Interpolation by Projections in C^*-Algebras

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Dedicated to the memory of Gert. K. Pedersen

Note from author L.G.B.: This paper was begun in 2002 and was mainly completed in that year. There were some possible small changes still under discussion. In this version I have made only very minor changes that I'm sure Gert would have approved of.

Summary. If x is a self-adjoint element in a unital C^*-algebra $\mathcal{A}$, and if p_δ and q_δ denote the spectral projections of x corresponding to the intervals $]\delta, \infty[$ and $]-\infty, -\delta[$, we show that there is a projection p in $\mathcal{A}$ such that $p_\delta \leq p \leq \mathbf{1} - q_\delta$, provided that $\delta > \operatorname{dist}\{x, A_{\rm sa}^{-1}\}$. This result extends to unbounded operators affiliated with a C^*-algebra, and has applications to certain other distance functions.

1 Introduction

1.1

Let x be an operator on a Hilbert space $\mathcal{H}$ with polar decomposition $x = v|x|$, and for each $\delta \geq 0$ let e_δ and f_δ denote the spectral projections of $|x|$ and $|x^*|$, respectively, corresponding to the interval $]\delta, \infty[$. Practically the first observation to be made in single operator theory is that e_δ and f_δ are Murray–von Neumann equivalent; in fact, $ve_\delta v^* = f_\delta$. The second observation is that $\mathbf{1} - e_\delta$ and $\mathbf{1} - f_\delta$ need not be equivalent if $\mathcal{H}$ is infinite dimensional; in fact, $(\mathbf{1} - e_0)\mathcal{H} = \ker x$ and $(\mathbf{1} - f_0)\mathcal{H} = \ker x^*$, and these spaces may have widely different dimensions. If, however, $\mathbf{1} - e_\delta = w^*w$ and $\mathbf{1} - f_\delta = ww^*$ for some partial isometry w, then $u = w + ve_\delta$ is a unitary conjugating e_δ to f_δ. Equivalently phrased, the operator xe_δ can now be written $xe_\delta = u|xe_\delta|$ with a unitary u.

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If x belongs to an algebra $\mathcal{A}$ of operators on $\mathcal{H}$ the questions above can all be reformulated, asking now whether the unitary u can be chosen in $\mathcal{A}$. In the case of a von Neumann algebra $\mathcal{A}$ this question was solved by C.L. Olsen in [Ols89], using the distance to the set $\mathcal{A}^{-1}$ of invertible elements,

$$\alpha(x) = \operatorname{dist}\{x, \mathcal{A}^{-1}\}.$$

The answer is that $xe_\delta = u|xe_\delta|$ for some unitary u in $\mathcal{A}$ when $\delta > \alpha(x)$.

If $\mathcal{A}$ is only a C^*-algebra (always assumed unital in this paper unless otherwise specified) some care must be taken to formulate the question, because the spectral projections of an element do not (necessarily) belong to the algebra. However, if $x = v|x|$ is the polar decomposition then the element $x_f = vf(|x|) \in \mathcal{A}$ for every continuous function f vanishing at zero. We can therefore ask whether $x_f = u|x_f|$ for some unitary u in $\mathcal{A}$, provided that f vanishes on some interval $[0, \delta]$. In fact, this is equivalent to the demand that $ue_\delta = ve_\delta$ (whence also $f_\delta u = f_\delta v$), so that the partial isometry ve_δ has a unitary extension u in A. Combining a couple of highly technical lemmas this problem was solved in [Ror88, Theorem 2.2] and [Ped87, Theorem 5] with the same answer as in the von Neumann algebra case: If $\delta > \alpha(x)$ then for any continuous function f vanishing on $[0, \delta]$ we have $x_f = u|x_f|$ for some unitary u in $\mathcal{A}$. If $\delta < \alpha(x)$ no extension is possible.

The limit case $\delta = \alpha(x)$ is left undecided: Sometimes a unitary extension exists, sometimes not. For von Neumann algebras the index of x is a natural obstruction, but in general the situation is more subtle. Closer investigation shows that (outside finite AW^*-algebras) it is very unlikely that every x in the closure of the invertible elements in some C^*-algebra can be written in the form $x = u|x|$ with a unitary u in $\mathcal{A}$, cf. [HR93] and [Ped89].

