Power variation analysis of some integral long-memory processes

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Summary. We show some results about the asymptotic behavior of the power variation and how they can be used for statistical purposes in the context of some integral long-memory processes. These processes are obtained as integrals with respect to a fractional Brownian motion with Hurst parameter $H > 1/2$.

1 Introduction

Let $\{Z_t, t \geq 0\}$ be a stochastic process. The realized power variation of order $p > 0$ is defined as

$$V_p^n(Z)_t = \sum_{i=1}^{[nt]} \left| Z_{i/n} - Z_{(i-1)/n} \right|^p .$$

For $p = 2$ we have the realized quadratic variation that has been widely used in statistics of random processes.

For any $p > 0$ the p-variation of a real valued function f on an interval $[a, b]$ is defined as

$$\mathrm{Var}_p(f; [a, b]) = \sup_\pi \left(\sum_{i=1}^n |f(t_i) - f(t_{i-1})|^p \right)^{1/p},$$

where the supremum runs over all partitions $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$.

Young (1936) proved that the Riemann–Stieltjes integral $\int_a^b f dg$ exists if f and g have finite p-variation and finite q-variation, respectively, in the interval $[a, b]$ and $\frac{1}{p} + \frac{1}{q} > 1$. Moreover, the following inequality holds

$$\begin{aligned} &\left| \int_a^b f dg - f(a)(g(b) - g(a)) \right| \\ &\leq c_{p,q} \mathrm{Var}_p(f; [a, b]) \mathrm{Var}_q(g; [a, b]), \end{aligned}$$

where $c_{p,q} = \zeta(\frac{1}{q} + \frac{1}{p})$, with $\zeta(s) := \sum_{n\geq 1} n^{-s}$.

We consider processes of the form $Z_t = \int_0^t u_s dB_s^H$, where B^H is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, and u is a stochastic process with paths of finite q-variation, $q < \frac{1}{1-H}$. The integral is a pathwise Riemann–Stieltjes integral and we are interested in the asymptotic behavior of the realized power variation conveniently scaled

$$n^{-1+pH} \sum_{i=1}^{[nt]} \left| Z_{i/n} - Z_{(i-1)/n} \right|^p = n^{-1+pH} \sum_{i=1}^{[nt]} \left| \int_{(i-1)/n}^{i/n} u_s dB_s^H \right|^p .$$

If $B^H = \{B_t^H, t \geq 0\}$ is a fBm with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$, then it is a zero mean Gaussian process with covariance function

$$E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), s, t \geq 0.$$

Also B^H is self-similar with exponent H and the fractional Gaussian noise: $\{B_n^H - B_{n-1}^H, n \in \mathbf{N}\}$ is a ergodic sequence with positive correlation function:

$$\rho_H(n) = \frac{(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}}{2} \sim cn^{2H-2}. \tag{1}$$

2 The results

Theorem 1. *Suppose that $u = \{u_t, t \in [0,T]\}$ is a stochastic process with finite q-variation, where $q < \frac{1}{1-H}$. Set $Z_t = \int_0^t u_s dB_s^H$. Then,*

$$n^{-1+pH} V_p^n(Z)_t \xrightarrow[n\to\infty]{P} c_p \int_0^t |u_s|^p ds.$$

where $c_p = E(|N(0,1)|^p) = \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(1/2)}$.

Proof. (A simple case) Assume first that $u_s \equiv 1$. Then $Z_t = B_t^H$ and

$$\begin{aligned} & n^{-1+pH} V_p^n(Z)_t \\ &= \left(\frac{1}{n}\right)^{1-pH} \sum_{i=1}^{[nt]} \left| B_{\frac{i}{n}}^H - B_{\frac{i-1}{n}}^H \right|^p \\ &= \frac{1}{n} \sum_{i=1}^{[nt]} \left| \frac{B_{\frac{i}{n}}^H - B_{\frac{i-1}{n}}^H}{\left(\frac{1}{n}\right)^H} \right|^p \\ &\sim \frac{1}{n} \sum_{i=1}^{[nt]} \left| B_i^H - B_{i-1}^H \right|^p \quad \text{(self-similarity)} \\ &\underset{L^1}{\overset{a.s}{\to}} tE(|B_1^H|^p) = c_p t \quad \text{(ergodicity)} \end{aligned}$$

