Theory and applications of infinite dimensional oscillatory integrals

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Summary. Theory and main applications of infinite dimensional oscillatory integrals are discussed, with special attention to the relations with the original work of K. Itô in this area. New developments related to polynomial interactions are also presented.

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1 Introduction

Professor K. Itô's work on the topic of infinite dimensional oscillatory integrals has been very germinal and stimulated much the subsequent research in this area. It is therefore a special honour and pleasure to be able to dedicate the present pages to him. We shall give a short exposition of the theory of a particular class of functionals, the oscillatory integrals:

$$I^{\frac{\Phi}{\epsilon}}(f) = \text{“} \int_{\Gamma} e^{i\frac{\Phi}{\epsilon}(\gamma)} f(\gamma) d\gamma \text{”} \tag{1}$$

where Γ denotes either a finite dimensional space (e.g. $\mathbb{R}^s$, or an s-dimensional differential manifold M^s), or an infinite dimensional space (e.g. a “path space”). $\Phi : \Gamma \to \mathbb{R}$ is called phase function, while $f : \Gamma \to \mathbb{C}$ is the function to be integrated and $\epsilon \in \mathbb{R} \setminus \{0\}$ is a parameter. The symbol $d\gamma$ denotes a “flat” measure. In particular, if $dim(\Gamma) < \infty$ then $d\gamma$ is the Riemann-Lebesgue volume measure, while if $dim(\Gamma) = \infty$ an analogue of Riemann-Lebesgue measure is not mathematically defined and $d\gamma$ is just an heuristic expression.

1.1 Finite dimensional oscillatory integrals

In the case where Γ is a finite dimensional vector space, i.e. $\Gamma = \mathbb{R}^s$, $s \in \mathbb{N}$, the expression (1)

$$\text{“}\int_{\mathbb{R}^s} e^{i\frac{\Phi}{\epsilon}(\gamma)} f(\gamma) d\gamma\text{”} \tag{2}$$

can be defined as an improper Riemann integral. The study of finite dimensional oscillatory integrals of the type (2) is a classical topic, largely developed in connection with several applications in mathematics (such as the theory of Fourier integral operators [48]) and physics. Interesting examples of integrals of the form (2) in the case $s = 1$, $\epsilon = 1$, $f = \chi_{[0,w]}$, $w > 0$, and $\Phi(x) = \frac{\pi}{2}x^2$, are the Fresnel integrals, that are applied in optics and in the theory of wave diffraction. If $\Phi(x) = x^3 + ax$, $a \in \mathbb{R}$ we obtain the Airy integrals, introduced in 1838 in connection with the theory of the rainbow.
Particular interest has been devoted to the study of the asymptotic behavior of integrals (2) when ϵ is regarded as a small parameter converging to 0. Originally introduced by Stokes and Kelvin and successively developed by several mathematicians, in particular van der Corput, the “stationary phase method” provides a powerful tool to handle the asymptotics of (2) as $\epsilon \downarrow 0$. According to it, the main contribution to the asymptotic behavior of the integral should come from those points $\gamma \in \mathbb{R}^s$ which belong to the critical manifold:

$$\Gamma_c^{\Phi} := \{\gamma \in \mathbb{R}^s, \mid \Phi'(\gamma) = 0\},$$

that is the points which make stationary the phase function Φ. Beautiful mathematical work on oscillatory integrals and the method of stationary phase is connected with the mathematical classification of singularities of algebraic and geometric structures (Coxeter indices, catastrophe theory), see, e.g. [31].

1.2 Infinite dimensional oscillatory integrals

The extension of the results valid for $\Gamma = \mathbb{R}^s$ to the case where Γ is an infinite dimensional space is not trivial. The main motivation is the study of the “Feynman path integrals”, a class of (heuristic) functional integrals introduced by R.P Feynman in 1942[3] in order to propose an alternative, Lagrangian, formulation of quantum mechanics. According to Feynman, the solution of the Schrödinger equation describing the time evolution of the state $\psi \in L^2(\mathbb{R}^d)$ of a quantum particle moving in a potential V

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{3}$$

[3] The first proposal going in the direction of Feynman's formulation can be found in work by P. Dirac in 1935, which inspired Feynman's own work.

(where $m > 0$ is the mass of the particle, $\hbar$ is the reduced Planck constant, $t \geq 0$, $x \in \mathbb{R}^d$) can be represented by a "sum over all possible histories", that is an integral over the space of paths γ with fixed end point

$$\psi(t,x) = \text{“} \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))d\gamma \text{”} \tag{4}$$

$S_t(\gamma) = S^0(\gamma) - \int_0^t V(s,\gamma(s))ds$, $S^0(\gamma) = \frac{m}{2}\int_0^t |\dot{\gamma}(s)|^2 ds$, is the classical action of the system evaluated along the path γ and $d\gamma$ a heuristic "flat" measure on the space of paths (see e.g. [40] for a physical discussion of Feynman's approach and its applications). The Feynman path integrals (4) can be regarded as oscillatory integrals of the form (1), where

$$\Gamma = \{ \text{ paths } \gamma : [0,t] \to \mathbb{R}^s,\ \gamma(t) = x \in \mathbb{R}^s\},$$

the phase function Φ is the classical action functional S_t, $f(\gamma) = \psi_0(\gamma(0))$, the parameter ϵ is the reduced Planck constant $\hbar$ and $d\gamma$ denotes heuristically

$$d\gamma = \text{“}C \prod_{s\in[0,t]} d\gamma(s) \text{”}, \tag{5}$$

$C :=$ “$(\int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_0(\gamma)}d\gamma)^{-1}$” being a normalization constant.

