|>\epsilon\}} \mu(dx\, d\theta\, ds) , \qquad t \in \mathbb{T} ,$$

be the corresponding cadlag process. We can see that

$$\mu(B \times \Delta) = \sum_{s<t_\omega\le u} 1_B(\rho_\omega) - \iint_\Delta L(B,\theta,t) \times M(d\theta\, dt), \tag{14}$$

where ρ_ω are the jumps of $\xi(t,\omega)$, $t \in \mathbb{T}$. Note that here the actually involved jumps

$$\rho_\omega = x_\omega : (x_\omega, \theta_\omega, t_\omega) \in B \times \Delta$$

are the same for all $\epsilon : 0 < \epsilon < r$. Cf. (13). Accordingly, formula (14) holds true for any $r > 0$ and $\epsilon = 0$, i.e. for

$$\rho_\omega = \xi(t_\omega,\omega) - \xi(t_\omega - 0,\omega)$$

as the jumps of the cadlag process

$$\xi(t) = \iiint_{(\mathbb{R}\setminus[0])\times\Theta\times(0,t]} 1_\Delta x \, \mu(dx\, d\theta\, ds) , \qquad t \in \mathbb{T} . \tag{15}$$

In fact the component

$$\xi_\epsilon(t) := \iiint_{(\mathbb{R}\setminus[0])\times\Theta\times(0,t]} 1_\Delta x \, 1_{\{|x|\le\epsilon\}} \, \mu(dx\, d\theta\, ds) , \qquad t \in \mathbb{T},$$

is negligible, for $\epsilon \to 0$, in the above process $\xi(t)$, $t \in \mathbb{T}$, with

$$\xi(t) = \iiint_{(\mathbb{R}\setminus[0])\times\Theta\times(0,t]} 1_\Delta x \, 1_{\{|x|>\epsilon\}} \mu(dx\, d\theta\, ds) + \iiint_{(\mathbb{R}\setminus[0])\times\Theta\times(0,t]} 1_\Delta x \, 1_{\{|x|\le\epsilon\}} \mu(dx\, d\theta\, dt).$$

2.4 Gaussian-Poisson stochastic measures.

Let us turn to the *Gaussian* stochastic measure μ^G on $\Theta \times \mathbb{T}$:

$$E\mu^G = 0, \qquad E(\mu^G)^2 = \sigma^2 \cdot M,$$

and the *Poisson* (centred) stochastic measure μ^P on $(\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T}$:

$$E\mu^P = 0, \qquad E(\mu^P)^2 = L \times M.$$

The formula

$$\mu(\Delta) := \iint_\Delta \mu^G(d\theta\, dt) + \iiint_{(\mathbb{R}\setminus[0])\times\Delta} x\, \mu^P(dx\, d\theta\, dt), \qquad \Delta \subseteq \Theta \times \mathbb{T}, \quad (16)$$

defines the stochastic measure μ on $\Theta \times \mathbb{T}$ characterized by the infinitely-divisible probability law (17) with the above parameters σ^2, L and M such that $E\mu = 0$ and $E\mu^2 = M$. Here, we treat μ^P as in the general framework of Example 2.2.

The Poisson (centred) stochastic measure μ^P can be determined as

$$\mu^P(B \times \Delta) = \sum_{s<t_\omega\leq u} 1_B(\rho^P_\omega) - \iint_\Delta L(B, \theta, t) \times M(d\theta\, dt), \qquad (17)$$

on the sets of form

$$B \times \Delta: \quad B \subseteq \{|x| > r\}, \ \ r > 0, \quad \Delta \subseteq \Theta \times (s, u] \, : \, M(\Delta) < \infty,$$

via the jumps $\rho^P_\omega := \xi^P(t_\omega, \omega) - \xi^P(t_\omega - 0, \omega)$ of the trajectories $\xi^P(t, \omega)$, $t \in \mathbb{T}$, of the processes of type

$$\xi^P(t) := \iiint_{(\mathbb{R}\setminus[0])\times\Theta\times(0,t]} 1_\Delta x\, \mu^P(dx\, d\theta\, ds), \qquad t \in \mathbb{T}.$$

Cf. (14)-(15). For any $\Delta \subseteq \Theta \times \mathbb{T}$: $M(\Delta) < \infty$, the above process is a component in

$$\xi(t) := \iint_{\Theta\times(0,t]} 1_\Delta \mu(d\theta\, ds) = \xi^G(t) + \xi^P(t), \qquad t \in \mathbb{T}, \qquad (18)$$

- cf. (16). Here, the other component

$$\xi^G(t) = \iint_{\Theta\times(0,t]} 1_\Delta\, \mu^G(d\theta\, ds), \qquad t \in \mathbb{T},$$

is a *Gaussian* process with independent increments having *continuous* variance - cf. (3). These Gaussian processes are similar to the Wiener process. In particular, their cadlag versions have actually *continuous* trajectories $\xi^G(t, \omega)$, $t \in \mathbb{T}$, for almost all $\omega \in \Omega$. Accordingly, the trajectories $\xi(t, \omega)$, $t \in \mathbb{T}$, of the processes (18) have jumps as

$$\rho_\omega := \xi(t_\omega,\omega) - \xi(t_\omega - 0,\omega) \equiv \xi^P(t_\omega,\omega) - \xi^P(t_\omega - 0,\omega) = \rho^P_\omega .$$

Hence, the stochastic measure μ^P can be determined through the stochastic processes (18) as

$$\mu^P(B\times\Delta) = \sum_{s<t_\omega\le u} 1_B(\rho_\omega) - \iint_\Delta L(B,\theta,t)\times M(d\theta\,dt), \quad \omega\in\Omega. \tag{19}$$

Cf. (17). So, the stochastic measures μ^G, μ^P in the representation (16) are *uniquely* determined by μ.

Moreover, let us consider the *Gaussian-Poisson mixture* $\mu^{G,P}$ on the space-time product

$$\mathbb{R}\times\Theta\times\mathbb{T} = \Big([0]\times\Theta\times\mathbb{T}\Big)\cup\Big((\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}\Big)$$

of the independent Gaussian stochastic measure μ^G on $\Theta\times\mathbb{T}$, identified with $[0]\times\Theta\times\mathbb{T}$, and the Poisson (centred) stochastic measure μ^P on $(\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}$ - cf. (10). We can see that

$$\mathfrak{A}_{\mathbb{R}\times\Theta\times(0,t]} \equiv \mathfrak{A}_{\Theta\times(0,t]}\,, \qquad t\in\mathbb{T}\,, \tag{20}$$

for the σ-algebras generated in the course of time by $\mu^{G,P}$ and μ, correspondingly. Cf. (15)-(16).

Let Θ be a general complete separable metric space. Within the framework described in (8)-(20), we obtain the following result. See e.g. [25], [51] in the case of stochastic processes.

Theorem 5. *The representation (16) holds for a general stochastic measure with independent values characterized by the probability law* (17)*.*

Proof. Let μ be a general stochastic measure with independent values characterized by the probability law (17). For an appropriate probability space $\widetilde{\Omega}$ there exist the independent Gaussian stochastic measure $\widetilde{\mu}^G$ and Poisson (centred) stochastic measure $\widetilde{\mu}^P$ for which

$$\widetilde{\mu}(\Delta) = \iint_\Delta \widetilde{\mu}^G(d\theta\,dt) + \iiint_{(\mathbb{R}\setminus[0])\times\Delta} x\,\widetilde{\mu}^P(dx\,d\theta\,dt)\,, \qquad \Delta\subseteq\Theta\times\mathbb{T}\,,$$

is a stochastic measure with the same probability law as μ. We have

$$\widetilde{\mathfrak{A}}_{(\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}} = \widetilde{\mathfrak{A}}_{\Theta\times\mathbb{T}}$$

for the σ-algebras generated by the Gaussian-Poisson mixture $\widetilde{\mu}^{G,P}$ over $(\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}$ and the stochastic measure $\widetilde{\mu}$ over $\Theta\times\mathbb{T}$ - cf. (20). For the random variables $\widetilde{\xi}$ on $\widetilde{\Omega}$, measurable with respect to the σ-algebra $\widetilde{\mathfrak{A}}_{\Theta\times\mathbb{T}}$ generated by $\widetilde{\mu}$, we have the linear isometry

$$L_2(\widetilde{\Omega}) \ni \widetilde{\xi} \Longrightarrow \xi \in L_2(\Omega)\,,$$

defined through the mapping

$$\widetilde{\xi} \;=\; F\Big(\widetilde{\mu}(\Delta_1), ..., \widetilde{\mu}(\Delta_m)\Big) \Longrightarrow F\Big(\mu(\Delta_1), ..., \mu(\Delta_m)\Big) \;=\; \xi$$

of all the functions of all the values of $\widetilde{\mu}$, μ. This mapping preserves the finite-dimensional probability distributions. Hence the above isometry yields

$$\mu^G(\Delta) \,:\, \widetilde{\mu}^G(\Delta) \Longrightarrow \mu^G(\Delta)\,, \qquad \Delta \subseteq \Theta \times \mathbb{T}\,,$$

and

$$\mu^P(B \times \Delta): \quad \widetilde{\mu}^P(B \times \Delta) \Longrightarrow \mu^P(B \times \Delta), \quad B \times \Delta \subseteq (\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T},$$

as the independent Gaussian and Poisson (centred) stochastic measures for which we have

$$\mu(\Delta) \;=\; \iint_{\Delta} \mu^G(d\theta\, dt) + \iiint_{(\mathbb{R}\setminus[0])\times\Delta} x\, \mu^P(dx\, d\theta\, dt), \quad \Delta \subseteq \Theta \times \mathbb{T}. \quad \square$$

3 The non-anticipating integral representation.

3.1 Multilinear polynomials and Itô multiple integrals.

For being able to model stochastic processes via stochastic integration in the course of time - cf. (1), it is fundamental to characterize the random variables $\xi \in L_2(\Omega)$ which admit the non-anticipating integral representation

$$\xi \;=\; E\xi + \iint_{\Theta\times\mathbb{T}} \varphi\, \mu(d\theta\, dt). \tag{1}$$

Let μ be a general stochastic measure of the type $E\mu = 0$, $E\mu^2 = M$ - cf. (9). In the sequel, we focus on the random variables which are limits in $L_2(\Omega)$ of multilinear polynomials of the values of μ (hereafter μ-values). By *multilinear polynomial* we mean a linear combination of the p-power ($p = 1, 2, ...$) multilinear forms

$$\xi \;=\; \prod_{j=1}^{p} \xi_j \;\text{ with } \quad \xi_j \;=\; \mu(\Delta_j), \quad j = 1, ..., p, \tag{2}$$

of the μ-values taken on the *disjoint* sets $\Delta_j \subseteq \Theta \times \mathbb{T} \,:\, M(\Delta_j) < \infty$, $j = 1, ..., p$, plus the constants (which formally correspond to $p = 0$).

