On Rørdam's classification of certain C^*-algebras with one non-trivial ideal

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Summary. We extend Rørdam's stable classification result for purely infinite C^*-algebras with exactly one non-trivial ideal to allow for the lifting of an isomorphism on the level of the invariants to a $*$-isomorphism, and to allow for unital isomorphism when the isomorphisms of the invariant respect the relevant classes of units.

1 Introduction

Rørdam in [14] establishes that the six term exact sequence

$$\begin{tikzcd} K_0(\mathfrak{A}) \arrow[r, "\iota_*"] & K_0(\mathfrak{E}) \arrow[r, "\pi_*"] & K_0(\mathfrak{E}/\mathfrak{A}) \arrow[d, "\partial"] \\ K_1(\mathfrak{E}/\mathfrak{A}) \arrow[u, "\partial"] & K_1(\mathfrak{E}) \arrow[l, "\pi_*"] & K_1(\mathfrak{A}) \arrow[l, "\iota_*"] \end{tikzcd} \tag{1}$$

is a complete invariant for stable isomorphism of C^*-algebras $\mathfrak{E}$ with precisely one non-trivial ideal $\mathfrak{A}$, provided that the ideal $\mathfrak{A}$ and the quotient $\mathfrak{E}/\mathfrak{A}$ are both in the class of purely infinite simple C^*-algebras classified by Kirchberg and Phillips ([9]).

Most classification results of C^*-algebras by K-theoretical invariants are established in such a way that one with little or no extra effort can prove that any isomorphism between a pair of invariants may be lifted to a $*$-isomorphism. It is often also easy to pass between results yielding stable isomorphism for general C^*-algebras and isomorphism of unital C^*-algebras in a certain class, by adding or leaving out the class of the unit in the invariant.

Rørdam's classification result forms a notable exception to these two rules. Indeed, there is no obvious way to extract from Rørdam's proof a way to establish these kinds of sligthly improved classification results. It is the purpose of this note to show that by invoking more recent results by Bonkat and Kirchberg, one may prove such results in the class considered by Rørdam.

Using the language promoted by Elliott ([6]) this proves that the *classification functor* used by Rørdam is indeed a *strong classification functor*.

We are first going to prove, by straightforward observations on central results in Bonkat's thesis, that every isomorphism among invariants of the type (1) – i.e., a 6-tuple of coherent group isomorphisms – lifts to a $*$-isomorphism. With this in hand we can then prove a unital classification result by appealing to a useful principle which we shall develop in a rather general context.

An update on the status of the work of Bonkat may be in order. Bonkat sets out to reprove the classification result of Rørdam using Kirchberg's results. However, the class classified by Bonkat is, a priori, smaller than the class classified by Rørdam, and to prove that they coincide in this case, Bonkat is forced to appeal to Rørdam's result. Fortunately, more recent results by Kirchberg or Toms and Winter ([16]) show in a direct way that the classes coincide, rendering Bonkat's proof truly independent of [14]. More details are given after Lemma 4.

2 Bonkat's method

We shall concentrate on C^*-algebras $\mathfrak{E}$ having exactly one non-trivial ideal $\mathfrak{A}$, noting that this is the case exactly when the extension

$$0 \longrightarrow \mathfrak{A} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \mathfrak{E}/\mathfrak{A} \longrightarrow 0 \tag{2}$$

is essential and the C^*-algebras $\mathfrak{A}$ and $\mathfrak{E}/\mathfrak{A}$ are simple. The primitive ideal spectrum of such C^*-algebras we denote by X_1; it has two points of which the closure of one is the whole space while the other point is closed.

Kirchberg proves in [7, Corollary N] (cf. [8, Folgerung 4.3]) a result which in the case of C^*-algebras with one non-trivial ideal specializes to the following:

Theorem 1. *Let $\mathfrak{E}$ and $\mathfrak{E}'$ be strongly purely infinite, separable, stable and nuclear C^*-algebras, each with exactly one non-trivial ideal. If $z \in KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}')$ is a $KK(\mathsf{X}_1; -, -)$-equivalence then there exists a $*$-isomorphism*

$$\phi : \mathfrak{E} \longrightarrow \mathfrak{E}'$$

with $[\phi] = z$.