1.2

If $\mathcal{A}_l^{-1}$ denotes the set of left invertible elements in a C^*-algebra $\mathcal{A}$ we can define the function

$$\alpha_l(x) = \operatorname{dist}\{x, \mathcal{A}_l^{-1}\}.$$

It was shown in [Ped91, Theorem 7.1] that if $\delta > \alpha_l(x)$ then any element $x_f = vf(|x|)$ can be written as $x_f = u|x_f|$ for some isometry u in $\mathcal{A}$, provided that f vanishes on $[0, \delta]$. The proof, however, is not very illuminating, since it quickly reduces to the regular case. Evidently there is also a symmetric result for the set $\mathcal{A}_r^{-1}$ of right invertible elements and co-isometries in $\mathcal{A}$, using the function $x \to \alpha_l(x^*)$.

A much more serious approach was needed to handle the set $\mathcal{A}_q^{-1}$ of quasi-invertible elements. Recall from [BP95] that $a \in \mathcal{A}_q^{-1}$ if $(\mathbf{1} - ba)\mathcal{A}(\mathbf{1} - ab) = 0$ for some b in $\mathcal{A}$. If we can choose $b = a^*$ then a is an extreme point in the unit ball of $\mathcal{A}$ and may be regarded as a partial isometry which is "maximally extended". A general quasi-invertible element always has the form $a = xuy$

with x, y in $\mathcal{A}^{-1}$ and u an extreme partial isometry, cf. [BP95, Theorem 1.1]. Now define

$$\alpha_q(x) = \operatorname{dist}\{x, \mathcal{A}_q^{-1}\}.$$

By [BP95, Theorem 2.2] we can then find an extreme partial isometry u in $\mathcal{A}$ such that $x_f = u|x_f|$, whenever $x_f = vf(|x|)$ and f is a continuous function vanishing on an interval $[0, \delta]$ with $\delta > \alpha_q(x)$. Equivalently, $ue_\delta = ve_\delta$ and $f_\delta u = f_\delta v$ if $\delta > \alpha_q(x)$.

1.3

Corresponding to the three distance functions mentioned above we have three classes of C^*-algebras, characterized by the norm density of the three subsets $\mathcal{A}^{-1}$, $\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$ and $\mathcal{A}_q^{-1}$. These are known, respectively, as C^*-algebras *of stable rank one*, *isometrically rich* C^*-algebras and *extremally rich* C^*-algebras. In such an algebra the polar decomposition of any element $x_f = vf(|x|)$ can be "upgraded", i.e. v can be replaced by a unitary, an isometry or a co-isometry, or an extreme partial isometry, if only f vanishes in some (small) neighbourhood of zero.

In [BP91] we introduced the class of C^*-algebras of *real rank zero* as those C^*-algebras $\mathcal{A}$ for which the set $\mathcal{A}_{\mathrm{sa}}^{-1}$ of invertible self-adjoint elements in the algebra was dense in $\mathcal{A}_{sa}$. (As for the other classes, a non-unital C^*-algebra has real rank zero if the unitized algebra fulfills the criterion.) Over the years a considerable theory has been developed for these classes of C^*-algebras, the real rank zero being the most "$AF-$like," the stable rank one algebras the most "finite."

One of the surprising phenomena (and the guiding principle in [BPa] and [BPb]) has been the patent, albeit subtle, similarity between C^*-algebras of stable rank one and C^*-algebras of real rank zero. For example, a theorem in $K-$theory that is valid for one class stands a very good chance also of being valid for the other class, but with a change of degree from $K_n(\mathcal{A})$ to $K_{n+1}(\mathcal{A})$. Related to this is the extension theory for the two classes. In both cases there is a known obstruction for an extension to be in the same class as the ideal and the quotient. For stable rank one algebras it is the lifting of unitaries from the quotient, for real rank zero the lifting of projections (equivalently, the lifting of self-adjoint unitaries).