Theorem 6. *The multilinear polynomials admit the non-anticipating integral representation (1).*

Proof. The result is immediate for the multilinear forms (2) of the μ-values on the sets $\Delta_j \subseteq \Theta \times (s_j, u_j]$, $j = 1, ..., p$, related to the *disjoint* time intervals $(s_j, u_j] \subseteq \mathbb{T}$, $j = 1, ..., p$, on $\mathbb{T} = (0, T]$. Indeed, taking these intervals ordered in time $0 < s_1 < u_1 \leq ... \leq s_p < u_p \leq T$, we can see that

$$\xi \; = \iint_{\Delta_p} \left(\prod_{j=1}^{p-1} \xi_j \right) \mu(d\theta \, dt) \, .$$

The range of the non-anticipating integral, as an isometric linear operator, is closed. Cf. (1)-(4). So, ξ admits the representation (1) if $\xi = \lim_{n\to\infty} \xi^{(n)}$ is the limit in $L_2(\Omega)$ of the linear combinations of the above type multilinear forms of the μ-values. Hereafter we refer to the limit $\xi = \lim_{n\to\infty} \xi^{(n)}$ as the *proper approximation*, when ξ is a general multilinear form (2) of the μ-values on the disjoint sets Δ_j, $j = 1, ..., p$ and $\xi^{(n)}$, $n = 1, 2, ...$, are multilinear forms which involve only the μ-values on the subsets in $\Delta = \sum_{j=1}^{p} \Delta_j$.

In general, for the limits $\xi_k = \lim_{n\to\infty} \xi_k^{(n)}$, $k = 1, ..., m$, with the independent approximations $\{\xi_k^{(n)}, \, n = 1, 2, ...\}$, $k = 1, ..., m$, we have

$$\prod_{k=1}^{m} \xi_k = \lim_{n\to\infty} \prod_{k=1}^{m} \xi_k^{(n)},$$
$$\text{i.e.} \quad \big\| \prod \xi - \prod \xi^{(n)} \big\| \leq const \cdot \max_k \big\| \xi_k - \xi_k^{(n)} \big\| \longrightarrow 0, \; n \to \infty.$$

Keeping this in mind, let us suppose that all multilinear forms of the power $p < q$ $(q > 1)$ admit a proper approximation. The claim trivially holds for $p = 1$. Considering $\xi = \prod_{k=1}^{q} \mu(\Delta_k)$ as a general q-power multilinear form (2) through the n^{th} -partitions

$$\Delta_k \; = \sum \Big(\Delta_k \cap \big(\Theta \times (s, u] \big) \Big) \, , \quad \sum (s, u] = \mathbb{T} \, : \, \max(u - s) \longrightarrow 0 \, , \quad n \to \infty \, ,$$

we can see that

$$\big\| \xi_0^{(n)} \big\|^2 := \sum \prod_{k=1}^{q} M \Big(\Delta_k \cap \big(\Theta \times (s, u] \big) \Big)$$
$$\leq const \cdot \max M \Big(\Delta_k \cap \big(\Theta \times (s, u] \big) \Big) \to 0, \quad n \to \infty,$$

for

$$\xi_0^{(n)} := \sum \prod_{k=1}^{q} \mu \Big(\Delta_k \cap \big(\Theta \times (s, u] \big) \Big), \quad n = 1, 2, ...$$

- cf. (5). We can also see that the differences

$$\xi^{(n)} := \xi - \xi_0^{(n)} = \prod_{k=1}^{q} \Big[\sum \mu\Big(\Delta_k \cap \big(\Theta \times (s,u]\big)\Big)\Big] - \sum \prod_{k=1}^{q} \mu\Big(\Delta_k \cap \big(\Theta \times (s,u]\big)\Big)$$

admit proper approximation. The same holds for ξ as the limit

$$\xi = \lim_{n\to\infty} \xi^{(n)}$$

in $L_2(\Omega)$. $\square$

Let us now turn to $H^p \subseteq L_2(\Omega)$ as the linear closure of all the p-power multilinear forms (2). The subspaces H^p $(p = 1, 2, ...)$ are orthogonal. Let us consider

$$H := \sum_{p=0}^{\infty} \oplus H^p \tag{3}$$

which is the standard orthogonal sum of H^p, $p = 1, 2, ...$, in $L_2(\Omega)$ where H^0 represents the set of all the constants. We remark that *all random variables* $\xi \in H$ *admit the non-anticipating integral representation (1).* Cf. Theorem 3.1.

The representation (1) of the elements in $\xi \in H^p$, $p > 1$, can be specified by means of the Itô type *multiple integrals* [27] (see also e.g. [53]). Here, we have in mind the *p-multiple integrals*

$$I^p\varphi_p = \int \dots \int_{\{t_1<\dots<t_p\}} \varphi_p \, \mu(d\theta_1\, dt_1) \times \dots \times \mu(d\theta_p\, dt_p)\,, \qquad p > 1\,, \tag{4}$$

over the indicated domain $\{t_1 < ... < t_p\}$ in the p-times product $(\Theta \times \mathbb{T})^p$ which consists of

$$(\theta_1, t_1, ..., \theta_p, t_p) \in (\Theta \times \mathbb{T})^p : \quad t_1 < ... < t_p.$$

The integrator in (4) is the standard type *stochastic measure with orthogonal values* defined on the product-sets

$$\Delta_1 \times ... \times \Delta_p \subseteq \{t_1 < ... < t_p\}$$

as the product $\mu(\Delta_1) \times ... \times \mu(\Delta_p)$:

$$E\Big(\mu(\Delta_1) \times ... \times \mu(\Delta_p)\Big) = 0\,,$$
$$E\Big(\mu(\Delta_1) \times ... \times \mu(\Delta_p)\Big)^2 = M(\Delta_1) \times ... \times M(\Delta_p)\,.$$

The integrands φ_p are the deterministic functions

$$\varphi_p = \varphi_p(\theta_1, t_1, ..., \theta_p, t_p), \quad (\theta_1, t_1, ..., \theta_p, t_p) \in \{t_1 < ... < t_p\},$$

in the standard (complex) L_2-space with the norm

$$\|\varphi_p\|_{L_2} = \Big(\int ... \int_{\{t_1<...<t_p\}} |\varphi_p|^2 \, M(d\theta_1\, dt_1) \times ... \times M(d\theta_p\, dt_p)\Big)^{1/2}.$$

So, in (4) we have the standard stochastic integral $I^p\varphi_p$: $\|I^p\varphi_p\| = \|\varphi\|_{L_2}$. The p-power multilinear forms (2) of the μ-values on $\Delta_j \subseteq \Theta\times(s_j, u_j]$, $j = 1, ..., p$, related to the *disjoint* time intervals (ordered according to $0 \le s_1 < u_1 \le ... \le s_p < u_p \le T$) are identical to the p-multiple integrals (4) with the indicators $\varphi_p = 1_{\Delta_1\times...\times\Delta_p}$ as integrands. Hence, following the proof of Theorem 3.1, we can see that *all* $\xi_p \in H^p$ are represented by *all* the p-multiple integrals $I^p\varphi_p$. For $\xi_p = I^p\varphi_p$, the non-anticipating integral representation (1) can be given as

$$\xi_p = \iint_{\Theta\times\mathbb{T}} I^{p-1}\varphi_p(\cdot,\theta,t)\, \mu(d\theta\, dt). \tag{5}$$

Here the function

$$\varphi_p(\cdot,\theta,t) = \varphi_p(\theta_1, t_1, ..., \theta_{p-1}, t_{p-1}, \theta, t),$$
$$(\theta_1, t_1, ..., \theta_{p-1}, t_{p-1}) \in \{t_1 < ... < t_{p-1}\},$$

with $(\theta, t) \in \Theta \times \mathbb{T}$ as parameter, is the integrand in the $(p-1)$-multiple integral.