For the general case we can consider two step sizes $1/m$ and $1/n$, nt integer, then for any $m \geq n$ we have the following decomposition

$$
\begin{aligned}
& m^{-1+pH} V_p^m(Z)_t - c_p \int_0^t |u_s|^p ds \\
= {} & m^{-1+pH} \sum_{j=1}^{[mt]} \left(\left| \int_{\frac{j-1}{m}}^{\frac{j}{m}} u_s dB_s^H \right|^p - \left| u_{\frac{j-1}{m}} (B_{\frac{j}{m}}^H - B_{\frac{j-1}{m}}^H) \right|^p \right) \\
& + m^{-1+pH} \sum_{j=1}^{[mt]} \left| u_{\frac{j-1}{m}} (B_{\frac{j}{m}}^H - B_{\frac{j-1}{m}}^H) \right|^p - \sum_{i=1}^{nt} \left| u_{\frac{i-1}{n}} \right|^p \sum_{j \in I(i)} \left| B_{\frac{j}{m}}^H - B_{\frac{j-1}{m}}^H \right|^p \\
& + m^{-1+pH} \sum_{i=1}^{nt} \left| u_{\frac{i-1}{n}} \right|^p \sum_{j \in I(i)} \left| B_{\frac{j}{m}}^H - B_{\frac{j-1}{m}}^H \right|^p - c_p n^{-1} \sum_{i=1}^{nt} \left| u_{\frac{i-1}{n}} \right|^p \\
& + c_p n^{-1} \sum_{i=1}^{nt} \left| u_{\frac{i-1}{n}} \right|^p - c_p \int_0^t |u_s|^p ds \\
= {} & A_m + B_{n,m} + C_{n,m} + D_n, \text{ where } I(i) = \{ j : \frac{j}{m} \in (\frac{i-1}{n}, \frac{i}{n}] \}, 1 \leq i \leq nt,
\end{aligned}
$$

and we have to show that each term goes to zero as n, m goes to infinity (see Corcuera *et al.* (2005) for the details).

Corollary 2. *Consider a stochastic process $Y = \{Y_t, t \geq 0\}$ such that*

$$n^{-1+pH} V_p^n(Y)_t \xrightarrow{P} 0$$

as n tends to infinity. Then

$$n^{-1+pH} V_p^n(Z+Y)_t \xrightarrow{P} c_p \int_0^t |u_s|^p ds,$$

as n tends to infinity.

Proof. (For $p \leq 1$). By the triangular inequality and the fact that

$$|V_p^n(Z+Y)_t - V_p^n(Z)_t| < V_p^n(Y)_t$$

For $H \in (\frac{1}{2}, \frac{3}{4}]$ the fluctuations of the power variation, properly normalized, have conditionally Gaussian asymptotic distributions. Set

$$v_1^2 := \lim_{n \to \infty} \operatorname{Var} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \left| B_i^H - B_{i-1}^H \right|^p \right).$$

It is not difficult to see that

$$v_1^2 = \delta_p + 2 \sum_{j \geq 1} \left(\gamma_p(\rho_H(j)) - \gamma_p(0) \right),$$

with $\rho_H(n)$ given by (1),

$$\delta_p = 2^p(\frac{1}{\sqrt{\pi}}\Gamma(p+\frac{1}{2}) - \frac{1}{\pi}\Gamma(\frac{p+1}{2})^2)$$

and $\gamma_p(x) = (1-x^2)^{p+\frac{1}{2}}\frac{2^p}{\pi}\Gamma(\frac{p+1}{2})^2\ {}_1\mathrm{F}_1(\frac{p+1}{2};\frac{1}{2};x^2)$, where ${}_1\mathrm{F}_1$ the confluent hyper-geometric function, that is

$${}_1F_1(a;b;z) = 1 + \frac{az}{b} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + ...$$

Theorem 3. *Fix $p > 0$. Assume $1/2 < H < 3/4$. Then*

$$(B_t^H, n^{-1/2+pH}\ V_p^n(B^H)_t - c_p t n^{1/2}) \xrightarrow{\mathcal{L}} (B_t^H, v_1 W_t),$$

as n tends to infinity, where $W = \{W_t, t \in [0,T]\}$ is a Brownian motion independent of the process B^H, and the convergence is in the space $\mathcal{D}([0,T])^2$ equipped with the Skorohod topology.

Proof. (Sketch of the proof) First we show the convergence of the finite dimensional distributions. Let $J_k = (a_k, b_k]$, $k = 1, \ldots, N$ be pairwise disjoint intervals contained in $[0,T]$. Define the random vectors $B = (B_{b_1}^H - B_{a_1}^H, \ldots, B_{b_N}^H - B_{a_N}^H)$ and $X^{(n)} = (X_1^{(n)}, \ldots, X_N^{(n)})$, where

$$X_k^{(n)} = n^{-1/2+pH} \sum_{[na_k]<j\le[nb_k]} \left| B_{j/n}^H - B_{(j-1)/n}^H \right|^p - n^{1/2} c_p |J_k|,$$