The distance function

$$\alpha_r(x) = \operatorname{dist}\{x, \mathcal{A}_{\mathrm{sa}}^{-1}\}$$

provides another parallel case. Thus we show in [BPa, Theorem 2.2] for a general (unital) C^*-algebra $\mathcal{A}$ that the self-adjoint part of the largest ideal $I_{\mathrm{RR0}}(\mathcal{A})$ of $\mathcal{A}$ of real rank zero consists precisely of elements x in $\mathcal{A}_{\mathrm{sa}}$ such that $\alpha_r(x + y) = \alpha_r(y)$ for every y in $\mathcal{A}_{\mathrm{sa}}$. This should be compared to Rørdam's characterization in [Ror88, Propositions 4.1 & 4.2] of the largest ideal $I_{\mathrm{sr1}}(\mathcal{A})$ of stable rank one in $\mathcal{A}$, as consisting precisely of those elements x in $\mathcal{A}$ such that $\alpha(x + y) = \alpha(y)$ for every y in $\mathcal{A}$.

1.4

The main result in this paper, Theorem 3, is the exact analogue of the three polar decomposition results mentioned above, but now for self-adjoint elements only. This may at first seem odd, because $x = x^*$ implies that $e_\delta = f_\delta$ for all δ, and if $x = v|x|$ then $v = v^*$, so that $v = p - q$ for a pair of orthogonal projections. But if $\mathcal{A}$ is only a C^*-algebra it is still meaningful and interesting to ask whether v can be extended to a self-adjoint unitary, i.e. a symmetry in the algebra. Evidently this is so if zero is an isolated point in $\mathrm{sp}(x)$, but there are many other cases. For von Neumann algebras there is no problem; but then von Neumann algebras all have real rank zero. For C^*-algebras not of real rank zero there may not be very many projections around, hence also not very many symmetries. Our result may serve to locate these projections and control their behaviour.

Our result can also be interpreted as an interpolation, and we shall most often phrase it as such: If p_δ and q_δ denote the spectral projections of x corresponding to the intervals $]\delta, \infty[$ and $]-\infty, -\delta[$ (so that $p_\delta + q_\delta = e_\delta$ in the previous terminology), we show that there is a projection p in the algebra such that

$$p_\delta \le p \le \mathbf{1} - q_\delta,$$

provided that $\delta > \alpha_r(x)$. For C^*-algebras of real rank zero, where $\alpha_r(x) = 0$ for every x, this result was obtained in [Bro91]. In fact it was proved in [Bro91, Theorem 1] that $\mathcal{A}$ has real rank zero (in the sense that it satisfies one of the equivalent conditions HP or FS from [BP91]) if and only if it has *interpolation of projections*, IP, in the sense that whenever $\overline{p}$ is a compact and p° an open projection in $\mathcal{A}^{**}$ with $\overline{p} \le p^\circ$, then $\overline{p} \le p \le p^\circ$ for some projection p in $\mathcal{A}$.

2 Main results

Lemma 1. *Let x be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and for $\delta > 0$ define the continuous functions*

$$c_\delta(t) = t, \qquad d_\delta(t) = (\delta^2 - t^2)^{1/2} \qquad \textit{for} \quad |t| \le \delta, \tag{1}$$

$$c_\delta(t) = \delta \,\mathrm{sign}\, t, \qquad d_\delta(t) = 0 \qquad \textit{for} \quad |t| \ge \delta. \tag{2}$$

Then $\mathrm{sp}(a) \cap]-\delta, \delta[= \emptyset$, where a is the operator matrix

$$a = \begin{pmatrix} c_\delta(x) & d_\delta(x) \\ d_\delta(x) & -x \end{pmatrix}.$$

Proof. If $\lambda \in \mathrm{sp}(a)$ then for some t in $\mathrm{sp}(x)$ we have

$$(c_\delta(t) - \lambda)(-t - \lambda) - d_\delta(t)^2 = 0.$$

If $\delta \le |t|$ this equation simply becomes $(\delta \,\mathrm{sign}\, t - \lambda)(t + \lambda) = 0$, with the solutions $\lambda = \delta \,\mathrm{sign}\, t$ and $\lambda = -t$. It follows that $|\lambda| \ge \delta$.