Analogously to the characterization of the bifunctor KK by universal means (cf. [1, Corollary 22.3.1] we may describe $KK(\mathsf{X}_1; -, -)$ by the universal property that any stable, homotopy invariant and split exact functor from the category of extensions of separable C^*-algebras into an additive category factorises uniquely through $KK(\mathsf{X}_1; -, -)$. Strong pure infiniteness is considered in [11], and it is shown that a separable, stable and nuclear C^*-algebra $\mathfrak{E}$ is strongly purely infinite if and only if $\mathfrak{E}$ absorbs $\mathcal{O}_\infty$, i.e. if and only if $\mathfrak{E} \cong \mathfrak{E} \otimes \mathcal{O}_\infty$.

This extremely general and powerful result should be considered as an *isomorphism theorem* allowing one to conclude from the existence of a very weak kind of isomorphism, at the level of ideal-preserving KK-theory, the existence of a genuine $*$-isomorphism at the level of C^*-algebras.

This result could be turned into a *bona fide* classification result for such algebras with one non-trivial ideal by a suitable universal coefficient theorem allowing one to lift an isomorphism at the level of K-theory to one at the level of ideal-preserving KK. And by a very generally applicable trick originating with Rosenberg and Schochet, cf. Lemma 3 below, all one seems to need is a surjective group homomorphism

$$KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}') \longrightarrow \mathrm{Hom}(\mathfrak{k}(\mathfrak{E}), \mathfrak{k}(\mathfrak{E}')).$$

where $\mathfrak{k}$ is an appropriately chosen variant of K-theory. The main challenge for carrying out such a program thus becomes to identify a feasible flavour of K-theory to use as $\mathfrak{k}(-)$, and to establish the existence of such an epimorphism. However, we are aware of no approach to doing so which does not also involve identifying the kernel of this map.

Indeed, this is exactly what Bonkat manages to do in his thesis work [2] in the case when Rørdam's classification result indicates that the correct flavor of K-theory is the class of six term exact sequences considered in [14], thus providing an alternative proof for many of the results there.

A UCT for Kirchberg's $KK(\mathsf{X}_1; -, -)$ is established in [2] as follows. Bonkat works in the category of 6-periodic complexes

$$\mathbf{G} \qquad \begin{tikzcd} G_0 \arrow[r, "\phi_0"] & G_1 \arrow[r, "\phi_1"] & G_2 \arrow[d, "\phi_2"] \\ G_5 \arrow[u, "\phi_5"] & G_4 \arrow[l, "\phi_4"] & G_3 \arrow[l, "\phi_3"] \end{tikzcd}$$

of abelian groups and group homomorphisms, which he establishes is additive. For two such complexes $\mathbf{G}$ and $\mathbf{G}'$ the natural notion of homomorphisms is the abelian group of coherent 6-tuples of group homomorphisms:

$$\mathrm{Hom}_{\hexagon}(\mathbf{G}, \mathbf{G}') = \{(\xi_i)_{i=0}^5 \mid \xi_i : G_i \longrightarrow G_i', \phi_i'\xi_i = \xi_{i+1}\phi_i\}$$

Note that any C^*-algebra $\mathfrak{E}$ with precisely one non-trivial ideal $\mathfrak{A}$ gives rise to a 6-periodic complex

$$\begin{tikzcd} K_0(\mathfrak{A}) \arrow[r, "\iota_*"] & K_0(\mathfrak{E}) \arrow[r, "\pi_*"] & K_0(\mathfrak{E}/\mathfrak{A}) \arrow[d, "\partial"] \\ K_1(\mathfrak{E}/\mathfrak{A}) \arrow[u, "\partial"] & K_1(\mathfrak{E}) \arrow[l, "\pi_*"] & K_1(\mathfrak{A}) \arrow[l, "\iota_*"] \end{tikzcd}$$

which we may denote $K_{\hexagon}(\mathfrak{E})$ without risk of confusion, and that any $*$-isomorphism $\phi : \mathfrak{E} \longrightarrow \mathfrak{E}'$ (as well as any other $*$-homomorphism sending $\mathfrak{A}$ into $\mathfrak{A}'$)

induces an element $\phi_* \in \mathrm{Hom}_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon}(\mathfrak{E}'))$. We let $K_{\hexagon+1}(\mathfrak{E})$ denote the complex obtained by shifting the groups by three indices.