The Feynman's path integral representation (4) for the solution of the Schrödinger equation is particularly suggestive. Indeed it creates a connection between the classical (Lagrangian) description of the physical world and the quantum one and makes intuitive the study of the semiclassical limit of quantum mechanics, that is the study of the detailed behavior of the wave function ψ in the case where the Planck constant $\hbar$ is regarded as a small parameter. According to an (heuristic) application of the stationary phase method, in the limit $\hbar \downarrow 0$ the main contribution to the integral (4) should come from those paths γ which make stationary the action functional S_t. These, by Hamilton's least action principle, are exactly the classical orbits of the system.

Despite its powerful physical applications, formula (4) lacks mathematical rigour, in particular the "flat" measure $d\gamma$ given by (5) has no mathematical meaning.

In 1949 Kac [54, 55] observed that, by considering the heat equation (with $m = \hbar = 1$ for simplicity)

$$\begin{cases} \frac{\partial}{\partial t}u = \frac{1}{2}\Delta u - Vu \\ u(0,x) = u_0(x) \end{cases} \tag{6}$$

instead of the Schrödinger equation and by replacing the oscillatory factor $e^{iS_t(\gamma)}d\gamma$ by the non oscillatory $e^{-1S_t(\gamma)}d\gamma$, one can give (for "good" V) a mathematical meaning to Feynman's formula in terms of a well defined Gaussian integral on the space of continuous paths: an integral with respect to the well known Wiener measure

$$u(t,x) = \text{``} \int e^{-S_t(\omega)} u_0(\omega(t)) d\omega \text{``} = \mathbb{E}\Big[e^{-\int_0^t V(\omega(s)+x)ds} u_0(\omega(t)+x) \Big] \quad (7)$$

(with E standing for expectation with respect to the standard Wiener process (mathematical Brownian motion) ω started at time 0 at the origin). Equation (7) is called Feynman-Kac formula.

In 1956 I.M. Gelfand and A.M. Yaglom [44] tried to realize Feynman's heuristic complex measure $e^{\frac{i}{\hbar}\Phi(\gamma)} d\gamma$ by means of a limiting procedure:

$$e^{\frac{i}{\hbar}\Phi(\gamma)} d\gamma := \lim_{\sigma \downarrow 0} e^{\frac{i}{\hbar - i\sigma}\Phi(\gamma)} d\gamma$$

In 1960 Cameron [34] proved however that the resulting measure cannot be $\sigma-$ additive and of bounded variation, even on very "nice" subsets of paths' space, and it is not possible to implement an integration in the Lebesgue's traditional sense (not even locally in space). As a consequence mathematicians tried to realize the integral (4) as a linear continuous functional on a suitable Banach algebra of integrable functions.

A particularly interesting approach can be found in the two pioneering papers by K. Itô [51, 52]. Itô was aware of the interest of Feynman's formula, as well as of the mathematical problems involved in it:
"It is easy to see that (4) *solves* (3) *unless we require mathematical rigour."* [51]
In the first paper in 1961 the author starts to study the problem by assuming that the potential V has a simple form, postponing the study of a more general case:
"It is our purpose to define the generalized measure $d\gamma$ (that, in our terms, is the integral $I^{\frac{\Phi}{\epsilon}}(f)$) *rigorously and prove* (4) *solves* (3) *in the case* $V \equiv 0$ *(case of no force) or* $V(x) = x$ *(case of constant force). We hope that this fact will be proven for a general* V *with some appropriate regularity conditions."*
Very shortly, what Itô does is to define rigorously the "generalized measure" (5), hence the heuristic integral (4), for V of above form and ψ_0 having a Fourier transform of compact support as a linear functional, taken to be the limit for $n \to \infty$ of finite dimensional approximations $I_n(\psi_0) = C_n \int_{L_x} e^{\frac{i}{2\hbar}\int_0^t \dot\gamma(s)^2 ds} \psi_0(\gamma(t)) P_n^{(x)}(d\gamma)$, with L_x the "translate by x of Cameron-Martin space", $P_n^{(x)}$ a suitable Gaussian measure associated with a certain compact operator T concentrated on L_x and $C_n \equiv \prod_j (1 - in\nu_j\hbar)^{\frac{1}{2}}$, $\{\nu_j\}$ being the eigenvalues of T. In the second paper [52] on the subject in 1967 K. Itô extended the class of potentials which can be handled and covers the case where the function $V : \mathbb{R}^d \to \mathbb{C}$ is the Fourier transform of a complex bounded variation measure on $\mathbb{R}^d$.
Itô's definition of the heuristic integral (4) is of the form

$$\lim_V \prod_{j=1}^{\infty} (1 - i\mu_j)^{\frac{1}{2}} E\left(e^{\frac{i}{2\hbar}\int_0^t \dot\gamma(s)^2 ds} \psi_0(\gamma(t)); a; V \right),$$