$k = 1, \ldots, N$ and $|J_k| = b_k - a_k$. We claim that

$$(B, X^{(n)}) \xrightarrow{\mathcal{L}} (B, V),$$

where B and V are independent and V is a Gaussian random vector with zero mean and with independent components of variances $v_1^2|J_k|$. By the self-similarity of the fBm, the convergence is equivalent to the convergence in distribution of $(B^{(n)}, Y^{(n)})$ to (B,V), where

$$B_k^{(n)} = n^{-H} \sum_{[na_k]<j\le[nb_k]} X_j,\ 1 \le k \le N$$

$$Y_k^{(n)} = \frac{1}{\sqrt{n}} \sum_{[na_k]<j\le[nb_k]} H(X_j),\ 1 \le k \le N.$$

$X_j = B_j^H - B_{j-1}^H$ and $H(x) = |x|^p - c_p$. The function $H(x)$ can be expanded in the form

$$H(x) = \sum_{m=2}^{\infty} c_m H_m(x),$$

where H_m is the mth Hermite polynomial. Let $\mathcal{H}_1$ be the closed subspace of L^2 generated by $\{X_j\}$, and for any $m \ge 2$ denote by $\mathcal{H}_m$ the closed subspace of L^2 generated by $H_m(X)$, where $X \in \mathcal{H}_1$, $E(X^2) = 1$. Let $\mathcal{H}_1^{\odot m}$ be the symmetric tensor product equipped with the norm $\sqrt{m!}\,\|\cdot\|_{\mathcal{H}_1^{\otimes m}}$. We know that the mapping

$$I_m : \mathcal{H}_1^{\odot m} \to \mathcal{H}_m$$

defined by $I_m(X^{\otimes m}) = H_m(X)$, is a linear isometry. We will denote by J_m the projection operator on $\mathcal{H}_m(X)$.

Now by the works of Nualart and Peccati (2005), Peccati and Tudor (2005) and Hu and Nualart (2005) we simply have to check that:

For any $m \geq 2$ and $k = 1, \ldots, N$, the limit $\lim_{n\to\infty} E(|J_m Y_k^{(n)}|^2) = \sigma_{m,k}^2$ exists and $\sum_{m=2}^{\infty} \sup_n E(|J_m Y_k^{(n)}|^2) < \infty$.

For any $m \geq 2$ and $k \neq h$ $\lim_{n\to\infty} E(J_m Y_k^{(n)} J_m Y_h^{(n)}) = 0$.

For any $m \geq 2$, $k = 1, \ldots, N$ and $1 \leq p \leq m-1$,

$$\lim_{n\to\infty} I_m^{-1} J_m Y_k^{(n)} \otimes_p I_m^{-1} J_m Y_k^{(n)} = 0,$$

where $\otimes_p$ denotes the contraction of p indices. (i), (ii) and (iii) are true because

$$\sum_{j=1}^{\infty} \rho_H(j)^m \sim \sum_{j=1}^{\infty} j^{(2H-2)m} < \infty$$

since $m \geq 2$ and $1/2 < H < 3/4$. For the tightness condition is sufficient to show that the sequence of processes

$$Z_t^{*(n)} = n^{-1/2+pH}\ V_p^n(B^H)_t - c_p[nt]/n^{1/2}.$$

is tight in $\mathcal{D}([0,T])$. Then we can compute for $s < t$

$$E(\left|Z_t^{*(n)} - Z_s^{*(n)}\right|^4) = n^{-2} E\left(\left|\sum_{j=[ns]+1}^{[nt]} H(X_j)\right|^4\right).$$

By Taqqu (1977) we know that, for all $N \geq 1$

$$\frac{1}{N^2} E\left(\left|\sum_{j=1}^{N} H(X_j)\right|^4\right) \leq K(\sum_{u=0}^{\infty} \rho_H^2(u))^2.$$

As a consequence,

$$\sup_n E(\left|Z_t^{*(n)} - Z_s^{*(n)}\right|^4) \leq C|t-s|^2,$$

and by Billingsley (1968) we get the desired tightness property.

As a consequence we have the following Theorem

Theorem 4. *Fix $p > 0$. Let B^H be a fBm with Hurst parameter $H \in (1/2, 3/4)$. Suppose that $u = \{u_t, t \in [0,T]\}$ is a stochastic process measurable with respect to $\mathcal{F}_T^H$, and with Hölder continuous trajectories of order $a > \frac{1}{2(p\wedge 1)}$. Set $Z_t = \int_0^t u_s dB_s^H$. Then*

$$n^{-1/2+pH} V_p^n(Z)_t - c_p\sqrt{n}\int_0^t |u_s|^p ds \xrightarrow{\mathcal{L}} v_1 \int_0^t |u_s|^p dW_s,$$

as n tends to infinity, where $W = \{W_t, t \ \in [0,T]\}$ is a Brownian motion independent of $\mathcal{F}_T^H$, and the convergence is stable *and in $\mathcal{D}([0,T])$.*

For the notion of *stable converge* see Aldous and Eagleson (1978). The following corollary gives the distributional effect of adding a process Y to the process Z, see also Corollary 2 above.