Bonkat identifies the projective objects in this category as those complexes which are exact and have projective, i.e. free, groups at each entry and proves that there are enough projectives so that the $\mathrm{Hom}_{\hexagon}$-functor of coherent 6-tuples defines left derived functors $\mathrm{Ext}^n_{\hexagon}$. It is then proved ([2, Korollar 7.2.14]) that all *exact* such periodic complexes have a projective resolution of length at most one, and by giving in [2, Abschnitt 7.4] a geometric resolution – i.e. a realization at the level of C^*-algebras – of this, Bonkat arrives at the following universal coefficient theorem

Theorem 2. *[2, Cf. Satz 7.5.3] Let $\mathfrak{E}$ and $\mathfrak{E}'$ be separable C^*-algebras each with exactly one non-trivial ideal $\mathfrak{A}$ and $\mathfrak{A}'$, respectively. Assume further that $\mathfrak{A}, \mathfrak{A}', \mathfrak{E}/\mathfrak{A}$ and $\mathfrak{E}'/\mathfrak{A}'$ lie in the UCT class $\mathcal{N}$. There is a short exact sequence*

$$\mathrm{Ext}^1_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon+1}(\mathfrak{E}')) \longrightarrow KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}') \xrightarrow{\Gamma} \mathrm{Hom}_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon}(\mathfrak{E}'))$$

Along the way Bonkat works in a different picture of $KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}')$; the differences are explained in [2, Abschnitt 5.6]. By naturality of the UCT one proves as in [15, Proposition 7.3]:

Lemma 3. *[2, Proposition 7.7.2] Let $\mathfrak{E}$ and $\mathfrak{E}'$ be as in Theorem 2. The element $z \in KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}')$ is an equivalence precisely when*

$$\Gamma(z) \in \mathrm{Hom}_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon}(\mathfrak{E}'))$$

is a 6-tuple of group isomorphisms.

Following Rørdam we say that a C^*-algebra is a *Kirchberg algebra* if it is purely infinite, simple, nuclear and separable. We need to use the following:

Lemma 4. *Let $\mathfrak{E}$ be an essential extension of two stable Kirchberg algebras from the UCT class $\mathcal{N}$. Then $\mathfrak{E}$ is strongly purely infinite.*

In Bonkat's thesis ([2, Satz 7.8.8]) this is established using Rørdam's classification, but more recent results by Kirchberg or (using the fact that strong purely infiniteness coincides with $\mathcal{O}_\infty$-stability in this case) by Toms and Winter [16, Theorem 4.3] this may be proved directly.

Theorem 5. *Let $\mathfrak{E}$ and $\mathfrak{E}'$ be C^*-algebras each with exactly one non-trivial ideal $\mathfrak{A}$ and $\mathfrak{A}'$, with the property that $\mathfrak{A}, \mathfrak{A}', \mathfrak{E}/\mathfrak{A}, \mathfrak{E}'/\mathfrak{A}'$ are all Kirchberg algebras in the UCT class $\mathcal{N}$. Any invertible element of $\mathrm{Hom}_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon}(\mathfrak{E}'))$ can be realized by a $*$-isomorphism $\phi : \mathfrak{E} \otimes \mathbb{K} \longrightarrow \mathfrak{E}' \otimes \mathbb{K}$.*

Proof. We may assume that $\mathfrak{E}$ and $\mathfrak{E}'$ are themselves stable. By Lemma 3 and Theorem 2 there exists an equivalence $\gamma \in KK(\mathsf{X}_1; -, -)$ realizing this 6-tuple of morpisms. Thus by Theorem 1 and Lemma 4 the map is realized by a $*$-isomorphism $\phi : \mathfrak{E} \longrightarrow \mathfrak{E}'$.

Corollary 6. *Let $\mathfrak{E}$ be a C^*-algebra with exactly one non-trivial ideal $\mathfrak{A}$, with the property that $\mathfrak{A}$ and $\mathfrak{E}/\mathfrak{A}$ are both stable Kirchberg algebras in the UCT class $\mathcal{N}$. The map*

$$\mathrm{Aut}(\mathfrak{E}) \longrightarrow \mathrm{Aut}_{\hexagon}(K_{\hexagon}(\mathfrak{E}))$$

is surjective.