with E meaning expectation with respect to the Gaussian measure with mean a in L_x and a nuclear covariance operator V with eigenvalues μ_j (lim being taken along the directed system of all such V's, being independent of a). Itô's method for the definition of the Feynman's functional applies also to the Wiener integral and to the path integral representation (7) of the solution of the heat equation: *"Our definition is also applicable to the Wiener integral; namely, using it, we shall prove that the solution of the heat equation* (6) *is given by*

$$u(t,x) = \int_{\Gamma} e^{-\int_0^t \left(\frac{\dot{\gamma}^2(s)}{2} + V(\gamma(s))\right) ds} u_0(\gamma(t)) d\gamma$$

for any bounded continuous function $V(x)$*.... This should be called the Feynman's version of Kac's theorem".*
"Now that Kac's theorem is well known to probabilists, no one bothers with its Feynman version. However it is interesting that Kac had the Feynman version ... in mind and formulated it as ... to make it rigorous".

1.3 Other examples of "Feynman type formulae"

The path integral representation (4) has been extended to more general dynamical systems. As we have already seen, its probabilistic version, i.e. the Feynman-Kac formula (7), is a representation of the solution of the heat equation. More generally probabilistic type integrals, which can be heuristically represented by expressions of the following form

$$\text{“}\int_{\Gamma} e^{-\frac{\Phi}{\epsilon}} f(\gamma) d\gamma\text{”} \tag{8}$$

(with the function $\Phi : \Gamma \to \mathbb{R}$ lower bounded and $\epsilon > 0$) have several applications, e.g. in stochastic analysis, statistical mechanics, hydrodynamics and in the theory of acoustic and electromagnetic waves.
The original Feynman path integral representation (4) for the solution of the Schrödinger equation and, more generally, heuristic oscillatory integrals of the type

$$\text{“}\int_{\Gamma} e^{i\frac{\Phi}{\epsilon}} f(\gamma) d\gamma\text{”} \tag{9}$$

can also be extended to the study of more general quantum systems. Feynman himself generalized formula (4) to a corresponding formula describing (relativistic) quantum fields. Recent applications of heuristic path integrals can be found in gauge theory (Yang-Mills fields), quantum gravity and in string theory.

Particularly interesting is the application to topological field theory, e.g. Chern-Simons' model. In this case the integration is performed on a space Γ of geometric objects, i.e. on the space of connection 1-forms on the principal fiber bundle over a 3-dimensional manifold M. The phase function Φ is the Chern-Simons action functional:

$$\Phi(\gamma) \equiv \frac{k}{4\pi} \int_M \Big(\langle \gamma \wedge d\gamma \rangle + \frac{1}{3} \langle \gamma \wedge [\gamma \wedge \gamma] \rangle \Big), \tag{10}$$

where γ denotes a $g-$valued connection 1-form, g being the Lie algebra of a compact Lie group G (the "gauge group"). Φ is metric independent. The function f to be integrated is given by

$$f(\gamma) := \prod_{i=1}^{n} \mathrm{Tr}(Hol(\gamma, l_i)) \in \mathbb{C}, \tag{11}$$

where $(l_1, ..., l_n)$, $n \in \mathbb{N}$, are loops in M whose arcs are pairwise disjoint and $Hol(\gamma, l)$ denotes the holonomy of γ around l. According to a conjecture by Witten [70] and Schwartz the integral $I^{\Phi}(f)$ should represent a topological invariant. In particular, if $M = S^3$ and $G = SU(2)$ resp. $G = SU(N)$ resp. $G = SO(N)$, $I^{\Phi}(f)$ gives the Jones polynomials, resp. the Homfly polynomials resp. the Kauffmann polynomials. In the next section we shall see how a good part of these statements can be rigorously implemented using an adequate mathematical definition of Feynman path integrals.

2 Mathematical definition of infinite dimensional oscillatory integrals

The heuristic Feynman integrals given by formula (4) and its generalization (9) have lead to fascinating and fruitful applications in physics and mathematics, even though as as they stand they do not have a well defined mathematical meaning. The present section is devoted to the description of the mathematical definition of the Feynman functional, and more generally of the infinite dimensional oscillatory integrals. In order to mirror the features of the heuristic Feynman measure, the Feynman functional should have some basic properties:

1. It should behave in a simple way under "translations and rotations in Γ", reflecting the fact that $d\gamma$ is a "flat" measure.
2. It should satisfy a Fubini type theorem, concerning iterated integrations along subspaces of Γ (allowing the construction, in physical applications, of a one-parameter group of unitary operators).
3. It should be approximable by finite dimensional oscillatory integrals, allowing a sequential approach in the spirit of Feynman's original work.
4. It should be related to probabilistic integrals with respect to the Wiener measure, allowing an "analytic continuation approach to Feynman path integrals from Wiener type integrals".
5. It should be sufficiently flexible to yield a rigorous mathematical implementation of an infinite dimensional version of the stationary phase method and the corresponding study of the semiclassical limit of quantum mechanics.