Of course, in the case $p = 1$, the non-anticipating integral representation (1) of the elements $\xi \in H^p$ is trivial:

$$\xi_1 = \iint_{\Theta\times\mathbb{T}} \varphi_1\, \mu(d\theta\, dt),$$

with the *deterministic* integrands φ_1: $\|\varphi_1\|_{L_2} = (\iint_{\Theta\times\mathbb{T}} |\varphi_1|^2 M(d\theta\, dt))^{1/2}$. In line with the case $p > 1$, we write $\xi_1 = I^1\varphi_1$: $\varphi_1 = I^0\varphi_1(\cdot,\theta,t)$, $(\theta,t) \in \Theta\times\mathbb{T}$, for the above stochastic integral. Clearly, the representation (1) of all $\xi \in H$ in the subspace $H \in L_2(\Omega)$ - cf. (3), is obtained via the orthogonal sum

$$\xi = \sum_{p=0}^{\infty} \oplus\, \xi_p : \qquad \xi_p \in H^p, \quad p = 0, 1, ...,$$

with $\xi_0 = E\xi$ and

$$\xi_p = \iint_{\Theta\times\mathbb{T}} I^{p-1}\varphi_p(\cdot,\theta,t)\, \mu(d\theta\, dt). \quad p = 1, 2, ...,$$

This yields

$$\xi = E\xi \oplus \iint_{\Theta\times\mathbb{T}} \Big[\sum_{p=1}^{\infty} \oplus I^{p-1}\varphi_p(\cdot,\theta,t)\Big]\, \mu(d\theta\, dt). \tag{6}$$

Here we refer to [16] and [28]. See also e.g. [42] for some results on Lévy processes.

3.2 Integral representations with Gaussian-Poisson integrators.

Let $\mu = \mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, be a general *Gaussian-Poisson mixture*, i.e. the mixture of the components μ_k, $k = 0, 1, \dots$, which are either Gaussian or Poisson (centred) stochastic measures multiplied by scalars - cf. (10) and (16)-(19). Let us consider the subspaces

$$H_q := \sum_{p=0}^{q} \oplus H^p, \quad q = 1, 2, ...,$$

in $L_2(\Omega)$. Cf. (3).

Theorem 7. *The q-power polynomials of the values of μ belong to H_q, $q = 1, 2, \dots$.*

Proof. The proof is quite similar to the one of Theorem 3.1. All the q-power multilinear polynomials belong to H_q $(q = 1, 2, ...)$. So $\xi \in H_q$ if it can be represented as limit $\xi = \lim_{n\to\infty} \xi^{(n)}$ in $L_2(\Omega)$ of the q-power multilinear polynomials $\xi^{(n)}$, $n = 1, 2, \dots$. Let ξ be a q-power polynomial of μ-values, we can treat ξ as the q-power polynomial of the values $\mu(\Delta_j)$, $j = 1, ..., m$, on appropriately choosen *disjoint* sets $\Delta_j \subseteq \Theta \times \mathbb{T}$, $j = 1, ..., m$. Accordingly, we refer to the limit $\xi = \lim_{n\to\infty} \xi^{(n)}$ of the q-power multilinear polynomials $\xi^{(n)}$, $n = 1, 2, ...$, of the values of μ just on the subsets in $\Delta = \sum_{j=1}^{m} \Delta_j$ as the *proper approximation*. The proper approximation holds for $p = 1$. Suppose it holds for the polynomials of power $p < q$, $(q > 1)$. Than we can see that the proper approximation does hold for all the q-power polynomials, if it holds for

$$\xi = \mu(\Delta)^q , \qquad \Delta \subseteq \Theta \times \mathbb{T}.$$

Moreover, note that here it is enough to consider the sets Δ where μ is either the Gaussian or the Poisson (centred) stochastic measure. Let us take the n^{th}-partitions

$$\Delta = \sum \Delta \cap (\Theta \times (s, u]), \qquad \sum (s, u] = \mathbb{T} : \max(u - s) \longrightarrow 0, \quad n \to \infty,$$

into account. We can see that the limit $\lim_{n\to\infty} \xi_0^{(n)} = \xi_0$ in $L_2(\Omega)$ with

$$\xi_0^{(n)} := \sum \mu\Big(\Delta \cap \big(\Theta \times (s, u]\big)\Big)^q , \qquad n = 1, 2, \dots ,$$

has the following form: $\xi_0 = \mu(\Delta)$, $q = 2$, or $\xi_0 = 0$, $q > 2$, if μ is Gaussian and $\xi_0 = \mu(\Delta) + M(\Delta)$ if μ is the Poisson (centred) stochastic measure. In all cases we can say that ξ_0 admits proper approximation. Following the arguments applied in the proof of Theorem 3.1, we can also see that the differences

$$\xi - \xi_0^{(n)} = \Big[\sum \mu\Big(\Delta \cap \big(\Theta \times (s, u]\big)\Big)\Big]^q - \sum \mu\Big(\Delta \cap \big(\Theta \times (s, u]\big)\Big)^q, \quad n = 1, 2, \dots ,$$

admit proper approximation as well. So, such approximation holds also for $\xi = \mu(\Delta)^q$ as the limit

$$\xi = \lim_{n\to\infty} \xi^{(n)} \text{ with } \quad \xi^{(n)} = (\xi - \xi_0^{(n)}) + \xi_0, \quad n = 1, 2, \dots,$$

in $L_2(\Omega)$. □

In the sequel it is important that the σ-algebra is generated by the stochastic measure μ, i.e.

$$\mathfrak{A} := \mathfrak{A}_{\Theta\times\mathbb{T}}$$

- cf. (12), and that the flow of events in the course of time is represented by the σ-algebras (15)

$$\mathfrak{A}_t := \mathfrak{A}_{\Theta\times(0,t]}, \quad t \in \mathbb{T}.$$

Note that the polynomials of the values $\mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, are dense in $L_2(\Omega)$ when μ is a general stochastic measure with independent values which obeys the probability law (17) and restricted by the condition

$$Ee^{\lambda\mu(\Delta)} = \exp \iint_{\Delta} \Big[\frac{\lambda^2}{2}\sigma^2(\theta,t) + \int_{\mathbb{R}\setminus[0]} (e^{\lambda x} - 1 - \lambda x)\, L(dx,\theta,t)\Big] M(d\theta\, dt) < \infty, \quad \lambda \in \mathbb{R}. \quad (7)$$

To explain, for the *complete system* of functions of the form

$$e^{i\sum_{k=1}^m \lambda_k\xi_k} \qquad (\lambda_k \in \mathbb{R},\ k = 1, ..., m),$$

with the values $\xi_k = \mu(\Delta_k)$, $k = 1, ..., m$, taken on all finite combinations of disjoint sets in $\Theta \times \mathbb{T}$, we have

$$\Big\| e^{i\sum_{k=1}^m \lambda_k\xi_k} - \sum_{p=0}^{q} \frac{(i\sum_{k=1}^m \lambda_k\xi_k)^p}{p!} \Big\| \longrightarrow 0, \quad q \to \infty.$$

In the following result we do consider that, for a general Gaussian-Poisson mixture μ, the polynomials of the values of μ are dense in $L_2(\Omega)$. See e.g. [10], [11], [16], [28], [42].

Theorem 8. *All the elements* $\xi \in L_2(\Omega)$*:*

$$L_2(\Omega) = \sum_{p=0}^{\infty} \oplus H^p, \quad (8)$$

admit the non-anticipating integral representation (1).

Proof. Cf. (6) and Theorem 3.2. □

3.3 Homogeneous integrators.

In the Theorem 3.3 we have seen that, for a given Gaussian-Poisson mixture and a flow of events generated by the values of this measure in the course of time, all the elements of the corresponding L_2-space $L_2(\Omega)$ admit integral representation (1). However, in general, for a given stochastic measure, though with homogeneous (see below) and independent values, and a filtration generated by the measure itself, we cannot claim that *all* the elements in the corresponding $L_2(\Omega)$ space admit the representation (1). This fact finds evidence and consequences in many applied situations, we can refer as an example to the incompleteness of certain well-known market models in mathematical finance. The next result addresses the issue of characterizing the stochastic measures for which it is possible that *all* the elements of the corresponding $L_2(\Omega)$ admit the representation (1).

Let us turn our attention to the measure $\mu = \mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, of the type $E\mu = 0$, $E\mu^2 = M$ satisfying (7), which is *homogeneous*, in the sense that all the values $\mu(\Delta)$ on the sets $\Delta \subseteq \Theta \times \mathbb{T}$ of the *same* measure $M(\Delta)$ obey the *same* probability law. Accordingly, they follow the infinetely-divisible law of the form (17) with parameters σ^2, L that do *not* depend on $(\theta, t) \in \Theta \times \mathbb{T}$. Namely we have

$$logEe^{i\lambda\mu(\Delta)} = \Big[-\frac{\lambda^2}{2}\sigma^2 + \int_{\mathbb{R}\setminus[0]} \big(e^{i\lambda x} - 1 - i\lambda x\big) L(dx)\Big] \cdot M(\Delta), \quad \lambda \in \mathbb{R}: \ (9)$$

$$\sigma^2 + \int_{\mathbb{R}\setminus[0]} x^2 L(dx) = 1$$

with σ^2 constant and $L(dx)$, $x \in \mathbb{R}\setminus[0]$, measure on $\mathbb{R}\setminus[0]$. Let Θ be a complete separable metric space. For the following result see e.g. [2], [6], [15].