It would be interesting to investigate when two such realizing $*$-isomorphisms ϕ, ϕ' were approximately unitarily equivalent. It is necessary that ϕ and ϕ' induce the same map on $K_*(\mathfrak{E}; \mathsf{Z}/n), K_*(\mathfrak{A}; \mathsf{Z}/n)$, and $K_*(\mathfrak{B}; \mathsf{Z}/n)$ for any $n \in \{2, 3 \dots\}$, and it is tempting to conjecture that this condition is also sufficient. It is, however, not even clear that any automorphism on a six term exact sequence of total K-theory lifts to a $*$-automorphism.

3 Unital classification

Using the main theorem of preceding section, Theorem 5, we will extend Rørdam's stable classification to allow for unital isomorphism when the isomorphisms of the invariant respect the relevant classes of units. This will be done by appealing to a useful principle which we shall develop in a rather general context. First we need some facts about properly infinite projections.

In [5] Cuntz considers C^*-algebras $\mathfrak{A}$ that contain a set $\mathscr{P}$ of projections satisfying the following conditions:

(Π_1) If $p, q \in \mathscr{P}$ and $p \perp q$, then $p + q \in \mathscr{P}$.
(Π_2) If $p \in \mathscr{P}$ and p' is a projection in $\mathfrak{A}$ such that $p \sim p'$, then $p' \in \mathscr{P}$.
(Π_3) For all $p, q \in \mathscr{P}$, there is $p' \in \mathscr{P}$ such that $p \sim p'$, $p' < q$ and $q - p' \in \mathscr{P}$.
(Π_4) If q is a projection in $\mathfrak{A}$, which majorizes an element of $\mathscr{P}$, then $q \in \mathscr{P}$.

If p is a projection in a C^*-algebra, then we let $[p]$ denote the Murray–von Neumann equivalence class of this projection. Cuntz shows in [5, Theorem 1.4] following theorem:

Theorem 7. *Let $\mathfrak{A}$ be a C^*-algebra with a non-empty set $\mathscr{P} \subseteq \mathfrak{A}$ of projections satisfying (Π_1), (Π_2) and (Π_3) above. Then $G = \{[p] \mid p \in \mathscr{P}\}$ is a group with the natural addition $[p]+[q] = [p'+q']$, where $p', q' \in \mathscr{P}$ are chosen such that $p \sim p'$, $q \sim q'$ and $p' \perp q'$ by (Π_3). Moreover, if $\mathfrak{A}$ is unital and $\mathscr{P}$ also satisfies (Π_4), then $G \ni [p] \mapsto [p]_0 \in K_0(\mathfrak{A})$ defines a group isomorphism.*

Recall that a projection p in a C^*-algebra $\mathfrak{A}$ is called *full* if $\mathfrak{A}$ is the only ideal in $\mathfrak{A}$ containing p, and p is called *properly infinite* if there exist projections $p_1, p_2 \leq p$ in $\mathfrak{A}$ such that $p_1 \perp p_2$ and $p_1 \sim p_2 \sim p$. See e.g. [10] and [11] for more on infinite projections and related topics.

Lemma 8. *Let $\mathfrak{A}$ be a C^*-algebra and let $\mathscr{P}$ be the set of full, properly infinite projections in $\mathfrak{A}$. Then $\mathscr{P}$ satisfies (Π_1), (Π_2), (Π_3) and (Π_4).*

Proof. (Π_1): Suppose there are given projections $p, q \in \mathscr{P}$ with $p \perp q$. Then there exist projections p_1, p_2, q_1, q_2 in $\mathfrak{A}$ such that

$$p_1, p_2 \leq p, \quad q_1, q_2 \leq q, \quad p_1 \perp p_2, \quad q_1 \perp q_2, \quad p_1 \sim p_2 \sim p, \quad q_1 \sim q_2 \sim q.$$

Put $r_1 = p_1 + q_1$, $r_2 = p_2 + q_2$ and $r = p + q$. It is easy to check that these are projections satisfying $r_1, r_2 \leq r$, $r_1 \perp r_2$ and $r_1 \sim r_2 \sim r$; i.e. r is properly infinite. Clearly r is full, so $r \in \mathscr{P}$.