2.1 Finite dimensional case

The first step is the definition of the oscillatory integrals on a finite dimensional space $\Gamma := \mathbb{R}^n$, whose elements will be denoted by $x \in \mathbb{R}^n$:

$$“\int_{\mathbb{R}^n} e^{i\frac{\Phi}{\epsilon}(x)} f(x)dx” \tag{12}$$

If the function $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue integrable, then the integral (12) is well defined in Lebesgue's sense. However, for suitable non integrable functions f, e.g. $f \equiv 1$, it is still possible to define expression (12) by exploiting the cancellations due to the oscillatory term $e^{i\frac{\Phi}{\epsilon}(x)}$. The following definition was proposed in [38] and is a modification of the one introduced by Hörmander [48].

Definition 1 *The oscillatory integral of a Borel function $f : \mathbb{R}^n \to \mathbb{C}$ with respect to a phase function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is well defined if and only if for each test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(0) = 1$ the integral*

$$I_\delta(f,\phi) := \int_{\mathbb{R}^n} (2\pi i\epsilon)^{-n/2} e^{i\frac{\Phi(x)}{\epsilon}} f(x)\phi(\delta x)dx$$

exists for all $\delta > 0$ and the limit $\lim_{\delta\to 0} I_\delta(f,\phi)$ exists and is independent of ϕ. In this case the limit is called the oscillatory integral of f with respect to Φ and denoted by

$$I^\Phi(f) \equiv \tilde{\int}_{\mathbb{R}^n} e^{i\frac{\Phi(x)}{\epsilon}} f(x)dx. \tag{13}$$

The symbol $\tilde{\int}$ recalls the normalization factor $(2\pi i\epsilon)^{-n/2}$ which makes the integral “normalized” in the case $\Phi(x) = \frac{|x|^2}{2}$, in the sense that $I^\Phi(1) = 1$ for such a Φ.

A “complete direct characterization” of the class of functions f and phases Φ for which the integral (13) is well defined is still an open problem. However, for suitable Φ, it is possible to find an interesting set of “integrable functions” f, for which the oscillatory integral $I^\Phi(f)$ is well defined and can be explicitly computed in terms of an absolutely convergent integral thanks to a Parseval-type equality.

We shall shortly introduce a setting first presented in [15]. Given a (finite or infinite dimensional) real separable Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$, let us denote by $\mathcal{M}(\mathcal{H})$ the Banach space of the complex bounded variation measures on $\mathcal{H}$, endowed with the total variation norm, that is:

$$\mu \in \mathcal{M}(\mathcal{H}), \qquad \|\mu\| = \sup \sum_i |\mu(E_i)|,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of $\mathcal{H}$, such that $\cup_i E_i = \mathcal{H}$. $\mathcal{M}(\mathcal{H})$ is a Banach algebra, where the product of two measures $\mu * \nu$ is by definition their convolution:

$$\mu * \nu(E) = \int_{\mathcal{H}} \mu(E-x)\nu(dx), \qquad \mu, \nu \in \mathcal{M}(\mathcal{H})$$

and the unit element is the Dirac point measure δ_0 (with support at the origin).
Let $\mathcal{F}(\mathcal{H})$ be the space of complex functions on $\mathcal{H}$ which are Fourier transforms of measures belonging to $\mathcal{M}(\mathcal{H})$, that is:

$$f : \mathcal{H} \to \mathbb{C} \qquad f(x) = \int_H e^{i\langle x, \beta\rangle} \mu_f(d\beta) \equiv \hat{\mu}_f(x).$$

$\mathcal{F}(\mathcal{H})$ is a Banach algebra of functions, where the product is the pointwise one; the unit element is the function 1, i.e. $1(x) = 1\ \forall x \in \mathcal{H}$ and the norm is given by $\|f\| = \|\mu_f\|$.
It is possible to prove [19] that if $\mathcal{H} \in \mathbb{R}^n$ and $f \in \mathcal{F}(\mathbb{R}^n)$, $f = \hat{\mu}_f$, and if the phase function Φ is such that $F^{\Phi} \equiv \frac{e^{i\frac{\Phi}{\epsilon}}}{(2\pi i\epsilon)^{n/2}}$ has a Fourier transform $\hat{F}^{\Phi}$ having the property that the integral

$$\int_{\mathbb{R}^n} \hat{F}^{\Phi}(\alpha) d\mu_f(\alpha)$$

exists, then the oscillatory integral $I^{\Phi}(f)$ exists (in the sense of definition 1) and it is given by the following "Parseval formula":

$$I^{\Phi}(f) = \int_{\mathbb{R}^n} \hat{F}^{\Phi}(\alpha) d\mu_f(\alpha) \tag{14}$$