Corollary 5. *Assume the same conditions as in the previous Theorem. Consider a stochastic process $Y = \{Y_t, t \in [0,T]\}$ such that*

$$n^{-\frac{1}{2}+pH} V_p^n(Y)_T \xrightarrow{P} 0,$$

as n tends to infinity. Then,

$$n^{-1/2+pH} V_p^n(Y+Z)_t - c_p\sqrt{n}\int_0^t |u_s|^p ds \xrightarrow{\mathcal{L}} v_1 \int_0^t |u_s|^p dW_s$$

as n tends to infinity, where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of the process B^H, and the convergence is stable and in $\mathcal{D}([0,T])$.

We can also derive the following convergence in distribution for the fluctuations of the power variation of stochastic integrals, in the case $H = \frac{3}{4}$.

Theorem 6. *Suppose that $H = 3/4$ and $u = \{u_t, t \in [0,T]\}$ is a stochastic process measurable with respect to $\mathcal{F}_T^H$ with Hölder continuous trajectories of the order $a > \frac{1}{2(p\wedge 1)}$. Then,*

$$(\log n)^{-1/2}\left(n^{-1/2+pH} V_p^n(Z)_t - c_p\sqrt{n}\int_0^t |u_s|^p ds\right) \xrightarrow{\mathcal{L}} v_2 \int_0^t |u_s|^p dW_s,$$

as $n \to \infty$, where $W = \{W_t, t \ \in [0,T]\}$ is a Brownian motion independent of $\mathcal{F}_T^H$ and v_2 is given by

$$v_2^2 := \lim_{n\to\infty} Var\left(\frac{1}{\sqrt{n\log n}}\sum_{i=1}^n \left|B_i^H - B_{i-1}^H\right|^p\right).$$

Where the converge is stable in the Skorohod space $\mathcal{D}([0,T])$.

If $H > \frac{3}{4}$, the fluctuations of the power variation converge to a process in the second chaos which is called the Rosenblatt process.

Theorem 7. *Fix $p > 0$ and assume that $\frac{3}{4} < H < 1$. Then, in $\mathcal{D}([0,T])$,*

$$n^{2-2H}(n^{-1+pH}V_p^n(B^H)_t - c_p t) \xrightarrow{\mathcal{L}} Z_t$$

where

$$Z_t = \frac{1}{\Gamma(2-2H)\cos((1-H)\pi)} d_p \times \int_0^{\infty} \int_0^{x_2} \frac{e^{i(x_1+x_2)t} - 1}{i(x_1+x_2)} |x_1|^{1/2-H} |x_2|^{1/2-H} dW_{x_1} dW_{x_2},$$

is the Rosenblatt process, $\{W_t, t \in [0,T]\}$ is a Brownian motion and

$$d_p = E(|B_1^H|^{2+p}) - E(|B_1^H|^p).$$

3 Applications

Many statistical analysis of financial and temperature data have shown the presence of significant power at low frequencies in their spectral analysis, which means long-range dependence (see Willinger *et al.* (1999), Cutland *et al.* (1995), Brody *et al.* (2002) and the references in Shiryaev (1999). But these investigations have produced controversies, specially because if we use these models to describe the evolution of stock prices, then the resulting market has arbitrage opportunities and the lack of arbitrage is a paradigm in the modern financial economics (see Rogers (1997)).

We have seen that, under certain assumptions, the values of $V_p^n(Z)_t$ oscillate around cn^{pH-1}. This can be used to give a consistent estimator of H. Then, we shall study the behavior of this estimator by simulating a fractional Brownian motion and a geometric fractional Brownian motion and we will try to corroborate the theoretical results of the previous section. In the case the model is not completely specified by H because the process u is unknown, we will estimate H by a regression of $\log V_p^n(Z)_t$ against $\log n$ for different values of n. We shall consider real data of the stocks prices in the Spanish financial market and we will compare the results with the power variation analysis with the results using the well known R/S analysis.

3.1 The method

Let $Z_{1/n}, Z_{2/n}, ..., Z_1$ be,a sample of n observations. Then by Theorem 1 we have

$$V_p^n(Z)_1 \sim c_p \left(\int_0^1 |u_s|^p ds \right) n^{1-pH}$$

Therefore a consistent estimator of H is given by

$$\hat{H} = \frac{1}{p} - \frac{\log V_p^n(Z)_1 - \log c_p - \log \int_0^1 |u_s|^p ds}{p \log n}.$$

Note that this estimator requires the process u to be observable to evaluate $\int_0^1 |u_s|^p ds$, this is not true in general, however if the process is, for instance, a solution of the stochastic differential equation

$$dZ_t = Z_t(bdt + \sigma dB_t^H)$$

with σ known then $u = \sigma Z$. Nevertheless Z is observed in discrete times $1/n, 2/n, ...$, so $\int_0^1 |u_s|^p ds$ has to be estimated by, for instance, $\frac{1}{n}\sum_{i=1}^n |Z_{\frac{i}{n}}|^p$. It can be easily seen that if $H > \frac{1}{2(p \wedge 1)}$ this estimation does not affect to the asymptotic distribution of $\hat{H}$.