(Π_2): Let there be given projections $p \in \mathscr{P}$ and $p' \in \mathfrak{A}$ such that $p \sim p'$. Then there exist orthogonal projections $p_1, p_2 \leq p$, such that $p_1 \sim p_2 \sim p$, and there exists a partial isometry $v \in \mathfrak{A}$ such that $p = vv^*$ and $p' = v^*v$. Define $p_1' = v^* p_1 v$ and $p_2' = v^* p_2 v$. Then one easily shows, that p_1' and p_2' are orthogonal projections such that $p_1', p_2' \leq p'$ and $p_1' \sim p_2' \sim p'$. From $p = p^2 = vv^*vv^* = vp'v^*$ it is clear that p' is full. Hence $p' \in \mathscr{P}$

(Π_4): Let q be a projection in $\mathfrak{A}$ such that $p \leq q$ for a $p \in \mathscr{P}$. Then $p \precsim q$, and hence q is properly infinite by [10, Lemma 3.8] (see Section 2 in the same paper for more on Cuntz comparison $\precsim$). From $p \leq q$ we immidiately get that $pq = p$, so q is clearly full. Thus we have shown that $q \in \mathscr{P}$.

(Π_3): Let $p, q \in \mathscr{P}$ be given projections. Then the ideal $\overline{\mathfrak{A}q\mathfrak{A}}$ generated by q is $\mathfrak{A}$ (q is full). According to [10, Proposition 3.5] we have $p \precsim q$, i.e. there exists a projection $p' \leq q$ such that $p \sim p'$. So there exist orthogonal projections $p_1', p_2' \leq p'$ in $\mathfrak{A}$ such that $p_1' \sim p_2' \sim p'$. The projection p is in $\mathscr{P}$, which by (Π_2) implies that $p_1', p_2' \in \mathscr{P}$. From $p_1' + p_2' \leq p' \leq q$ we deduce that $p_2' \leq q - p_1' < q$. From (Π_4) we get $q - p_1' \in \mathscr{P}$, because $p_2' \in \mathscr{P}$.

Analogous to Brown's result ([3, Corollary 2.7]) one easily proves the following theorem:

Theorem 9. *Let p be a full projection in a separable C^*-algebra $\mathfrak{A}$. Then the embedding $\iota\colon p\mathfrak{A}p \to \mathfrak{A}$ induces an isomorphism $K_0(\iota)\colon K_0(p\mathfrak{A}p) \to K_0(\mathfrak{A})$.*

Proposition 10. *Let p and q be full, properly infinite projections in a separable C^*-algebra $\mathfrak{A}$. Then $[p]_0 = [q]_0$ if and only if p is Murray–von Neumann equivalent to q.*

Proof. Let p and q be full, properly infinite projections in a separable C^*-algebra $\mathfrak{A}$. Assume that $[p]_0 = [q]_0$. We want to show, that $p \sim q$. By (Π_3) we can w.l.o.g. assume that $p \perp q$. Put $r = p + q$.

The hereditary corner algebra $r\mathfrak{A}r$ of $\mathfrak{A}$ is unital. The set $\mathscr{P}$ of full, properly infinite projections in $r\mathfrak{A}r$ contains p and q. By Theorem 9, $[p]_0 = [q]_0$ in $K_0(r\mathfrak{A}r)$. By Cuntz' result is $p \sim q$ (in $r\mathfrak{A}r$).

The claims in this proposition are stated several places in the literature for unital C^*-algebras without the separability condition, but the proofs do not readily generalize to the non-unital case. It is likely that one can get by without the separability condition – it may even be a known result – but we will not need this here.

We can use Cuntz' argument in the proof of [13, Theorem 6.5] to prove the following meta-theorem:

Theorem 11. *Let $\mathscr{C}$ be a subcategory of the category of C^*-algebras, and let $\mathsf{F}\colon \mathscr{C} \to \mathscr{D}$ be a covariant functor defined on this subcategory. Assume that*

(i) For every C^*-algebra $\mathfrak{A}$ in $\mathscr{C}$, $\mathfrak{A}\otimes\mathbb{K}$ belongs to $\mathscr{C}$, and the $*$-homomorphism $\mathfrak{A} \ni a \mapsto a\otimes e \in \mathfrak{A}\otimes\mathbb{K}$ induces an isomorphism from $\mathsf{F}(\mathfrak{A})$ onto $\mathsf{F}(\mathfrak{A}\otimes\mathbb{K})$, where e is a minimal projection in $\mathbb{K}$.