Theorem 9. *The non-anticipating integral representation (1) holds for all $\xi \in L_2(\Omega)$ if and only if μ is either Gaussian or Poisson (centred) stochastic measure multiplied by a scalar.*

Proof. Let us treat μ as

$$\mu(\Delta) \ = \ \iint_\Delta \mu^G(d\theta\, dt) + \iiint_{\mathbb{R}\setminus[0]\times\Delta} x\, \mu^P(dx\, d\theta\, dt), \qquad \Delta \in \Theta \times \mathbb{T}\,,$$

in relation to the Gaussian-Poisson mixture $\mu^{G,P}$. Cf. (16)-(20) and Theorem 2.1. For all $\xi \in L_2(\Omega)$, the non-anticipating integral representation has the form

$$\xi \ = \ E\,\xi \oplus \iint_{\Theta\times\mathbb{T}} \varphi_G\, \mu^G(d\theta\, dt) \oplus \iiint_{(\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}} \varphi_P\, \mu^P(dx\, d\theta\, dt) \qquad (10)$$

which is here considered with respect to $\mu^{G,P}$ as integrator Cf. Theorem 3.3. For all those ξ which admit the non-anticipating integral representation

$$\xi \,=\, E\,\xi + \iint_{\Theta\times\mathbb{T}} \varphi\mu(d\theta\,dt)$$
$$= E\,\xi \oplus \iint_{\Theta\times\mathbb{T}} \varphi\mu^G(d\theta\,dt) \oplus \iint_{\Theta\times\mathbb{T}} \varphi\cdot x\,\mu^P(dx\,d\theta\,dt)$$

with respect to μ, the identity

$$\varphi(\theta,t) \,\equiv\, \varphi_G(\theta,t) \,\equiv\, x^{-1}\varphi_P(x,\theta,t)\,, \qquad (x,\theta,t)\in(\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T},$$

must hold for all the integrands φ_G and φ_P. Clearly, this identity can only hold for *all* the *different* integrands φ_G, φ_P as elements of the corresponding functional L_2-spaces (related to the measures $\sigma^2\cdot M$ and $L\times M$) if either $\sigma^2=1$, $L=0$ or $\sigma^2=0$ and L is concentrated at the *single* point $x\in\mathbb{R}\setminus[0]$, with $L(x)=x^{-2}$. In other terms it means that either $\mu=\mu^G$ or $\mu=x\mu^P$. Here μ^P is the Poisson (centred) measure concentrated on the product $[x]\times\Theta\times\mathbb{T}$, which can be identified with $\Theta\times\mathbb{T}$. $\square$

Let us now consider the cadlag processes of type

$$\xi(t)=\iint_{\Theta\times(0,t]} 1_\Delta\mu(d\theta dt), \quad t\in\mathbb{T}$$

($\Delta\subseteq\Theta\times\mathbb{T}$), for the stochastic measure μ the values of which follow the probability law (9). The jumps of the trajectories of these processes are

$$\rho_\omega=\xi(t_\omega,\omega)-\xi(t_\omega-0,\omega), \quad t_\omega\in\mathbb{T}.$$

In relation to these jumps we can define the stochastic measure $\mu^F=\mu^F(\Delta)$, $\Delta\subseteq\Theta\times\mathbb{T}$: $M(\Delta)<\infty$, as

$$\mu^F(\Delta):=\sum_{t_\omega\in\mathbb{T}} F(\rho_\omega)-\int_{\mathbb{R}\setminus[0]} F\cdot L(dx)\cdot M(\Delta) \tag{11}$$

by means of the deterministic real function $F=F(x)$, $x\in\mathbb{R}$, such that $F(0)=0$ and

$$\int_{\mathbb{R}\setminus[0]} |F(x)|^p L(dx)<\infty, \quad p=1,2.$$

Here μ^F is a homogeneous stochastic measure with independent values of form

$$\mu^F(\Delta)=\iiint_{(\mathbb{R}\setminus[0])\times\Delta} F\mu^P(dxd\theta dt), \quad \Delta\subseteq\Theta\times\mathbb{T},$$

where μ^P is the Poisson (centred) stochastic component of μ. Accordingly we have that the random variable $\mu^F(\Delta)$ has distribution characterized by

$$logEe^{i\lambda\mu^F(\Delta)}=\int_{\mathbb{R}\setminus[0]} \big(e^{i\lambda F(x)}-1-i\lambda F(x)\big)L(dx)\cdot M(\Delta), \quad \lambda\in\mathbb{R}.$$

To explain the statement above it is enough to observe that

$$F(1_\Delta x) = F(x) 1_\Delta(\theta, t), \quad (x, \theta, t) \in (\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T},$$

and refer to the argument used in (16)-(19).

Now let us consider an *orthogonal basis* F_k, $k = 1, 2, ...$, in the standard (complex) space $L_2(\mathbb{R} \setminus [0])$:

$$\|F\|_{L_2(\mathbb{R}\setminus[0])} = \Big(\int_{\mathbb{R}\setminus[0]} |F|^2 L(dx) \Big)^{1/2}.$$

For each F_k, one can apply the arguments above and define the stochastic measures

$$\mu^{F_k} = \mu^{F_k}(d\theta dt), \quad (\theta, t) \in \Theta \times \mathbb{T}, \ k = 1, 2, ...$$

Theorem 10. *Let $\xi \in L_2(\Omega)$. In the representation* (10) *the integral with respect to the Poisson (centred) component μ^P can be written as*

$$\iiint_{(\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}} \varphi_P \mu^P(dx d\theta dt) = \sum_{k=1}^{\infty} \iint_{\Theta\times\mathbb{T}} \varphi_k \mu^{F_k}(d\theta dt). \tag{12}$$

Proof. The integrands φ_P in the stochastic integral with respect to the Poisson (centred) stochastic measure μ_P are elements of the subspace

$$L_2^I((\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T} \times \Omega) \subseteq L_2(\mathbb{R} \setminus [0] \times \Theta \times \mathbb{T} \times \Omega)$$

in the standard L_2-space related to the integrator

$$L(dx) \times M(d\theta dt) \times P(d\omega), \quad (x, \theta, t, \omega) \in (\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T} \times \Omega$$

- cf. (20) and (2)-(5). The elements of the form

$$F_k \cdot \Phi : \quad \Phi \in L_1(\Theta \times \mathbb{T} \times \Omega), \ k = 1, 2, ...,$$

constitute a complete system in the above space. Hence, the orthogonal projections

$$F_k \cdot \varphi : \quad \varphi = E\big(\Phi(\theta, t)|\mathfrak{A}_t\big), \ (\theta, t) \in \Theta \times \mathbb{T},$$

of the elements $F_k \cdot \Phi$, $k = 1, 2, ...$, on the subspace $L_2^I((\mathbb{R} \setminus [0]) \times \Theta \times \mathbb{T} \times \Omega)$ constitute a complete system in this subspace. Any linear combination of the above elements is represented as the orthogonal sum

$$\sum_k \oplus \varphi_k \cdot F_k : \quad \varphi_k \in L_2(\Theta \times \mathbb{T} \times \omega).$$

Thus any integrand φ_P for μ^P can be represented as the orthogonal series

$$\varphi_P = \sum_{k=1}^{\infty} \oplus \varphi_k \cdot F_k$$

in $L_2^I((\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}\times\Omega)$ and this yields the representation (12), i.e.

$$\iiint_{(\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}}\varphi_P\mu^P(dxd\theta dt) = \sum_{k=1}^{\infty}\oplus\iiint_{(\mathbb{R}\setminus[0])\times\Theta\times\mathbb{T}}(\varphi_k\cdot F_k)\mu^P(dxd\theta dt)$$
$$= \sum_{k=1}^{\infty}\oplus\iint_{\Theta\times\mathbb{T}}\varphi_k\mu^{F_k}(d\theta dt),$$

as the standard orthogonal series (11) in $L_2(\Omega)$. □

In the sequel we will introduce the *non-anticipating derivative*. Here we would however note straightaway that the integrands in the representation (12) are the non-anticipating derivatives $\varphi_k = D_k\xi$ of $\xi\in L_2(\Omega)$ with respect to the stochastic measure μ^{F_k}, $k=1,2,\ldots$.

We also would like to note that in the case the stochastic measure μ has no Gaussian component μ^G, i.e. μ is following the probability law (9) with $\sigma^2=0$, then the representation (12) can be applied directly with $\mu^{F_1}=\mu$ and with the μ^{F_k}, $k=2,3,\ldots$, given by (11). Here $F_1=x$, $x\in\mathbb{R}\setminus[0]$ and F_k, $k=2,3,\ldots$, constitute an orthogonal system in $L_2(\mathbb{R}\setminus[0])$. The same arguments used in the proof of Theorem 3.5 lead to the following result. See also [42].