Equation (14) holds for smooth phase functions Φ of at most even polynomial growth at infinity (see [19] for more details). It is worthwhile to recall that $I^{\Phi}(f)$ can be defined for more general f as proved in [48], but in this case formula (14) is no longer valid in general.
It is interesting to analyze two particular cases, which we in sect. 3 shall extend to the infinite dimensional case. Let us assume that $\varepsilon > 0$ and Φ has one of the following forms

$$\Phi(x) := \frac{1}{2}\langle x, Qx\rangle - V(x), \tag{15}$$

$$\Phi(x) := \frac{1}{2}\langle x, Qx\rangle - V(x) - \lambda P(x), \tag{16}$$

where $V \in \mathcal{F}(\mathbb{R}^n)$, $Q : \mathbb{R}^N \to \mathbb{R}^n$ is a linear, symmetric invertible operator, $\lambda < 0$ and P is an homogeneous 4-degree polynomial. Since the function $e^{iV} f$ belongs to $\mathcal{F}(\mathbb{R}^n)$ for $V, f \in \mathcal{F}(\mathbb{R}^n)$, we do not loose generality in setting $V \equiv 0$ (which simplifies notations). If Φ is of the type (15), then formula (14) assumes the following form [38, 6]:

$$I^{\Phi}(f) = \det Q^{-1/2} \int_{\mathbb{R}^n} e^{-i\frac{\epsilon}{2}\langle \alpha, Q^{-1}\alpha\rangle} d\mu_f(\alpha) \tag{17}$$

while in the case where Φ is of the type (16), with $Q>0$, then formula (14) is still valid with $\hat{F}^{\Phi}$ given by

$$\hat{F}^{\Phi}(\alpha)=(2\pi\epsilon)^{-n/2}\int_{\mathbb{R}^n} e^{ie^{i\frac{\pi}{4}}\langle\alpha,x\rangle}e^{-\frac{1}{2\epsilon}\langle x,Qx\rangle}e^{i\frac{\lambda}{\epsilon}P(x)}dx$$
$$=\mathbb{E}(e^{ie^{i\frac{\pi}{4}}\langle\alpha,x\rangle}e^{\frac{1}{2\epsilon}\langle x,(I-Q)x\rangle}e^{i\frac{\lambda}{\epsilon}P(x)}), \quad (18)$$

$\alpha\in\mathbb{R}^n$, where the expectation is taken with respect to the standard Gaussian measure $N(0,\epsilon I_{\mathbb{R}^n})$. Moreover under some analyticity assumptions on the function f, the integral $I^{\Phi}(f)$ can be computed by means of the following formula:

$$I^{\Phi}(f)=\mathbb{E}(f(e^{i\frac{\pi}{4}}x)e^{\frac{1}{2\epsilon}\langle x,(I-Q)x\rangle}e^{i\frac{\lambda}{\epsilon}P(x)}). \quad (19)$$

The r.h.s. of formula (19) extends to an analytic function of the variable λ, for $Im(\lambda)>0$, and is still continuous for $Im(\lambda)=0$ [20, 21].
The leading idea of the proof is the computation of the Fourier transform F^{Φ} by means of a rotation of $\pi/4$ in counterclock direction of the integration contour. This operation maps the quadratic part $e^{i\frac{|x|^2}{2\epsilon}}$ of $e^{i\frac{\Phi}{\epsilon}}$ into the Gaussian density $e^{-\frac{|x|^2}{2\epsilon}}$ while the quartic part $e^{-i\frac{\lambda P(x)}{\epsilon}}$ of $e^{i\frac{\Phi}{\epsilon}}$ remains bounded, going over to $e^{i\frac{\lambda P(x)}{\epsilon}}$. For more details see [20]

2.2 Infinite dimensional case

The results of the previous section can be partially extended to the case where Γ is an infinite dimensional real separable Hilbert space $(\mathcal{H},\langle\ ,\ \rangle)$. An infinite dimensional oscillatory integral can be defined as the limit of a sequence of finite dimensional approximations, as proposed in [38, 6].

Definition 2 *A function $f:\mathcal{H}\to\mathbb{C}$ is said to be integrable with respect to the phase function $\Phi:\mathcal{H}\to\mathbb{R}$ if for any sequence P_n of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n\leq P_{n+1}$ and $P_n\to 1$ strongly as $n\to\infty$ (1 being the identity operator in $\mathcal{H}$), the finite dimensional approximations*

$$\widetilde{\int}_{P_n\mathcal{H}} e^{i\frac{\Phi(P_nx)}{\epsilon}}f(P_nx)d(P_nx),$$

are well defined (in the sense of definition 1) and the limit

$$\lim_{n\to\infty}\widetilde{\int}_{P_n\mathcal{H}} e^{i\frac{\Phi(P_nx)}{\epsilon}}f(P_nx)d(P_nx) \quad (20)$$

exists and is independent of the sequence $\{P_n\}$.
In this case the limit is called oscillatory integral of f with respect to the phase function Φ and is denoted by

$$I^{\Phi}(f)\equiv\widetilde{\int_{\mathcal{H}}}e^{i\frac{\Phi(x)}{\epsilon}}f(x)dx.$$