(ii) For all stable C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathscr{C}$, every isomorphism from $\mathsf{F}(\mathfrak{A})$ to $\mathsf{F}(\mathfrak{B})$ is induced by a $*$-isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(iii) There exists a covariant functor G from $\mathscr{D}$ into the category of abelian groups such that $\mathsf{G}\circ\mathsf{F} = K_0$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital, properly infinite, separable C^*-algebras from $\mathscr{C}$. If there exists an isomorphism ρ from $\mathsf{F}(\mathfrak{A})$ onto $\mathsf{F}(\mathfrak{B})$, such that $\mathsf{G}(\rho)$ maps $[\mathbb{1}_{\mathfrak{A}}]_0$ onto $[\mathbb{1}_{\mathfrak{B}}]_0$, then the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are $*$-isomorphic. (If $\mathfrak{A}\otimes\mathbb{K}$ and $\mathfrak{B}\otimes\mathbb{K}$ have the cancellation property, we may omit the assumption of the algebras being properly infinite.)

Proof. Let $\rho\colon \mathsf{F}(\mathfrak{A}) \to \mathsf{F}(\mathfrak{B})$ be an isomorphism such that $\alpha = \mathsf{G}(\rho)$ maps $[\mathbb{1}_{\mathfrak{A}}]_0$ onto $[\mathbb{1}_{\mathfrak{B}}]_0$, i.e. $\alpha([\mathbb{1}_{\mathfrak{A}}]_0) = [\mathbb{1}_{\mathfrak{B}}]_0$. Let e denote a minimal projection in $\mathbb{K}$. The homomorphisms $\mathfrak{A} \ni a \mapsto a\otimes e \in \mathfrak{A}\otimes\mathbb{K}$ and $\mathfrak{B} \ni b \mapsto b\otimes e \in \mathfrak{B}\otimes\mathbb{K}$ induce isomorphisms from $\mathsf{F}(\mathfrak{A})$ to $\mathsf{F}(\mathfrak{A}\otimes\mathbb{K})$ and from $\mathsf{F}(\mathfrak{B})$ to $\mathsf{F}(\mathfrak{B}\otimes\mathbb{K})$, resp. Therefore we get an induced isomorphism $\tilde\rho$ from $\mathsf{F}(\mathfrak{A}\otimes\mathbb{K})$ to $\mathsf{F}(\mathfrak{B}\otimes\mathbb{K})$, with $\tilde\alpha = \mathsf{G}(\tilde\rho)$ being an isomorphism from $K_0(\mathfrak{A}\otimes\mathbb{K})$ to $K_0(\mathfrak{B}\otimes\mathbb{K})$ such that $\tilde\alpha([\mathbb{1}_{\mathfrak{A}}\otimes e]_0) = [\mathbb{1}_{\mathfrak{B}}\otimes e]_0$.

By assumption, $\tilde\rho$ (and therefore also $\tilde\alpha$) is induced by a $*$-isomorphism $\phi\colon \mathfrak{A}\otimes\mathbb{K} \to \mathfrak{B}\otimes\mathbb{K}$. So

$$[\phi(\mathbb{1}_{\mathfrak{A}}\otimes e)]_0 = K_0(\phi)([\mathbb{1}_{\mathfrak{A}}\otimes e]_0) = \tilde\alpha([\mathbb{1}_{\mathfrak{A}}\otimes e]_0) = [\mathbb{1}_{\mathfrak{B}}\otimes e]_0.$$

The projections $\phi(\mathbb{1}_{\mathfrak{A}}\otimes e)$ and $\mathbb{1}_{\mathfrak{B}}\otimes e$ are full and properly infinite – we show this only for $\mathbb{1}_{\mathfrak{B}}\otimes e$ (ϕ is a $*$-isomorphism). It is clear that $\mathbb{1}_{\mathfrak{B}}\otimes e$ is a full projection. The projection $\mathbb{1}_{\mathfrak{B}}$ is properly infinite, so there exist partial isometries u_1 and u_2 such that $u_1u_1^* = u_2u_2^* = \mathbb{1}_{\mathfrak{B}}$ and $u_1^*u_1 \perp u_2^*u_2$; from this we see that $(u_1\otimes e)(u_1\otimes e)^* = \mathbb{1}_{\mathfrak{B}}\otimes e = (u_2\otimes e)(u_2\otimes e)^*$ and $(u_1\otimes e)^*(u_1\otimes e)(u_2\otimes e)^*(u_2\otimes e) = u_1^*u_1u_2^*u_2\otimes e = 0$. We have thus shown that the projection is properly infinite. By Proposition 10, therefore $\phi(\mathbb{1}_{\mathfrak{A}}\otimes e)$ is Murray–von Neumann equivalent to $\mathbb{1}_{\mathfrak{B}}\otimes e$. So there exists a partial isometry v such that