Corollary 11. *Let stochastic measure μ follow the probability law* (9) *with $\sigma^2=0$. All the elements $\xi\in L_2(\Omega)$ admit the following representation via the orthogonal sum*

$$\xi = E\xi\oplus\sum_{k=1}^{\infty}\oplus\iint_{\Theta\times\mathbb{T}}\varphi_k\mu^{F_k}(d\theta dt): \quad \mu^{F_1}=\mu. \tag{13}$$

4 The non-anticipating derivative.

4.1 A general definition and related properties.

Let us consider the non-anticipating integral as the *isometric* linear operator I:

$$L_2^I(\Theta\times\mathbb{T}\times\Omega)\ni\varphi\Longrightarrow I\varphi\in L_2(\Omega)\,,$$

on the subspace of the non-anticipating functions φ - cf. (2). In relation to I, we can define the *non-anticipating derivative as the adjoint linear operator* $D=I^*$:

$$L_2(\Omega)\ni\xi\Longrightarrow D\xi\in L_2^I(\Theta\times\mathbb{T}\times\Omega)\,. \tag{1}$$

Note that we have

$$\|D\| \,=\, \|I\| \,=\, 1$$

for the operator norm of the adjoint linear operators $D=I^*$, $I=D^*$. It is $D\xi=0$ for ξ orthogonal to all the non-anticipating integrals

$$\iint_{\Theta\times\mathbb{T}} \varphi\,\mu(d\theta\,dt)\,, \qquad \varphi \in L_2^I(\Theta\times\mathbb{T}\times\Omega).$$

Accordingly, for any random variable $\xi \in L_2(\Omega)$, the non-anticipating derivative provides the best approximation

$$\hat{\xi} \,=\, \iint_{\Theta\times\mathbb{T}} D\xi\mu(d\theta\,dt) \tag{2}$$

to ξ in $L_2(\Omega)$ by non-anticipating integrals, i.e.

$$\left\|\xi - \hat{\xi}\right\| \,=\, \min_{\varphi\in L_2^I(\Theta\times\mathbb{T}\times\Omega)} \left\|\xi - \iint_{\Theta\times\mathbb{T}} \varphi\mu(d\theta\,dt)\right\|.$$

We obtain the following result - cf. [15] and [14], see also [22]. We can also refer to [52] for some results in this direction in the case of the Wiener process as integrator and to [45] for the space-time Brownian sheet.

Theorem 12. *For all $\xi \in L_2(\Omega)$, the non-anticipating differentiation can be carried through via the limit*

$$D\xi \,=\, \lim_{n\to\infty} \sum E\Big[\frac{1}{M(\Delta)} E\left(\xi\cdot\mu(\Delta)\big|\mathfrak{A}_s\right)\Big]\cdot 1_\Delta \tag{3}$$

in $L_2^I(\Theta\times\mathbb{T}\times\Omega)$. Here the sum is on the n^{th}-series sets $\Delta\subseteq\Theta\times(s,u]$ of some (which can be any) partition in $\Theta\times\mathbb{T}$ - cf. (3).

Proof. In the representation

$$\xi \,=\, \xi^0 \,\oplus \iint_{\Theta\times\mathbb{T}} \varphi\,\mu(d\theta\,dt)$$

with $\varphi = D\xi$ - cf.(2), the component ξ^0 is orthogonal to all the non-anticipating integrals (thus $D\xi^0 = 0$). This implies

$$E\left(\left(1_{\mathcal{A}}\mu(\Delta)\right)\cdot\xi^0\right) \,=\, 0, \qquad \Delta\subseteq\Theta\times(s,u]\,, \qquad \mathcal{A}\in\mathfrak{A}_s$$

- cf.(3), thus it is $E(\xi^0\cdot\mu(\Delta)|\mathfrak{A}_s) = 0$. With the use of

$$E\Big(\iint_{\Theta\times\mathbb{T}} \varphi\,\mu(d\theta\,dt)\cdot\mu(\Delta)\Big|\mathfrak{A}_s\Big) \,=\, E\Big(\iint_{\Delta} \varphi\,M(d\theta\,dt)\Big|\mathfrak{A}_s\Big), \quad \Delta\subseteq\Theta\times(s,u]$$

- cf.(6) and (7), we can see that the limit (3) for $\varphi = D\xi$ is identical to the limit of the approximations $\varphi^{(n)}$, $n = 1, 2...$, characterized in (24) - cf. Theorem 1.3. □

Example 4.1. Let μ be the mixture of the stochastic measures $\mu^k = \mu^k(\Delta)$, $\Delta\subseteq\Theta_k\times\mathbb{T}$, $k = 1, 2, ...$ - cf. (10). Then, whatever $\xi\in L_2(\Omega)$ be, the non-anticipating derivative $D\xi$ is

$$D\xi = \sum_k \oplus D_k\xi \, 1_{\Theta_k \times \mathbb{T}}$$

where, for any k, $D_k\xi$ is the non-anticipating derivative with respect to the measure μ^k.

Example 4.2. For a general μ following the law (17), the non-anticipating derivative of the element $\xi \in L_2(\Omega)$:

$$\xi = D\xi \oplus \sum_{p=1}^{\infty} \oplus I^p \varphi_p$$

can be determined by the formula

$$D\xi = \sum_{p=1}^{\infty} \oplus I^{p-1} \varphi_p(\cdot, \theta, t), \quad (\theta, t) \in \Theta \times \mathbb{T}.$$

4.2 Differentiation formulae.

The random variables $\xi \in L_2(\Omega)$ are functions of the values $\mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, of the stochastic measure. In fact the elements ξ are measurable with respect to the σ-algebra $\mathfrak{A} = \mathfrak{A}_{\Theta \times \mathbb{T}}$ generated by the values of μ - cf. (12). Let us now turn to the ξ which can be treated as functions of a *finite* number of values $\mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$. Any such random variable admits the representation

$$\xi = F(\xi_1, ..., \xi_m) \tag{4}$$

as a function of the values $\xi_k = \mu(\Delta_k)$, $k = 1, ..., m$, on the appropriately chosen *disjoint* sets Δ_k, $k = 1, ..., m$, in $\Theta \times \mathbb{T}$. Of course, the representation (4) *is not* unique. So, for *any* finite number of *any* particular group of *disjoint* sets

$$\Delta_k \subseteq \Theta \times \mathbb{T} : \quad M(\Delta_k) < \infty, \quad k = 1, ..., m,$$

we consider $\xi = F$ - cf. (4), for the functions

$$F = F(\xi_1, ..., \xi_m), \quad (\xi_1, ..., \xi_m) \in \mathbb{R}^m,$$

which are characterized as follows. First of all, we assume that $F \in C^1(\mathbb{R}^m)$, and we write

$$\partial_k^x F := \begin{cases} \frac{\partial}{\partial \xi_k} F(..., \xi_k, ...), & x \neq 0, \\ \frac{1}{x}\big[F(..., \xi_k + x, ...) - F(..., \xi_k, ...)\big], & x = 0. \end{cases}$$

According to the characterization of the stochastic measure μ by the infinitely-divisible law - cf. (17), we define

$$\mathcal{D}\xi(\theta,t) := \sum_{k=1}^{m} \Big[\partial_k^0 F \cdot \sigma^2(\theta,t) + \int_{\mathbb{R}\setminus[0]} \partial_k^x F \cdot x^2 L(dx,\theta,t)\Big] \cdot 1_{\Delta_k}(\theta,t), \quad (\theta,t) \in \Theta \times \mathbb{T}, \quad (5)$$

for the elements $\xi = F$ of the type above. And we assumed that

$$\mathcal{D}\xi = \mathcal{D}\xi(\theta,t), \quad (\theta,t) \in \Theta \times \mathbb{T},$$

satisfy the condition

$$|||\mathcal{D}\xi|||^2 := \sum_{k=1}^{m} \iint_{\Delta_k} \Big[\|\partial_k^0 F\|^2 \cdot \sigma^2(\theta,t) + \int_{\mathbb{R}\setminus[0]} \|\partial_k^x F\|^2 \cdot x^2 L(dx,\theta,t)\Big] M(d\theta\, dt) < \infty\,. \quad (6)$$

Hence we have in particular that $\mathcal{D}\xi \in L_2(\Omega \times \Theta \times \mathbb{T})$, since

$$\|\mathcal{D}\xi\|_{L_2} \le |||\mathcal{D}\xi|||. \quad (7)$$

The following result was first published in [18].

Theorem 13. *The non-anticipating derivative of the random variable* $\xi = F$ *of type* (4) *defined by the limit* (3) *can be computed by*

$$D\,\xi(\theta,t) \;=\; E\Big(\mathcal{D}\xi(\theta,t)\big|\mathfrak{A}_t\Big), \qquad (\theta,t) \in \theta \times \mathbb{T}. \quad (8)$$

Proof. The proof is subdivided in several steps in which the statement is shown for more and more general random variables ξ, see steps A, B, C. Finally an appropriate approximation argument leads to the conclusion, see step D.

A. Let us take $\xi = F = F(\xi_i, ..., \xi_m)$ with

$$F(\xi_1, ..., \xi_m) = e^{i\sum_{k=1}^{m} \lambda_k \xi_k}, \qquad (\lambda_k \in \mathbb{R}, \quad k = 1, ..., m) \quad (9)$$

into account. In this case formula (5) gives

$$\mathcal{D}\,\xi(\theta,t) = \xi \sum_{k=1}^{m} \Big[i\lambda_k \sigma^2(\theta,t) + \int_{\mathbb{R}\setminus[0]} \big(e^{i\lambda_k x} - 1\big) x\, L(dx,\theta,t)\Big] 1_{\Delta_k}(\theta,t), \quad (\theta,t) \in \Theta \times \mathbb{T}.$$

We consider $\xi_k = \mu(\Delta_k)$, $k = 1, ..., m$, with the *disjoint simple sets* Δ_k, $k = 1, ..., m$. Then, for $n \to \infty$, any set $\Delta \subseteq \Theta \times (s,u]$ of the n^{th}-series of the (3)-partitions either belongs to some Δ_k or it is disjoint with all Δ_k, $k = 1, ..., m$. In this last case we have

$$E\Big(\xi\,\mu(\Delta)|\mathfrak{A}_{]\Delta[}\Big) = \xi\,E\,\mu(\Delta) = 0$$