Again it is important to find classes of functions f and phases Φ, for which $I^{\Phi}(f)$ is well defined. All basic properties valid in the finite dimensional case remain valid in the infinite dimensional case. Indeed the fundamental space is again $\mathcal{F}(\mathcal{H})$, the space of functions which are Fourier transforms of complex bounded variation measures on $\mathcal{H}$. K. Itô was first in understanding the important role of this space in connection with the mathematical definition of Feynman path integrals. He introduced $\mathcal{F}(\mathcal{H})$ in his second paper on the topic [52], where he generalized the results of [51] to the case where the potential V belongs to $\mathcal{F}(\mathbb{R}^d)$. Itô's results were extensively developed by S. Albeverio and R. Høegh-Krohn [15, 16] and later by D. Elworthy and A. Truman [38]. In the case where the phase function Φ is of the form

$$\Phi(x) = \frac{1}{2}\langle x, Qx\rangle + \langle a, x\rangle + \Phi_{int}(x), \tag{21}$$

where $Q : \mathcal{H} \to \mathcal{H}$ linear invertible self-adjoint operator, $I - Q$ trace class, $a \in \mathcal{H}$ and $\Phi_{int} \in \mathcal{F}(\mathcal{H})$, and, moreover, $f \in \mathcal{F}(\mathcal{H})$, these authors prove that $I^{\Phi}(f)$ is well defined and can be explicitly computed in terms of a well defined absolutely convergent integral with respect to a bounded variation measure by means of a Parseval-type equality similar to (17). Some time later the definition of $I^{\Phi}(f)$ was generalized to unbounded functions Φ_{int} that are Laplace transforms of complex bounded variation measures on $\mathcal{H}$ [8, 59, 17]. More recently a breakthrough in handling the case where Φ_{int} is a fourth-order polynomial has been achieved [20, 21]. In fact formula (19) valid in the finite dimensional case has been generalized to the infinite dimensional case. Let us describe in more details this newer development, because of its relevance for applications (the quartic potential model is one of the most discussed ones in the physical literature).

Given a real separable infinite dimensional Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$ with norm $|\ |$, let ν be the finitely additive cylinder measure on $\mathcal{H}$, defined by its characteristic functional $\hat{\nu}(x) = e^{-\frac{1}{2}|x|^2}$. Let $\|\ \|$ be a "measurable" norm on $\mathcal{H}$, that is $\|\ \|$ is such that for every $\delta > 0$ there exist a finite-dimensional projection $P_\delta : \mathcal{H} \to \mathcal{H}$, such that for all $P \perp P_\delta$ one has $\nu(\{x \in \mathcal{H} |\ \|P(x)\| > \delta\}) < \delta$, where P and P_δ are called orthogonal ($P \perp P_\delta$) if their ranges are orthogonal in $(\mathcal{H}, \langle\ ,\ \rangle)$. One can easily verify that $\|\ \|$ is weaker than $|\ |$. Denoting by $\mathcal{B}$ the completion of $\mathcal{H}$ in the $\|\ \|$-norm and by i the continuous inclusion of $\mathcal{H}$ in $\mathcal{B}$, one proves that $\mu \equiv \nu \circ i^{-1}$ is a countably additive Gaussian measure on the Borel subsets of $\mathcal{B}$. The triple $(i, \mathcal{H}, \mathcal{B})$ is called an *abstract Wiener space* (in the sense of L. Gross). Let us consider a phase function of the following form:

$$\Phi(x) = \frac{1}{2}\langle x, Qx\rangle - \lambda P(x), \tag{22}$$

with $Q : \mathcal{H} \to \mathcal{H}$ a self-adjoint strictly positive operator such that $I - Q$ is trace class, and $P : \mathcal{H} \to \mathbb{R}$ is given by $P(x) = B(x, x, x, x)$, with $B : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ a completely symmetric positive covariant tensor operator on $\mathcal{H}$ such that the map $V : \mathcal{H} \to \mathbb{R}^+$, $x \mapsto V(x) \equiv B(x, x, x, x)$ is continuous in the $\| \|$ norm. Under these assumptions, it is possible to prove that the functions on $\mathcal{H}$ defined by

$$x \in \mathcal{H} \mapsto \langle k, x \rangle \text{ resp. } \langle x, (I-Q)x \rangle \text{ resp. } P(x),$$

can be lifted to random variables on $\mathcal{B}$, denoted by

$$\omega \in \mathcal{B} \mapsto \langle n(k)(\omega) \rangle \text{ resp. } \langle \omega, (I-Q)\omega \rangle \text{ resp. } P(\omega) \text{ (with } k \in \mathcal{H})$$

Moreover the following holds [20]:

Theorem 1 *Let $f : \mathcal{H} \to \mathbb{C}$ be the Fourier transform of a measure $\mu_f \in \mathcal{M}(\mathcal{H})$, $f \equiv \hat{\mu}_f$, satisfying the following assumption*

$$\int_{\mathcal{H}} e^{\frac{\epsilon}{4}\langle k, Q^{-1}k \rangle} |\mu_f|(dk) < +\infty. \tag{23}$$

Then the infinite dimensional oscillatory integral

$$\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2\epsilon}\langle x, Qx \rangle} e^{-i\frac{\lambda}{\epsilon}P(x)} f(x) dx \tag{24}$$

exists and is given by:

$$\int_{\mathcal{H}} \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\epsilon}\langle \omega, (I-Q)\omega \rangle} e^{i\frac{\lambda}{\epsilon}P(\omega)}] \mu_f(dk) \tag{25}$$

It is also equal to:

$$\mathbb{E}[e^{\frac{1}{2\epsilon}\langle \omega, (I-Q)\omega \rangle} e^{i\frac{\lambda}{\epsilon}P(\omega)} f(e^{i\pi/4}\omega)] \tag{26}$$

$\mathbb{E}$ denotes the expectation value with respect to the Gaussian measure μ on $\mathcal{B}$ (described before the statement of the theorem).