Asymptotic behavior of $\hat{H}$

By Theorem 4 and assuming that the estimation of $\int_0^1 |u_s|^p ds$ does not affect the asymptotic behavior, we have the conditional converge

$$\frac{n^{-1/2+pH} V_p^n(Z)_1 - c_p \frac{1}{\sqrt{n}} \sum_{i=1}^n |u_{\frac{i}{n}}|^p}{\sqrt{v_1^2 \frac{1}{n} \sum_{i=1}^n |u_{\frac{i}{n}}|^{2p}}} \xrightarrow{\mathcal{L}} N(0,1),$$

from here it is easy to obtain the approximate confidence interval of coefficient γ

$$\hat{H} \pm \frac{k_\gamma v_1}{p c_p \log n} \frac{\sqrt{\sum_{i=1}^n |u_{\frac{i}{n}}|^{2p}}}{\sum_{i=1}^n |u_{\frac{i}{n}}|^p} \tag{2}$$

where $k_\gamma = \Phi^{-1}(\frac{1-\gamma}{2})$ and Φ denotes the c.d.f. of the standard normal distribution.

Note that v_1 depends on p as it is shown in Figure 1.

The following table gives the values of $\hat{H}$ for different values of p and H in the case we have a sample of size $n = 2000$ of equally spaced observations of fBm, in parenthesis we have the radius of a confidence interval with $\gamma = 0.95$.

	H		
p	0.6	0.65	0.7
0.75	0.602(0.005)	0.651(0.005)	0.701(0.007)
1	0.603(0.004)	0.652(0.005)	0.702(0.006)
2	0.605(0.005)	0.654(0.004)	0.703(0.004)

The following table gives the values of $\hat{H}$ for different values of p and H in the case that Z is a geometrical fBm with $\sigma = 1$ and $b = 0$, the sample size of equally spaced observations is $n = 2000$. In parenthesis we have the radius of a confidence interval with $\gamma = 0.95$.

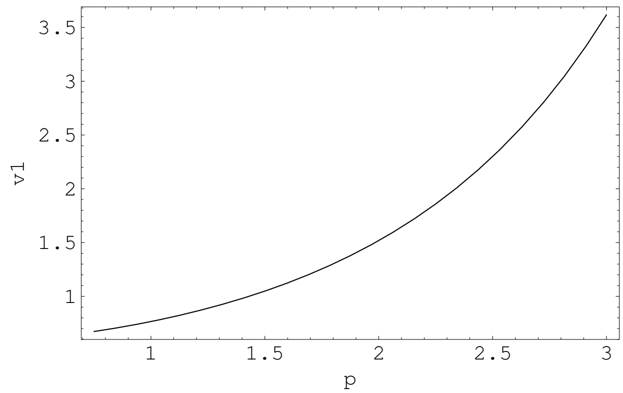

Fig. 1. Behavior of v_1 with p for $H = 0.7$

	H		
p	0.6	0.65	0.7
0.75	0.602(0.005)	0.651(0.005)	0.701(0.007)
1	0.603(0.004)	0.652(0.005)	0.702(0.006)
2	0.605(0.005)	0.654(0.004)	0.703(0.004)

3.2 Estimation of H when u is unknown

In case of the process u is not known we can consider the statistics

$$V_p^{[n/m]}(Z)_1 = \sum_{i=1}^{[n/m]} \left| Z_{im/n} - Z_{(i-1)m/n} \right|^p$$

for different values of m, $1 \le m \le m_u$. The results of the previous section imply that, under the assumptions on the process $Z_t = \int_0^t u_s dB_s^H, 0 \le t \le 1$,

$$V_p^{[n/m]}(Z)_1 \sim c_p \left(\int_0^1 |u_s|^p ds \right) (\frac{m}{n})^{pH-1}$$

whenever n/m is large enough. This is why we only consider values of $m \le m_u$. Then this can be used to estimate H by a log-log plot also called *pox plot*. We shall denote $\tilde{H}$ the corresponding estimator.

Figure 2 depicts the pox plot of the power variation of order 1 corresponding to a sample of size 2.000 of a fractional Brownian motion of Hurst parameter 0.7 series and results in an estimate of H=0.698. Figure 3 is similar but considering a geometrical fractional Brownian motion of Hurst parameter 0.7 and results also in an estimate of H=0.697. In both cases we have taken $m_u = 40$ and $p = 1$. We have use the method of Davies and Harte (1987) for simulating a Gaussian stationary sequence.