- cf.(27). Otherwise, if $\Delta \subseteq \Delta_k$, for some k, it is

$$\begin{aligned}
&E\Big(\xi\,\mu(\Delta)\big|\mathfrak{A}_{]\Delta[}\Big) = e^{-i\lambda_k\mu(\Delta)}\xi\,E\,\Big(\mu(\Delta)e^{i\lambda_k\mu(\Delta)}\Big)\\
=&e^{-i\lambda_k\mu(\Delta)}\xi\,Ee^{i\lambda_k\mu(\Delta)}\iint_\Delta\big[i\lambda_k\sigma^2(\theta,t)\\
&+\int_{\mathbb{R}\setminus[0]}\big(e^{i\lambda_k x}-1\big)x\,L(dx,\theta,t)\big]M(d\theta dt)\\
=&E\Big(\iint_\Delta \mathcal{D}\xi(\theta,t)\,M(d\theta\,dt)\big|\mathfrak{A}_{]\Delta[}\Big).
\end{aligned}$$

According to Theorem 1.4 the stochastic function $\mathcal{D}\xi$ admits the representation

$$\mathcal{D}\xi = \lim_{n\to\infty}\sum\frac{1}{M(\Delta)}E\Big(\xi\cdot\mu(\Delta)\big|\mathfrak{A}_{]\Delta[}\Big)1_\Delta \tag{10}$$

as a limit in $L_2(\Omega\times\Theta\times\mathbb{T})$. Here the sum refers to all the elements of the same n^{th}-series of partitions of $\Theta\times\mathbb{T}$. By use of an appropriate sub-sequence we have convergence in $L_2(\Omega)$ for almost all $(\theta,t)\in\Theta\times\mathbb{T}$:

$$D\xi(\theta,t) \;=\; \lim_{n\to\infty}\frac{1}{M(\Delta)}E\,\Big(\xi\cdot\mu(\Delta)\big|\mathfrak{A}_s\Big),\quad (\theta,t)\in\Delta$$

and

$$\mathcal{D}\xi(\theta,t) = \lim_{n\to\infty}\frac{1}{M(\Delta)}E\Big(\xi\cdot\mu(\Delta)\big|\mathfrak{A}_{]\Delta[}\Big),\quad (\theta,t)\in\Delta$$

for $\Delta\subseteq\Theta\times(s,u]$: $\Delta\ni(\theta,t)$. Moreover taking $t^-<t$, we obtain

$$E\Big(D\xi(\theta,t)\big|\mathfrak{A}_{t^-}\Big) = \lim_{n\to\infty}\frac{1}{M(\Delta)}E\Big(\xi\mu(\Delta)\big|\mathfrak{A}_{t^-}\Big) = E\Big(\mathcal{D}\xi(\theta,t)\big|\mathfrak{A}_{t^-}\Big).$$

Let $t^-\to t$ in the above relations then we have

$$\begin{aligned}
D\xi(\theta,t) = E\big(D\xi(\theta,t)|\mathfrak{A}_t\big) &= \lim_{t^-\to t}E\big(D\xi(\theta,t)|\mathfrak{A}_{t^-}\big)\\
&= \lim_{t^-\to t}E\big(\mathcal{D}\xi(\theta,t)|\mathfrak{A}_{t^-}\big) = E\big(\mathcal{D}\xi(\theta,t)|\mathfrak{A}_t\big),
\end{aligned}$$

since $\lim_{t^-\to t}\mathfrak{A}_{t^-}=\mathfrak{A}_t$ - cf. (16). Thus, formula (8) holds for $\xi=F$ with F of form (9) and the $\xi_k=\mu(\Delta_k)$ with Δ_k, $k=1,...,m$, as disjoint simple sets.
B. Indeed the above result holds for any group of measurable *disjoint* sets $\Delta_1,...,\Delta_m$. In fact it is enough to apply an approximation argument with $\Delta_k=\lim_{n\to\infty}\Delta_k^{(n)}$, $k=1,...,m$, by disjoint simple sets $\Delta_k^{(n)}$, $k=1,...,m$ $(n=1,2,...)$, such that $\mu(\Delta_k)=\lim_{n\to\infty}\mu(\Delta_k^{(n)})$ holds true in $L_2(\Omega)$ and for almost all $\omega\in\Omega$. Cf. (6) and (8). Accordingly for ξ and $\xi^{(n)}$ of type (9)

with $\xi_k = \mu(\Delta_k)$, $k = 1, ..., m$, and $\xi_k^{(n)} = \mu(\Delta_k^{(n)})$, $k = 1, ..., m$, respectively we have also $\xi = \lim_{n\to\infty} \xi^{(n)}$ in $L_2(\Omega)$ and

$$D\xi = \lim_{n\to\infty} D\xi^{(n)}, \qquad \mathcal{D}\xi = \lim_{n\to\infty} \mathcal{D}\xi^{(n)} \tag{11}$$

in $L_2(\Theta \times \mathbb{T} \times \Omega)$. Thus

$$D\xi(\theta, t) = E\Big(\mathcal{D}\xi(\theta, t)\,\big|\,\mathfrak{A}_t\Big)$$

for almost all $(\theta, t) \in \Theta \times \mathbb{T}$, i.e. formula (8) holds for $D\xi$ as the element in $L_2^I(\Theta \times \mathbb{T} \times \Omega) \subseteq L_2(\Theta \times \mathbb{T} \times \Omega)$.

C. Clearly, formula (8) is valid for *all* $\xi = F$ which are linear combinations of functions (9) with $\xi_k = \mu(\Delta_k)$ on disjoint measurable sets Δ_k, $k = 1, ..., m$.

D. The formula (8) can be extended on all the functions characterized in the scheme (4)-(7). Let us define the scalar functions

$$\mathbb{D}\xi := \sum_{k=1}^{m} \partial_k^x F \cdot 1_{\Delta_k}$$

on the product space $\mathbb{R} \times \Delta \times \Omega$: $\Delta = \sum_{k=1}^m \Delta_k$, equipped with the *finite* product-type measure

$$L_0(dx, \theta, t) \times M(d\theta dt) \times P(d\omega), \quad (x, \theta, t, \omega) \in \mathbb{R} \times \Delta \times \Omega.$$

Here $L_0(dx, \theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$, is equal to $\sigma^2(\theta, t)$ at the atom $x = 0$ and to $x^2 L(dx, \theta, t)$ on $\mathbb{R} \setminus [0]$. The functions

$$\mathbb{D}\xi = \mathbb{D}\xi(x, \theta, t, \omega), \quad (x, \theta, t, \omega) \in \mathbb{R} \times \Delta \times \Omega,$$

are elements of the standard space $L_2(\mathbb{R} \times \Delta \times \Omega)$ with norm

$$\big\|\mathbb{D}\xi\big\|_{L_2} := \Big(\iiint_{\mathbb{R}\times\Delta\times\Omega} |\mathbb{D}\xi|^2 L_0(dx, \theta, t) \times M(d\theta dt) \times P(d\omega)\Big)^{1/2}.$$

We have

$$\|\mathbb{D}\xi\|_{L_2} = |||\mathcal{D}\xi|||$$

for

$$\mathcal{D}\xi = \int_{\mathbb{R}} \mathbb{D}\xi L_0(dx, \theta, t), \quad (\theta, t, \omega) \in \Delta \times \Omega.$$

Cf. (5)-(7). The key-point of the approximation argument which will be applied is that, for $\xi = F$ and $\xi^{(n)} = F^{(n)}$, $n = 1, 2, ...$, the convergences

$$\|\xi - \xi^{(n)}\| \longrightarrow 0 \quad \text{and} \quad \|\mathbb{D}\xi - \mathbb{D}\xi^{(n)}\|_{L_2} \longrightarrow 0, \qquad n \to \infty, \tag{12}$$

imply the limits

$$D\xi = \lim_{n\to\infty} D\xi^{(n)} \quad \text{and} \quad \mathcal{D}\xi = \lim_{n\to\infty} \mathcal{D}\xi^{(n)}$$

in $L_2(\Theta\times\mathbb{T}\times\Omega)$. Note that in the coming considerations, we apply dominated point-wise with the appropriate corresponding majorants in order to prove the convergences (12). To simplify the notation, we give the proof in the case $k=1$ $(m=1)$, i.e. for $\xi_1 = \mu(\Delta_1)$.

For $F \in C_0^\infty(\mathbb{R})$, the convergence (12) holds with $\xi^{(n)} = F^{(n)}$ given by the partial sums $F^{(n)} = \Phi_n(F)$ of the Fourier series of F on $|\xi_1| \le h_n$ ($h_n \to \infty$, $n\to\infty$). In fact note that, for $n\to\infty$, we have

$$\partial_1^x F^{(n)} := \partial_1^x \Phi_n(F) = \Phi_n(\partial_1^x F)$$

whatever $x\in\mathbb{R}$ be. Next, for $\xi = F$: $F \in C_0^1(\mathbb{R})$ the convergence (12) holds with $\xi^{(n)} = F^{(n)}$:

$$F^{(n)} := F * \delta_n = \int_{\mathbb{R}} F(\xi_1 - x_1)\delta_n(x_1)dx_1 \in C_0^\infty(\mathbb{R})$$

with $\delta_n \in C_0^\infty(\mathbb{R})$ as the standard approximations to the delta-function. Here we have

$$\partial_1^x F^{(n)} := \partial_1^x(F * \delta_n) = (\partial_1^x F) * \delta_n, \quad n = 1,2,...$$

In general, for $\xi = F$: $F \in C^1(\mathbb{R})$, the convergence (12) holds with $\xi^{(n)} = F^{(n)}$ as the truncations $F^{(n)} = F\cdot w_n \in C_0^1(\mathbb{R})$. Here w_n is an appropriate approximation $w_n \in C_0^1(\mathbb{R})$ of the unit. Note that

$$\partial_1^x F^{(n)} := \partial_1^x(F\cdot w_n) = (\partial_1^x F)\cdot w_n + F\cdot(\partial_1^x w_n), \quad n=1,2,... \quad \square$$