If $|t| \leq \delta$ we obtain the equation $(t-\lambda)(-t-\lambda)-(\delta^2-t^2)=0$, or $\lambda^2-\delta^2=0$, with the solutions $\lambda=\pm\delta$; so that again $|\lambda|\geq\delta$. $\square$

Definition 2. *As usual, cf. [BP95, Section 1], given a self-adjoint element x in a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ we define the constant*

$$\begin{aligned} m(x) &= \sup\{\varepsilon \geq 0 \mid]-\varepsilon,\varepsilon[\cap \operatorname{sp}(x)=\emptyset\} \\ &= \inf\{\|x\xi\| \mid \xi\in\mathcal{H}:\ \|\xi\|=1\} \\ &= \operatorname{dist}\{x, \left(A_{\mathrm{sa}}\setminus A_{\mathrm{sa}}^{-1}\right)\}. \end{aligned}$$

Note that $m(x)=m(|x|)$, so that for a general (non self-adjoint) element x in $\mathcal{A}$ we can define $m(x)=m(|x|)$. Alternatively, we can use the second expression, which makes sense for all operators. It is an easy consequence of the open mapping theorem that x is invertible if and only if $m(x)>0$ and $m(x^*)>0$ [since then $\ker x=0$ *and $x(\mathcal{H})=\mathcal{H}$].*

Theorem 3. *Let x be a self-adjoint element in a unital C^*–algebra $\mathcal{A}$, and for $\delta\geq 0$ denote by p_δ and q_δ the spectral projections of x (in $\mathcal{A}^{**}$) corresponding to the intervals $]\delta,\infty[$ and $]-\infty,-\delta[$, respectively. If $\delta>\alpha_r(x)$ there is a projection p in $\mathcal{A}$ such that*

$$p_\delta \leq p \leq \mathbf{1}-q_\delta.$$

Equivalently, for any continuous function f vanishing on the interval $[-\delta,\delta]$ and such that $f(t)\operatorname{sign}t\geq 0$ for all t we have $f(x)=(2p-\mathbf{1})|f(x)|$ in $\mathcal{A}$.

If $\delta<\alpha_r(x)$ there are no projections p in $\mathcal{A}$ such that $p_\delta\leq p\leq\mathbf{1}-q_\delta$, and no symmetries u in $\mathcal{A}$ such that $f(x)=u|f(x)|$ if we choose $f(t)=$ $\operatorname{sign}t\ (|t|-\delta)_+$.

Proof. By assumption we can find y in $\mathcal{A}_{\mathrm{sa}}^{-1}$ with $\|x-y\|<\delta$. With c_δ and d_δ as in Lemma 1 this means that the operator matrix

$$b=\begin{pmatrix} c_\delta(x) & d_\delta(x) \\ d_\delta(x) & -y \end{pmatrix}$$

is still invertible (in $\mathbb{M}_2(\mathcal{A})$), because $m(b)\geq\delta-\|x-y\|>0$, cf. Definition 2. Consequently also the matrix

$$\begin{pmatrix} \mathbf{1} & d_\delta(x)y^{-1} \\ 0 & \mathbf{1} \end{pmatrix} b \begin{pmatrix} \mathbf{1} & 0 \\ y^{-1}d_\delta(x) & \mathbf{1} \end{pmatrix} = \begin{pmatrix} c_\delta(x)+d_\delta(x)y^{-1}d_\delta(x) & 0 \\ 0 & -y \end{pmatrix}$$

is invertible in $\mathbb{M}_2(\mathcal{A})$, from which we conclude that the self-adjoint element

$$z=c_\delta(x)+d_\delta(x)y^{-1}d_\delta(x)$$

is invertible in $\mathcal{A}$.

By construction we have

$$p_\delta d_\delta(x) = q_\delta d_\delta(x) = 0, \quad p_\delta c_\delta(x) = \delta p_\delta, \quad q_\delta c_\delta(x) = -\delta q_\delta.$$

Therefore $p_\delta z = \delta p_\delta$ and $q_\delta z = -\delta q_\delta$. If p denotes the spectral projection of z corresponding to the interval $]0,\infty[$, then $p \in \mathcal{A}$ since $0 \notin \mathrm{sp}(z)$. From the equations above we see that $p_\delta p = p_\delta$ and $q_\delta p = 0$, whence $p_\delta \le p \le \mathbf{1} - q_\delta$.