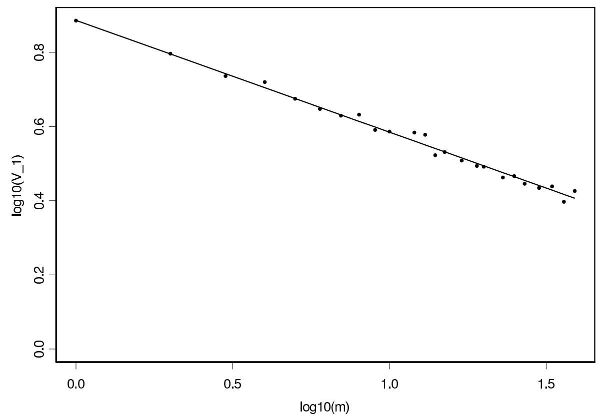

Fig. 2. Data corresponding to a fBm with H=0.7, $\tilde{H} = 0.698$.

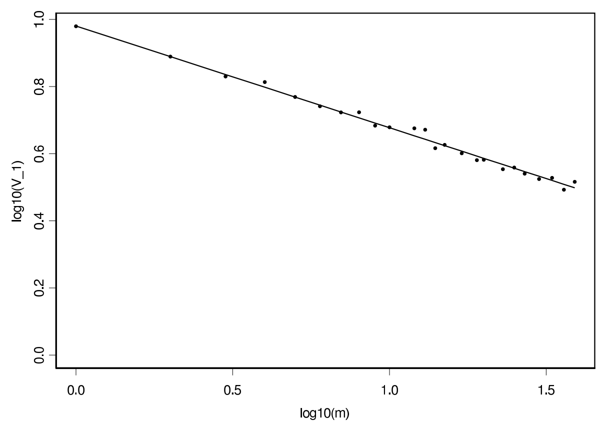

Fig. 3. Data corresponding to a geometrical fBm with $H = 0.7$, $\tilde{H} = 0.697$

In the following examples we consider the prices of certain stocks in the Spanish market as the process Z, and we estimate H by a log-log plot. Figure 4 corresponds to the index IBEX35, the estimation of H is $\tilde{H} = 0.533$, Figure 5 corresponds to the shares of the bank BBVA, resulting in $\tilde{H} = 0.517$, and Figure 6 to the shares of the Spanish Telephone company "Telefónica" and $\tilde{H} = 0.513$.

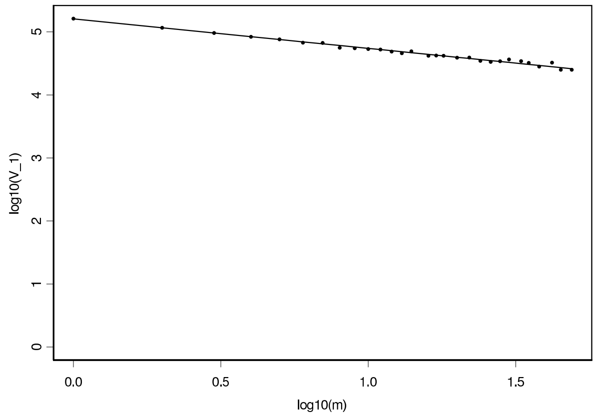

Fig. 4. Data corresponding to the Spanish index Ibex35 from 1992-2001, $n = 2465$, $\tilde{H} = 0.533$

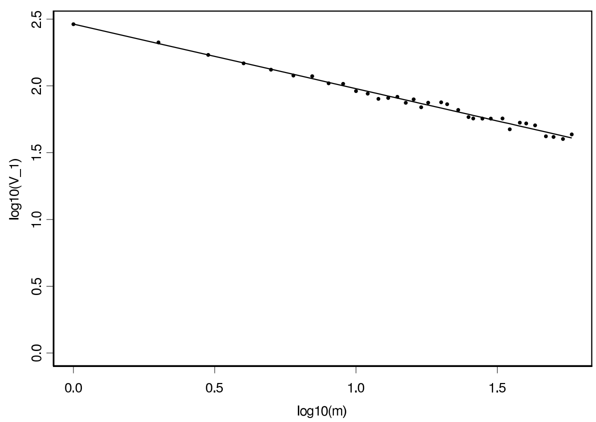

Fig. 5. Daily data for the price of the shares of the Spanish bank Bbva, from 1990-2001, $n = 2909$, $\tilde{H} = 0.517$

Asymptotic behavior of $\tilde{H}$

The problem with the previous method is that it provides only a point estimation of H but we do not have a confidence interval. However we can try to relate $\tilde{H}$ with the estimations $\hat{H}_n$ for different values of n and from here we can get a confidence interval for $\tilde{H}$.