Example 4.3. Let μ be a mixture of the Gaussian stochastic measure on $\Theta_0\times\mathbb{T}$ and Poisson (centred) stochastic measures (multiplied by the different scalars $x\ne 0$) on the corresponding space-time products $\Theta_x\times\mathbb{T}$. So, μ is a stochastic measure on the space-time product

$$\Theta\times\mathbb{T} = \big(\Theta_0\times\mathbb{T}\big) \cup \sum_{x\ne 0}\big(\Theta_x\times\mathbb{T}\big)$$

- cf. (10). For $\xi = F$ as a function in $C^1(\mathbb{R}^m)$ of the values $\xi_k = \mu(\Delta_k)$, $k=1,...,m$, on the *disjoint* sets Δ_k, $k=1,...,m$, in $\Theta\times\mathbb{T}$, the formula (8) can be written

$$\mathcal{D}\xi = \sum_{k=1}^m \Big[\partial_k^0 F 1_{\Theta_0\times\mathbb{T}} + \sum_{x\ne 0}\partial_k^x F 1_{\Theta_x\times\mathbb{T}}\Big]\cdot 1_{\Delta_k}.$$

Remark 4.1. The formula (8) is in general *not* valid if in $\xi = F$ the function F is evaluated on values of μ which are not on *disjoint* sets.

5 The anticipating derivative and integral.

5.1 Definition and related properties.

Following the discussion of the previous section we can now turn our attention to *all* the random variables $\xi = F$ where F is a linear combination of the random variables considered in the scheme (4)-(7). Also these linear combinations fit the scheme (4)-(7). Moreover we write

$$dom\mathcal{D} \subseteq L_2(\Omega) \tag{1}$$

for the *linear* domain of all elements in $L_2(\Omega)$ of the type $\xi = F$ characterized in (4)-(7), plus the limits $\xi = \lim_{n\to\infty} \xi^{(n)}$ in $L_2(\Omega)$ of the above type elements $\xi^{(n)}$, $n = 1, 2, ...$, for which the corresponding limits

$$\mathcal{D}\xi := \lim_{n\to\infty} \mathcal{D}\xi^{(n)}$$

exist $L_2(\Theta \times \mathbb{T} \times \Omega)$.

Note that whatever the representation $\xi = F$ (4) be, the corresponding stochastic function $\mathcal{D}\xi$ is a *unique* well-defined element of $L_2(\Theta \times \mathbb{T} \times \Omega)$. Moreover, we obtain the following result. See [19].

Theorem 14. *For all $\xi \in domD$, the stochastic functions $\mathcal{D}\xi$ are given by the well defined closed linear operator $\mathcal{D}$:*

$$L_2(\Omega) \supseteq dom\mathcal{D} \ni \xi \implies \mathcal{D}\xi \in L_2(\Theta \times \mathbb{T} \times \Omega), \tag{2}$$

with domain $dom\mathcal{D}$ dense in $L_2(\Omega)$.

Proof For *some* (which can be *any*) partitions of $\Theta \times \mathbb{T}$, let us fix a family of the elements in $L_2(\Omega)$ which are of type (9) with the μ-values taken on disjoint simple sets in $\Theta \times \mathbb{T}$. Any linear combination of these elements admits the representation

$$\xi = F(\xi_1, ..., \xi_m)$$

with F as linear combination of the *different* elements

$$e^{i\sum_{k=1}^m \lambda_k \xi_k}, \quad \xi_k = \mu(\Delta_k), \quad k = 1, ..., m,$$

Accordingly, we can see that for *all* these linear combinations the corresponding formula (5) is given by the limit

$$\mathcal{D}\xi = \lim_{n\to\infty} \sum \frac{1}{M(\Delta)} E\big(\xi\mu(\Delta)|\mathfrak{A}_{]\Delta[}\big) 1_\Delta \tag{3}$$

taken in $L_2(\Theta \times \mathbb{T} \times \Omega)$. The sum is here taken on the sets Δ of the n^{th}-series of the partitions of $\Theta \times \mathbb{T}$. Cf. (10). Clearly, this limit defines the *linear operator* $\mathcal{D}$:

$$dom\mathcal{D} \ni \xi \implies \mathcal{D}\xi \in L_2(\Theta \times \mathbb{T} \times \Omega) \tag{4}$$

on the *linear* domain $dom\mathcal{D} \subseteq L_2(\Omega)$. Let us show that *this* linear operator $\mathcal{D}$ is *closable*. Let $\Delta_0 \subseteq \Theta \times \mathbb{T}$ be a *simple set* and $\varphi_0 \cdot 1_{\Delta_0}$ be a simple function the element $\varphi_0 \in L_2(\Omega)$ as $\mathfrak{A}_{]\Delta_0[}$-measurable values on Δ_0. The limit (3) implies that

$$\iint_{\Theta\times\mathbb{T}} E\big[(\varphi_0 1_{\Delta_0})\mathcal{D}\xi\big] M(d\theta dt)$$
$$= \lim_{n\to\infty} \sum_{\Delta\subseteq\Delta_0} E\big[\varphi_0 E\big(\xi\mu(\Delta)|\mathfrak{A}_{]\Delta[}\big)\big] = E\big[\big(\varphi_0\mu(\Delta_0)\big)\xi\big]. \tag{5}$$

Hence, for the elements $\xi^{(n)} = F^{(n)}$, $n = 1, 2, ...$ in $dom\mathcal{D}$ such that $\lim_{n\to\infty} \xi^{(n)} = 0$ in $L_2(\Omega)$ and $\lim_{n\to\infty} \mathcal{D}\xi^{(n)} = \varphi$ in $L_2(\Theta \times \mathbb{T} \times \Omega)$, we have

$$\iint_{\Theta\times\mathbb{T}} E\left[\big(\varphi_0 \cdot 1_{\Delta_0}\big) \cdot \varphi\right] M(d\theta dt) \,=\, 0\,.$$

The considered simple functions $\varphi_0 \cdot 1_{\Delta_0}$ constitute a complete system in $L_2(\Theta\times\mathbb{T}\times\Omega)$. Cf. Theorem 1.4. Hence, the above equation implies that $\varphi = 0$. Thus the linear operator (3)-(4) is closable. Hence it admits the standard extension on *all* the random variables $\xi = F$ as functions of the type (9) involving all the *disjoint* sets $\Delta_k \subseteq \Theta \times \mathbb{T}$: $M(\Delta_k) < \infty$, $k = 1, ..., m$. Cf. (11). The next standard extension of the closable linear operator (3)-(4) up to the closed linear operator (1)-(2) is done by approximation arguments with respect to the limits (11)-(12), see the proof of Theorem 4.2. □

Note that formula (8) holds for all the elements $\xi \in dom\mathcal{D}$ in the domain of the closed linear operator (1)-(2) and, to repeat, it is

$$D\xi(\theta, t) = E\left[\mathcal{D}\xi(\theta, t)|\mathfrak{A}_t\right], \quad (\theta, t) \in \Theta \times \mathbb{T}. \tag{6}$$

According to this relationships with the non-anticipating derivative D, we call $\mathcal{D}$ the *anticipating derivative*.

Remark 5.1. In general, the formula (5) for the anticipating derivative $\mathcal{D}\xi$ of the random variable (4): $\xi = F$ as a function of the values of μ, is *non valid* if these values are taken on sets in $\Theta \times \mathbb{T}$ which are *not disjoint*.

In addition to the scheme (4)-(7), let us consider the functions $\xi = F(\xi_1, ..., \xi_m)$ with $F \in C^1(\mathbb{R}^m)$ and where ξ_k, $k = 2, ..., m$, are the stochastic integrals

$$\xi_k \,=\, \iint_{\Theta\times\mathbb{T}} \varphi_k \mu(d\theta\, dt)$$

with the deterministic integrands φ_k having the *disjoint* supports $\Delta_k = \{(\theta, t) : \varphi_k(\theta, t) \neq 0\}$, $k = 1, ..., m$. We introduce the stochastic functions

$$\mathcal{D}\xi := \sum_{k=1}^{m} \Big[\partial_k^0 F \, \sigma^2(\theta,t) + \int_{\mathbb{R}\setminus[0]} \partial_k^x F \, x^2 L(dx,\theta,t)\Big] 1_{\Delta_k}(\theta,t), \quad (\theta,t) \in \Theta \times \mathbb{T}. \tag{7}$$

with a new definition for the functions

$$\partial_k^x F := \begin{cases} \frac{\partial}{\partial \xi_k} F(..., \xi_k, ...) \, \varphi_k(\theta,t), & x \neq 0, \\ \frac{1}{x}\big[F(..., \xi_k + x\varphi(\theta,t), ...) - F(..., \xi_k, ...)\big], & x = 0. \end{cases}$$

The stochastic functions $\mathcal{D}\xi$ introduced above satisfy the condition (6) with the *newly* defined components $\partial_k^x F$, $k = 1, ..., m$. In this setting Theorem 5.1 implies the following result.