If f is a continuous function vanishing on $[-\delta, \delta]$ such that $f(t)\,\mathrm{sign}\,t \ge 0$ for all t, then by spectral theory

$$\begin{aligned} f(x) &= (p_\delta + q_\delta) f(x) = (p_\delta - q_\delta)|f(x)| \\ &= (2p - \mathbf{1})(p_\delta + q_\delta)|f(x)| = (2p - \mathbf{1})|f(x)|. \end{aligned}$$

Assume now that for some δ we have a projection p in $\mathcal{A}$ such that $p_\delta \le p \le \mathbf{1} - q_\delta$. Put $u = 2p - \mathbf{1}$. Then with $f(t) = \mathrm{sign}\,t\ (|t| - \delta)_+$ and $\varepsilon > 0$ consider the element $y = u\,((|x| - \delta)_+ + \varepsilon\mathbf{1})$. Evidently $y \in \mathcal{A}_{\mathrm{sa}}^{-1}$ and $\|f(x) - y\| \le \varepsilon$. Consequently

$$\|x - y\| \le \|x - f(x)\| + \varepsilon \le \delta + \varepsilon.$$

Since ε is arbitrary we conclude that $\alpha_r(x) \le \delta$. This proves the last statement in the theorem.

Corollary 4. *For every self-adjoint element x in a unital C^*-algebra $\mathcal{A}$ put $\alpha = \alpha_r(x)$ and define $x_\alpha = c_\alpha(x)$, where $c_\alpha(t) = \mathrm{sign}\,t\ (|t| \wedge \alpha)$ as in Lemma 1. Then $x - x_\alpha \in (\mathcal{A}_{\mathrm{sa}}^{-1})^=$, $\|x - x_\alpha\| = \|x\| - \alpha$ and $\|x_\alpha\| = \alpha = \alpha_r(x_\alpha)$.* □

Example 5. It is easy to find examples where no projections exist in the limit $\delta = \alpha_r(x)$. If $\Omega = [-1, 1] \cup \{1 + 1/n \mid n \in \mathbb{N}\} \cup \{-1 - 1/n \mid n \in \mathbb{N}\}$ and $\mathcal{A} = C(\Omega)$, then with $x = \mathrm{id}$ we obtain a self-adjoint element with $\alpha_r(x) = 1$ (but $\|x\| = 2$). The spectral projections p_1 and q_1 correspond to the characteristic functions for the sets $\{1 + 1/n \mid n \in \mathbb{N}\}$ and $\{-1 - 1/n \mid n \in \mathbb{N}\}$, respectively, so there is no projection p in $\mathcal{A}$ such that $p_1 \le p \le \mathbf{1} - q_1$.

Definition 6 (Unbounded Operators). *Let x be an unbounded self-adjoint operator in a Hilbert space $\mathcal{H}$. We say that x is affiliated with a non-unital C^*-algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ if $(x - \lambda\mathbf{1})^{-1} \in \mathcal{A}$ for every λ outside $\mathrm{sp}(x)$. Equivalently, $(x - it\mathbf{1})^{-1} \in \mathcal{A}$ whenever $t \ne 0$. It follows that $f(x) \in \mathcal{A}$ for every f in $C_0(\mathbb{R})$. In addition we demand that the subalgebra $\{f(x) \mid f \in C_0(\mathbb{R})\}$ contain an approximate unit for $\mathcal{A}$, which is equivalent to the demand that $(\mathbf{1} + x^2)^{-1}$ be a strictly positive element in $\mathcal{A}$. From this extra condition we conclude that the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$ contains every element of the form $f(x)$ where $f \in C_b(\mathbb{R})$.*

The set $\mathcal{A}^{\mathrm{aff}}$ of affiliated operators is not an algebra (in general), not even a vector space, but $x - a \in \mathcal{A}^{\mathrm{aff}}$ for every x in $\mathcal{A}^{\mathrm{aff}}$ and a in $M(\mathcal{A})_{\mathrm{sa}}$. To see this we take $|t| > \|a\|$. Then

$$\begin{aligned} (x - a - it\mathbf{1})^{-1} &= \big((x - it\mathbf{1})(\mathbf{1} - (x - it\mathbf{1})^{-1}a)\big)^{-1} \\ &= \sum_{n=0}^{\infty} \big((x - it\mathbf{1})^{-1}a\big)^n (x - it\mathbf{1})^{-1} \in \mathcal{A}. \end{aligned}$$