If we consider different values of n, $n_1 \leq n_2 ... \leq n_r$ we have that

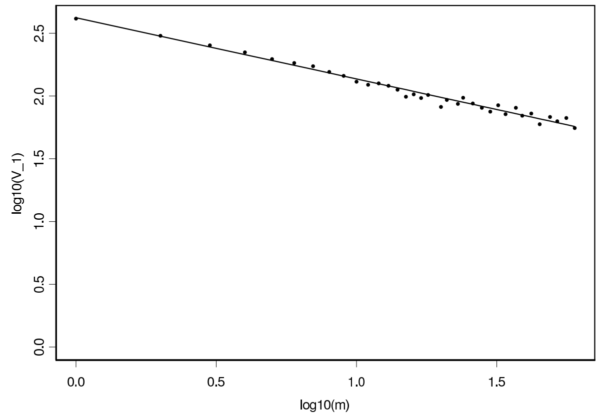

Fig. 6. Daily data for the price of shares of "Telefónica", from 1990-2001, $n = 3009$, $\tilde{H} = 0.513$

$$\hat{H}_{n_i} = \frac{1}{p} - \frac{\log V_p^{n_i}(Z)_1 - \log c_p - \log \int_0^1 |u_s|^p ds}{p \log n_i}.$$

On the other hand

$$1 - p\tilde{H} = \frac{\sum_{i=1}^{r} (\log V_p^{n_i}(Z)_1 - \log V_p^{\bar{n}}(Z)_1)(\log n_i - \overline{\log} n)}{\sum_{i=1}^{r} (\log n_i - \overline{\log} n)^2},$$

where the bar denotes the mean. Then, by straightforward calculations, we obtain that

$$\tilde{H} = \sum_{i=1}^{r} \hat{H}_{n_i} \frac{(\log n_i - \overline{\log} n)^2}{\sum_{i=1}^{r} (\log n_i - \overline{\log} n)^2} + \overline{\log} n \sum_{i=1}^{r} \hat{H}_{n_i} \frac{(\log n_i - \overline{\log} n)}{\sum_{i=1}^{r} (\log n_i - \overline{\log} n)^2}$$

and consequently

$$\tilde{H} - H = \sum_{i=1}^{r} (\hat{H}_{n_i} - H) \frac{(\log n_i - \overline{\log} n)^2}{\sum_{i=1}^{r} (\log n_i - \overline{\log} n)^2} + \overline{\log} n \sum_{i=1}^{r} (\hat{H}_{n_i} - H) \frac{(\log n_i - \overline{\log} n)}{\sum_{i=1}^{r} (\log n_i - \overline{\log} n)^2}$$

Then if Δ_{n_1} is the radius of a confidence interval for $\hat{H}_{n_1}$, and we take only two values of n, $n_1 = [n/2]$ and $n_2 = n$, we obtain

$$\Delta = \frac{\log n}{\log 2} (\Delta_{[n/2]} + \Delta_n).$$

and Δ_{n_1} can be obtained from (2) if we know u except for a scale factor. This is the case of a geometric fractional Brownian motion. Note that we have to use the union-intersection principle to determine the confidence of the interval.

By considering that the data follow a geometric fractional Brownian motion and taking logarithms to estimate H we obtain the following estimations and, in parenthesis, we estimate the radius of a confidence interval with $\gamma = 0.95$. The data obtained for a simulation of a fractional Brownian motion with $H = 0.7$ serve as a control.

Ibex35	Telefónica	Bbva	Fbm ($H = 0.7$)
0.538 (0.123)	0.550 (0.112)	0.574 (0.115)	0.703 (0.179).

Then, here data do not show evidence in favor of the "fractality", but the intervals we obtain are very conservative.

3.3 The R/S method

The graphical implementation of the classical R/S-statistic given by

$$\frac{\max_{k\leq n}(\sum_{i=1}^{k} Z_i - \frac{k}{n}\sum_{i=1}^{n} Z_i) - \min_{k\leq n}(\sum_{i=1}^{k} Z_i - \frac{k}{n}\sum_{i=1}^{n} Z_i)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} Z_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)^2}},$$

exploit the fact that if Z is a fractional Gaussian noise with $H > 1/2$ for large n its values oscillate around cn^H and we can also use this to estimate H by a log-log plot .

More specifically, given sample of n observations is subdivided into k blocks, each of size $[n/k]$. Then, for each *lag* $n_i, n_i \leq n$, estimates $R(k_m, n_i)/S(k_m, n_i)$ of $R(n_i)/S(n_i)$ are computed by starting at the points, $k_m = (m-1)[n/k] + 1, m = 1, 2, ..., k$, and such that $k_m + n_i \leq nN$. Thus, for any given m, all the data points before $k_m = (m-1)[n/k]+1$ are ignored. For values of n_i smaller than $[n/k]$, there are k different estimates of $R(n)/S(n)$; for values of n_i approaching n, there are fewer values, as few as 1 when $n \geq n-[n/k]$.