$$vv^* = \mathbb{1}_{\mathfrak{B}}\otimes e \quad \text{and} \quad v^*v = \phi(\mathbb{1}_{\mathfrak{A}}\otimes e).$$

Then $x\otimes e \mapsto v\phi(x\otimes e)v^*$ is a $*$-isomorphism from $\mathfrak{A}\otimes\mathbb{C}e$ onto $\mathfrak{B}\otimes\mathbb{C}e$. Because it is

- *well-defined:* For all $x\in\mathfrak{A}$ is

$$v\phi(x\otimes e)v^* = (\mathbb{1}_{\mathfrak{B}}\otimes e)v\phi(x\otimes e)v^*(\mathbb{1}_{\mathfrak{B}}\otimes e) \in \mathfrak{B}\otimes\mathbb{C}e.$$

- *a homomorphism:* The map $x \otimes e \mapsto v\phi(x \otimes e)v^*$ is clearly linear and $*$-preserving. For $x, y \in \mathfrak{A}$ is
$$v\phi(xy \otimes e)v^* = v\phi(x \otimes e)\phi(\mathbb{1}_{\mathfrak{A}} \otimes e)\phi(y \otimes e)v^* = v\phi(x \otimes e)v^* v\phi(y \otimes e)v^*.$$
- *surjective:* Let $y \in \mathfrak{B}$ be given. Then there exists $x \in \mathfrak{A} \otimes \mathbb{K}$ such that $\phi(x) = v^*(y \otimes e)v$. So
$$v\phi((\mathbb{1}_A \otimes e)x(\mathbb{1}_{\mathfrak{A}} \otimes e))v^* = v\phi(x)v^* = vv^*(y \otimes e)vv^* = y \otimes e.$$
Because $(\mathbb{1}_A \otimes e)x(\mathbb{1}_{\mathfrak{A}} \otimes e) \in \mathfrak{A} \otimes \mathbb{C}e$ the homomorphism is surjective.
- *injective:* Let $x, y \in \mathfrak{A}$. If $v\phi(x \otimes e)v^* = v\phi(y \otimes e)v^*$, then
$$\begin{aligned} \phi(x \otimes e) &= \phi(\mathbb{1}_{\mathfrak{A}} \otimes e)\phi(x \otimes e)\phi(\mathbb{1}_{\mathfrak{A}} \otimes e) = v^* v\phi(x \otimes e)v^* v \\ &= v^* v\phi(y \otimes e)v^* v = \phi(\mathbb{1}_{\mathfrak{A}} \otimes e)\phi(y \otimes e)\phi(\mathbb{1}_{\mathfrak{A}} \otimes e) = \phi(y \otimes e) \end{aligned}$$
and, consequently, $x = y$.

Corollary 12. *Let $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}$ and $\mathfrak{B}'$ be Kirchberg algebras from the UCT class $\mathcal{N}$, and assume that $\mathfrak{E}$ and $\mathfrak{E}'$ are unital, essential extensions:*
$$0 \longrightarrow \mathfrak{A} \xrightarrow{\ \iota\ } \mathfrak{E} \xrightarrow{\ \pi\ } \mathfrak{B} \longrightarrow 0$$
$$0 \longrightarrow \mathfrak{A}' \xrightarrow{\ \iota'\ } \mathfrak{E}' \xrightarrow{\ \pi'\ } \mathfrak{B}' \longrightarrow 0.$$
Then $\mathfrak{E} \cong \mathfrak{E}'$ if and only if there exists an isomorphism between the six term exact sequences from K-theory mapping $[\mathbb{1}_{\mathfrak{E}}]_0$ onto $[\mathbb{1}_{\mathfrak{E}'}]_0$.