Corollary 15. *The elements $\xi = F \in L_2(\Omega)$, defined here above belong to dom$\mathcal{D}$ and the formula (7) represents the anticipating derivative $\mathcal{D}\xi$.*

Proof. In the case $F \in C_b^1(\mathbb{R}^m)$ and the integrands φ_k, $k = 1, ..., m$, are linear combinations of indicators of the *disjoint* sets Δ_{jk}, $j = 1, ..., m_k$ $(k = 1, ..., m)$, i.e.

$$\varphi_k = \sum_{j=1}^{m_k} c_{jk} \cdot 1_{\Delta_{jk}},$$

the formula (7) gives the anticipating derivative $\mathcal{D}\xi$ of $\xi = F(\xi_1, ..., \xi_m)$ as function of $\xi_{jk} = \mu(\Delta_{jk})$, $j = 1, ..., m_k$ $(k = 1, ..., m)$. Cf. (5). By standard approximation arguments with respect to the limit (12) - cf. also (1), formula (7) admits extension on all elements characterized by the condition (6) involving the newly here above defined components $\partial_k^x F$, $k = 1, ..., m$. □

Example 5.1. Let $I^p \varphi_p$, $p = 1, 2, ...$, be the Itô p-multiple integrals with respect to a general stochastic measure μ in the scheme (3)-(5). The anticipating derivative is

$$\mathcal{D} I^p \varphi_p = I^{p-1} \hat{\varphi}_p(\cdot, \theta, t), \quad (\theta, t) \in \Theta \times \mathbb{T},$$

with the integrands

$$\hat{\varphi}_p := \sum_{j=1}^{p} \varphi_p(..., \theta, t, ...)$$

depending on $(\theta, t) \in \Theta \times \mathbb{T}$ as parameter. The couple (θ, t) comes in at the place of the corresponding couples (θ_j, t_j), $j = 1, ..., p$. Here we have

$$\|\mathcal{D}\xi\|_{L_2} = p^{1/2} \|\xi\|, \quad p = 1, 2, ...$$

All the elements

$$\xi = \sum_{p=0}^{\infty} \oplus \, \xi_p : \qquad \xi_0 = E\xi, \; \xi_p = I^p \varphi_p, \; p = 0, 1, ...,$$

with

$$\sum_{p=1}^{\infty} p \left\|\xi_p\right\|^2 < \infty$$

belong to the domain $dom\mathcal{D}$ of the anticipating derivative and

$$\mathcal{D}\xi = \sum_{p=1}^{\infty} \oplus I^{p-1} \hat{\varphi}_p. \tag{8}$$

See e.g. [16], [19].

Example 5.2. For μ as a general Gaussian-Poisson mixture, the elements $\xi \in dom\mathcal{D}$ characterized in Example 5.1 represent the whole domain $dom\mathcal{D}$. A key-point to show this is that, for all random variables of the form $\xi = e^{i \sum_{k=1}^{m} \lambda_k \xi_k}$ - cf. (9), the approximations

$$\xi = \lim_{q \to \infty} \xi^{(q)}, \qquad \mathcal{D}\xi = \lim_{q \to \infty} \mathcal{D}\xi^{(q)}$$

hold with the polynomials

$$\xi^{(q)} = \sum_{p=0}^{q} \Big(i \sum_{k=1}^{m} \lambda_k \xi_k \Big)^p \in dom\mathcal{D}, \qquad q = 1, 2, ...,$$

of the values of μ. Cf. Theorem 3.2 and Theorem 3.3.

The anticipating derivative (1)-(2) and its relationship (6) with the non-anticipating derivative can be regarded as in the same line as the *Malliavin derivative* [38] and the *Clark-Haussmann-Ocone* formula [7], [23], [43] within the stochastic calculus for the Wiener process. Here we would like also to refer to e.g. [1], [2], [4], [9], [12], [20], [21], [32], [37], [40], [41], [44], [46], [48], [47] and references therein, for some further developments of the Malliavin calculus with respect to the Wiener process and the Poisson process and Poisson (centred) random measure. See [35] for some results in the case of Lévy processes.

5.2 The closed anticipating extension of the Itô non-anticipating integral.

In this final section we present some results of anticipating calculus. In the framework of Wiener processes, we have the Skorohod integral [50] as the adjoint operator to the Malliavin derivative Similar arguments can be achieved in the case of Poisson (centred) random measures. We can refer e.g. [12], [29], [31], [40], [41] and references therein.

Here as usual in this paper we consider a general stochastic measure of type (17) on $\Theta \times \mathbb{T}$. See [19], see also [17].

Theorem 16. *The closed linear operator* $\mathfrak{J} = \mathcal{D}^*$*, adjoint to the anticipating derivtive* (1)-(2)*:*

$$L_2(\Theta \times \mathbb{T} \times \Omega) \supseteq dom\mathfrak{J} \ni \varphi \Longrightarrow \mathfrak{J}\varphi \ni L_2(\Omega) \tag{9}$$

represents the extension

$$\mathfrak{J}\varphi = \iint_{\Theta\times\mathbb{T}} \varphi\, \mu(d\theta\, dt)$$

of the Itô non-anticipating integral on all the stochastic functions φ in the domain $dom\mathfrak{J}$ of $\mathfrak{J}$ dense in $L_2(\Theta \times \mathbb{T} \times \Omega)$.

Proof. The key-point of the proof is equation (5) which can be extended by the standard approximation arguments to hold on all the simple functions of the form $\varphi\cdot 1_\Delta$ with $\mathfrak{A}_{]\Delta[}$-measurable values φ on the indicated sets $\Delta \subseteq \Theta \times \mathbb{T}$: $M(\Delta) < \infty$, i.e.

$$\iint_{\Theta\times\mathbb{T}} E\Big[(\varphi \cdot 1_\Delta)\cdot \mathcal{D}\xi\Big] M(d\theta\, dt) \;=\; E\Big[\big(\varphi\cdot\mu(\Delta)\big)\cdot\xi\Big], \qquad \xi \in dom\mathcal{D}. \tag{10}$$

The linear combinations of the above type simple functions are dense in $L_2(\Theta \times \mathbb{T} \times \Omega)$. Cf. Theorem 1.4. Equation (10) shows that the simple functions $\varphi \cdot 1_\Delta$ belong to the domain $dom\mathfrak{J}$ of the adjoint linear operator $\mathfrak{J} = \mathcal{D}^*$ and that

$$\mathfrak{J}(\varphi\cdot 1_\Delta) = \varphi\cdot\mu(\Delta). \tag{11}$$

Clearly, $\mathfrak{J}$ coincides with the non-anticipating integral I on the *non-anticipating* simple functions and also on any non-anticipating function thanks to the limit

$$\varphi = \lim_{n\to\infty} \varphi^{(n)}, \text{ i.e.} \quad \|\varphi - \varphi^{(n)}\|_{L_2} \longrightarrow 0, \quad n\to\infty,$$

in $L_2(\Theta \times \mathbb{T} \times \Omega)$ of the non-anticipating simple functions $\varphi^{(n)}$, $n = 1, 2, \ldots$. The corresponding limit

$$I\varphi \;=\; \lim_{n\to\infty} I\varphi^{(n)} \;=\; \lim_{n\to\infty} \mathfrak{J}\varphi^{(n)}$$

in $L_2(\Omega)$ implies that φ belongs to the domain of the *closed* linear operator $\mathfrak{J}$ and that $\mathfrak{J}\varphi = I\varphi$. Cf. (3)-(4). □

In relation to the anticipating derivative $\mathcal{D}$, we call $\mathfrak{J} = \mathcal{D}^*$ the *anticipating integral*. Note that $\mathcal{D} = \mathfrak{J}^*$ is the adjoint linear operator to $\mathfrak{J}$. Cf. the duality $D = I^*$ for the non-anticipating derivative D and the Itô non-anticipating integral $I = D^*$.

Example 5.3. In case μ is the Gaussian-Poisson mixture, the anticipating integral $\mathfrak{J}$ can be completety characterized in terms of the multiple Itô integrals $I^p\varphi_p$, $p = 2, \ldots$, and the anticipating derivatives

$$\mathcal{D}I^p\varphi_p \;=\; I^{p-1}\hat{\varphi}_p, \quad p = 1, 2, \ldots.$$

Cf. Example 5.1 and Example 5.2. Namely, the domain $dom\mathfrak{J}$ consists of all stochastic functions admitting the representation

$$\varphi = \varphi^0 \oplus \sum_{p=1}^{\infty} \oplus \mathcal{D}\xi_p,$$

$$\xi_p = I^p \varphi_p, \quad p = 1, 2, ... \text{ such that } \quad \sum_{p=1}^{\infty} p^2 \|\xi_p\|^2 < \infty.$$

Here the components φ^0 are orthogonal to the range of the anticipating derivative $\mathcal{D}$. Correspondingly, we have

$$\mathfrak{J}\varphi = \sum_{p=1}^{\infty} \oplus p\xi_p.$$

See [16], [19].

Remark 5.2. In general, for an integrand of the form $\varphi \cdot 1_\Delta$ where its value φ on $\Delta \subseteq \Theta \times \mathbb{T}$ is an element in $L_2(\Omega)$ which is *not* $\mathfrak{A}_{]\Delta[}$-measurable, it occurs that

$$\mathfrak{J}\,(\varphi \cdot 1_\Delta) \neq \varphi \cdot \mu(\Delta).$$

To illustrate we can mention that

$$\mathfrak{J}\left(\mu(\Delta) \cdot 1_\Delta\right) = \mu(\Delta)^2 - M(\Delta),$$

if μ is Gaussian, and

$$\mathfrak{J}\left(\mu(\Delta) \cdot 1_\Delta\right) = \mu(\Delta)^2 - \mu(\Delta) - M(\Delta),$$

if μ is a Poisson (centred) stochastic measure. Remind that μ is of the type $E\mu = 0$, $E\mu^2 = M$.

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