2.3 Properties and comparison with other approaches

The infinite dimensional oscillatory integral $I^{\Phi}(f)$, with $\Phi \equiv \frac{|x|^2}{2}$ and $f \in \mathcal{F}(\mathcal{H})$, was originally defined [15] by "duality" by means of the Parseval type equality (17). The more recent definition of $I^{\Phi}(f)$ (see definition 2, based on [37], [38]) by means of finite dimensional approximations maintains this property: indeed for suitable Φ the application $f \mapsto I^{\Phi}(f)$ is a linear continuous functional on $\mathcal{F}(\mathcal{H})$.

The realization of the integral $I^{\Phi}(f)$ by means of a duality relation is typical of several approaches to the definition of the Feynman path integral. In other words one tries to define the Feynman density $e^{i\Phi(\gamma)}$ as an "infinite dimensional distribution". Besides [15] origins of this idea can be found in work by C. DeWitt-Morette (see, e.g. [35], see also e.g. [57]). It was systematically developed in the framework of white noise calculus by T.Hida and L.Streit

[47, 59]. In the latter setting the integral $I^{\Phi}(f)$ is realized as the pairing $\langle T_{\Phi}, f\rangle$ with respect to the standard Gaussian measure $N(0, I_{L^2(\mathbb{R}^n)})$ of a white noise distribution $T_\phi \in (S')$ (which, heuristically, can be interpreted as $e^{\frac{i}{2}\Phi(\gamma)+\frac{1}{2}\langle\gamma,\gamma\rangle}$) and a regular $f \in (S)$, where $(S), (S')$ are elements of the Gelfand triple $(S) \subset L^2(N(0, I_{L^2(\mathbb{R}^n)})) \subset (S')$ (see [47] for details).

It is interesting to note that formula (26) shows a deep connection between infinite dimensional oscillatory integrals and probabilistic Gaussian integrals. Indeed, under suitable assumptions on the function f that is integrated and on the phase function Φ, the oscillatory integral of f with respect to Φ is equal to a Gaussian integral. On the other hand one of the first approaches to the rigorous mathematical definition of Feynman path integrals was by means of analytic continuation of Gaussian Wiener integrals [34, 64, 51, 53, 69, 56, 37, 62, 67]. The leading idea of this approach is the analogy between Schrödinger and heat equation on one hand, and between the rigorous Feynman-Kac formula (7) and the heuristic Feynman representation (4) on the other hand. By introducing in the heat equation (6) and in the corresponding path integral solution (7) a suitable parameter λ, proportional for instance to the time, or to the mass, or to the Planck constant, and by allowing λ to assume complex values, then one gets, at least heuristically, the Schrödinger equation and its solution. This procedure can be made completely rigorous under suitable conditions on the potential V and initial datum ψ_0.

Another approach to the mathematical definition of Feynman path integrals, which is very close to Feynman's original derivation, is the "sequential approach". It was originally proposed by A. Truman [68] and further extensively developed by D. Fujiwara and N. Kumano-go [41, 42, 43]. In this approach the paths γ in formula (4) are approximated by piecewise linear paths and the Feynman path integral is correspondingly approximated by a finite dimensional integral.

Two other alternative approaches to the mathematical definition of Feynman path integrals are based on Poisson measures respectively on nonstandard analysis. The first one was originally proposed by A.M. Chebotarev and V.P. Maslov [63] and further developed by several authors as S. Albeverio, Ph. Blanchard, Ph. Combe, R. Høegh-Krohn, M. Sirugue [3, 2] and V. Kolokol′tsov [58]. The second was proposed in the 80's by S. Albeverio, J.E. Fenstad, R. Høegh-Krohn and T. Lindstrøm [13], but it has not been much further developed yet.

2.4 The method of stationary phase

One of the main motivations for the rigorous mathematical definition of the infinite dimensional oscillatory integrals is the implementation of a corresponding infinite dimensional version of the method of the stationary phase and its application to the study of the asymptotic behavior of the expressions in formula (4) in the limit $\hbar \to 0$. The first results were obtained in [16] and further developed in [65] [6] (see also e.g. [3]-[5]). Up to now, only the case where the phase function Φ is of the form (21) has been handled rigorously. In this case the detailed asymptotic expansion of the infinite dimensional oscillatory integral has been computed, and, in the case where the phase function has a unique stationary point, the Borel summability of the expansion has been proved [65].

For results on the study of the asymptotic behavior of infinite dimensional probabilistic integrals and its connection with the semiclassical limit of Schrödinger equation see e.g. [33, 50, 61, 66, 26, 27, 18, 32].