Proof. By [14, Proposition 4.1] $\mathfrak{E}$ and $\mathfrak{E}'$ are properly infinite. This Corollary follows now directly from the Theorems 5 and 11 (where the objects of the subcategory are the C^*-algebras, which are essential extensions of Kirchberg algebras from the UCT class $\mathcal{N}$, and the morphisms are the $*$-homomorphisms mapping the non-trivial essential ideal into the non-trivial essential ideal).

Let $0 \to \mathfrak{A} \to \mathfrak{E} \to \mathfrak{B} \to 0$ be an essential extension of (non-zero) Kirchberg algebras. It is well known that Kirchberg algebras are either stable or unital. This forces $\mathfrak{A}$ to be stable. Then, as pointed out in [14], there are three kinds of extensions: (i) $\mathfrak{E}$ (and hence $\mathfrak{B}$) is unital, (ii) $\mathfrak{B}$ is unital but $\mathfrak{E}$ has no unit, and (iii) $\mathfrak{B}$ (and hence $\mathfrak{E}$) has no unit. In the latter case, both $\mathfrak{E}$ and $\mathfrak{B}$ are stable. Assuming that the algebras belong to the UCT class $\mathcal{N}$, we have classified the algebras of the first type up to $*$-isomorphism, while Rørdam has classified the algebras of the third type up to $*$-isomorphism. What remains is to classify the algebras in the intermediate case, where $\mathfrak{E}$ is neither unital nor stable[1].

[1] Note added in proof: This problem has been solved by the second named author and Efren Ruiz in *On Rørdam's classification of certain C^*-algebras with one non-trivial ideal, II*, preprint, 2006. The range question for the case considered in the present paper is also addressed there.

Corollary 13. *Let A and A' be non-degenerate $\{0,1\}$-matrices in the following block form*

$$A = \begin{pmatrix} M & 0 \\ X & N \end{pmatrix}, \quad A' = \begin{pmatrix} M' & 0 \\ X' & N' \end{pmatrix},$$

where N and N' are irreducible non-permutation matrices, the maximal non-degenerate principal submatrices of M and M' are irreducible non-permutationmatrices, $X \neq 0$, and $X' \neq 0$. So the matrices A and A' satisfy condition (II) of Cuntz ([4]) and the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A'}$ have exactly one non-trivial closed ideal.

Then $\mathcal{O}_A \cong \mathcal{O}_{A'}$ if and only if there exist isomorphisms

$$\gamma_1\colon \ker(I - N^{\mathsf{T}}) \to \ker(I - {N'}^{\mathsf{T}}),$$

$$\alpha_0\colon \operatorname{cok}(I - M^{\mathsf{T}}) \to \operatorname{cok}(I - {M'}^{\mathsf{T}}),$$

$$\beta_0\colon \operatorname{cok}(I - A^{\mathsf{T}}) \to \operatorname{cok}(I - {A'}^{\mathsf{T}})$$

such that

$$\begin{tikzcd}
\ker(I - N^{\mathsf{T}}) \arrow[r, "y \mapsto {[X^{\mathsf{T}} y]}"] \arrow[d, "\gamma_1", "\cong"'] & \operatorname{cok}(I - M^{\mathsf{T}}) \arrow[r, "{[x] \mapsto [\begin{smallmatrix} x \\ 0 \end{smallmatrix}]}"] \arrow[d, "\alpha_0", "\cong"'] & \operatorname{cok}(I - A^{\mathsf{T}}) \arrow[d, "\beta_0", "\cong"'] \\
\ker(I - {N'}^{\mathsf{T}}) \arrow[r, "y \mapsto {[{X'}^{\mathsf{T}} y]}"] & \operatorname{cok}(I - {M'}^{\mathsf{T}}) \arrow[r, "{[x] \mapsto [\begin{smallmatrix} x \\ 0 \end{smallmatrix}]}"] & \operatorname{cok}(I - {A'}^{\mathsf{T}})
\end{tikzcd}$$

commutes and $\beta_0([1\,1 \cdots 1]^{\mathsf{T}}) = [1\,1 \cdots 1]^{\mathsf{T}}$.

Proof. This follows from the previous Corollary combined with the paper [12] – the invariant there also asks for an isomorphism between the K_0-groups of the quotients, but here the existence is automatic, and no other commutative diagrams are required (because we have only *one* non-trivial ideal).

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