The graphical R/S-approach consists of calculating $R(k_m, n_i)/S(k_m, n_i)$ for logarithmically spaced values of n_i, and plotting $\log R(k_m, n_i)/S(k_m, n_i)$ versus $\log(n_i)$, for all starting points k_m. This results in the *rescaled adjusted range plot,* also known as the *pox plot of R/S*. See Willinger *et al.* (1999) for more details. The problem with the R/S method is that we do not have a distributional theory to give confidence intervals. The next figures show this method applied to the same data we used in the power variation analysis. We also denote by $\hat{H}$ the R/S estimator. Note that the results are quite similar to that using the power variation.

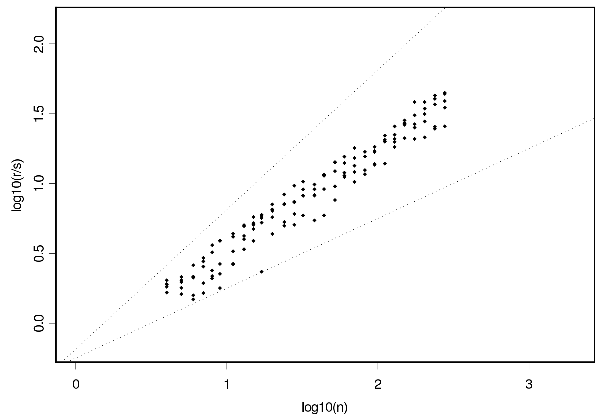

Fig. 7. Data corresponding to a fBm with $H = 0.7$, $\hat{H} = 0.716$

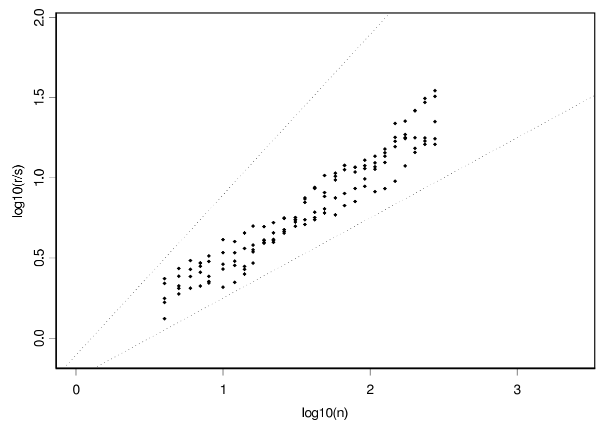

Fig. 8. Data corresponding to the Spanish Index Ibex35, $\hat{H} = 0.588$

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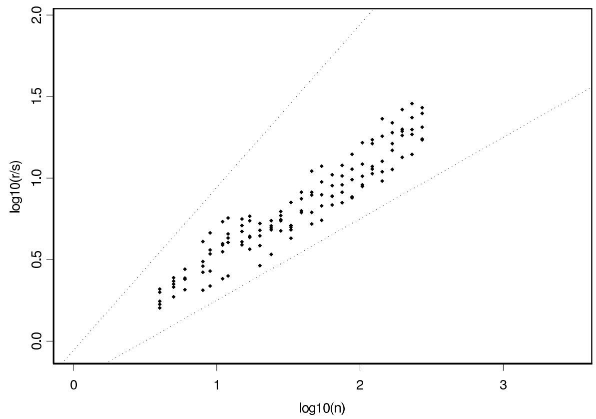

Fig. 9. Data corresponding to the shares of the Spanish Telephone company "Telefónica", $\hat{H} = 0.561$

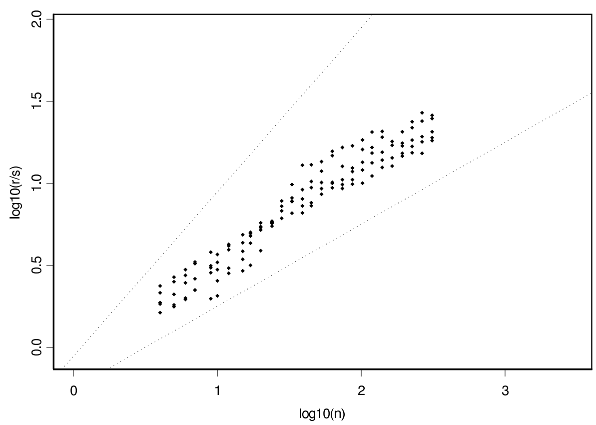

Fig. 10. Data corresponding to the shares of the Spanish Bank "Bbva", $\hat{H} = 0.582$

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