3 Applications

3.1 The Schrödinger equation

Infinite dimensional oscillatory integrals, as defined in section 1.2, provided a rigorous mathematical realization of the heuristic Feynman path integral representation (4) for the solution of the following Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = H\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{27}$$

where

$$H = -\frac{\hbar^2}{2m}\Delta + \frac{1}{2}xA^2x + V(x) + \lambda P(x),$$

$V, \psi_0 \in \mathcal{F}(\mathbb{R}^d)$, $A : \mathbb{R}^d \to \mathbb{R}^d$ is a symmetric positive operator, $\lambda \geq 0$, P is an homogeneous fourth order polynomial. In other words, under the assumptions above, the heuristic path integral (4) can be realized as a well defined infinite dimensional oscillatory integral on a suitable Hilbert space $\mathcal{H}$ with parameter $\epsilon \equiv \hbar$. We describe here the result in the case $\lambda = 0$ [15, 16, 38, 6], recalling that the general case with $\lambda \neq 0$ has been recently handled in [20]

Let us consider the Cameron-Martin space $(\mathcal{H}_t, \langle\ ,\ \rangle)$, i.e. the Hilbert space of absolutely continuous paths $\gamma : [0,t] \to \mathbb{R}^d$ such that $\gamma(t) = 0$ and $\dot\gamma \in L_2([0,t];\mathbb{R}^d)$, endowed with the inner product

$$\langle\gamma_1, \gamma_2\langle = \int_0^t \dot\gamma_1(s)\dot\gamma_2(s)ds.$$

From now on we shall assume for notational simplicity that $m = 1$. Let us consider the operator L on $\mathcal{H}_t$ given by

$$\langle\gamma, L\gamma\rangle \equiv \int_0^t \gamma(s)A^2\gamma(s)ds,$$

and the function $v : \mathcal{H}_t \to \mathbb{C}$

$$v(\gamma) \equiv \int_0^t V(\gamma(s)+x)ds + 2xA^2\int_0^t \gamma(s)ds\tau, \qquad \gamma \in \mathcal{H}_t.$$

By analyzing the spectrum of the operator L (see [38]) one can easily verify that L is trace class and $I - L$ is invertible. The following holds:

Theorem 2 *Under the assumptions above, the function* $f : \mathcal{H}_t \to \mathbb{C}$ *given by*

$$f(\gamma) := e^{-\frac{i}{\hbar}v(\gamma)}\psi_0(\gamma(0)+x)$$

is the Fourier transform of a complex bounded variation measure μ_f *on* $\mathcal{H}_t$ *and the infinite dimensional oscillatory integral of the function* $g(\gamma) = e^{-\frac{i}{2\hbar}\langle\gamma, L\gamma\rangle} f(\gamma)$

$$\widetilde{\int} \mathcal{H}_t e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle} e^{-\frac{i}{\hbar}v(\gamma)}\psi_0(\gamma(0)+x)d\gamma. \tag{28}$$

is well defined (in the sense of definition (2)*) and it is equal to*

$$\det(I-L)^{-1/2}\int_{\mathcal{H}_t} e^{\frac{-}{i\hbar}2\langle\gamma,(I-L)^{-1}\gamma\rangle} d\mu_f(\gamma),$$

$\det(I-L)$ *being the Fredholm determinant of the operator* $(I-L)$.
Moreover it is a representation of the solution of the Schrödinger equation (27) *evaluated at* $x \in \mathbb{R}^d$ *at time* t.

For a proof see [38]. An extension of this result to the case of the presence of a a polynomial potential (i.e. $\lambda \neq 0$) has been obtained in [20] (on the basis of theorem 1) In this case the Borel summability of the asymptotic expansion of $I^{\Phi}(f)$ in powers of the coupling constant λ has also been proven.

The method of the stationary phase in infinite dimensions has been applied to the study of the asymptotic behavior of the integral (28) in the limit $\hbar \to 0$, in the case $\lambda = 0$ [16, 6, 65] (for other methods leading to similar results, see e.g. [32]).
The result of theorem 2 has been recently generalized to the case where the potential V, the matrix A and the coupling constant λ are explicitly time dependent [23, 25, 30]. The case $\lambda \neq 0$ requires special attention because of the growth of the term $\lambda P(x)$ at infinity which excludes the possibility of applying the usual methods of the theory of hyperbolic evolution equations.

Let us also mention that infinite dimensional oscillatory integrals are a flexible tool and provide a rigorous mathematical realization for other large classes of Feynman path integral representations, such as the "phase space Feynman path integrals" [10] and the "Feynman path integrals with complex phase functions" that are applied to the solution of a stochastic Schrödinger equation [11, 12]. Other interesting applications are the solution of the Schrödinger equation with a magnetic field [7], the trace formula for the Schrödinger group [4, 5] (which includes a rigorous proof of "Gutzwiller's trace formula", of basic importance in the study of quantum chaos, and with interesting connections with number theory), the dynamics of Dirac systems [49] and of quantum open systems [9, 39].

3.2 The Chern-Simons model

The application of the infinite dimensional oscillatory integrals to the mathematical definition of the Chern-Simons functional integral described in section

1.3 has been realized in [28] in the case where the gauge group G is abelian. It has been proven in particular that if $H^1(M) = 0$ then $I^{\Phi}(f)$ gives the linking numbers. The same results were obtained in [60] in the framework of white noise calculus. These result have been extended to the case where G is not abelian and $M = \mathbb{R}^3$ in [29, 45] by means of white noise analysis (see also [14] for a detailed exposition of this topic). The case $M = S^1 \times S^2$ has been recently handled in [46]. There is certainly still a large gap between the extensive and productive heuristic use of Feynman type integrals in this area (and in related areas connected with quantum gravity and string theory) and what can be achieved rigorously. This is a great challenge for the future.

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