On the other hand, if $(x-a-it\mathbf{1})^{-1}\in\mathcal{A}$ for some t then also

$$(x-a-is\mathbf{1})^{-1}=\big((x-a-it\mathbf{1})(\mathbf{1}-(x-a-it\mathbf{1})^{-1})i(s-t)\big)^{-1}\in\mathcal{A}$$

for $|s-t|<|t|$, since $\|(x-a-it\mathbf{1})^{-1}\|\le|t|^{-1}$. Taken together this means that $(x-a-it\mathbf{1})^{-1}\in\mathcal{A}$ for all $t\neq 0$, whence $x-a\in\mathcal{A}^{\mathrm{aff}}$.

For each x affiliated with $\mathcal{A}$ and every $\gamma>0$ we define the cut-down operator $x_\gamma=c_\gamma(x)$ in $M(\mathcal{A})$, where $c_\gamma(t)=\operatorname{sign}t\,(|t|\wedge\gamma)$ as in Lemma 1 and Corollary 4. On these operators we apply the function $\alpha_r(\cdot)$ relative to the unital C^*-algebra $M(\mathcal{A})$.

Lemma 7. *If $\alpha_r(x_\gamma)<\gamma$ for some $\gamma>0$ then $\alpha_r(x_\beta)=\alpha_r(x_\gamma)$ for all $\beta>\gamma$.*

Proof. By assumption we can find δ such that $\alpha_r(x_\gamma)<\delta<\gamma$, and then consider the spectral projections p_δ and q_δ of x. However, p_δ and q_δ can also be regarded as spectral projections of x_γ and of x_β, still corresponding to the intervals $]\delta,\infty[$ and $]-\infty,-\delta[$.

It is therefore easy to deduce the result from Theorem 3. □

Definition 8. *If x is a self-adjoint operator affiliated with a non-unital C^*-algebra $\mathcal{A}$ we define $\alpha_r(x)$ to be the infimum of numbers $\|a\|$, where $a\in M(\mathcal{A})_{\mathrm{sa}}$ such that $x-a$ is invertible (whence $(x-a)^{-1}\in\mathcal{A}$). If no such a exists we set $\alpha_r(x)=\infty$. Loosely speaking we may refer to $\alpha_r(x)$ as the distance between x and the invertible operators affiliated with $\mathcal{A}$. At least we see that $\alpha_r(x+b)\le\alpha_r(x)+\alpha_r(b)$ for every x in $\mathcal{A}^{\mathrm{aff}}$ and b in $M(\mathcal{A})_{\mathrm{sa}}$.*

Theorem 9. *Let x be a self-adjoint operator affiliated with a non-unital C^*-algebra $\mathcal{A}$. For every $\delta>\alpha_r(x)$ there is then a projection p in $M(\mathcal{A})$ interpolating the spectral projections p_δ and $\mathbf{1}-q_\delta$ of x. Moreover,*

$$\alpha_r(x)=\inf\{\gamma>0\mid\alpha_r(x_\gamma)<\gamma\}.$$

Proof. If $\alpha_r(x)<\delta<\infty$ we can find a in $M(\mathcal{A})_{\mathrm{sa}}$ with $\|a\|<\delta$ such that $y=x-a$ is invertible (and $y^{-1}\in\mathcal{A}$). With c_δ and d_δ as in Lemma 1 we obtain an operator matrix, where now the $(2,2)-$corner is unbounded; but it is still true that the spectrum of the matrix misses the open interval $]-\delta,\delta[$. This means that when in the proof of Theorem 3 we define the operator matrix b, where the $(2,2)-$ corner is $-y$, we have an unbounded, but invertible operator. The element $z=c_\delta(x)+d_\delta(x)y^{-1}d_\delta(x)$ is therefore again invertible, but also bounded, and $z\in M(\mathcal{A})$. The rest of the proof proceeds as before to prove that there is a projection p in $M(\mathcal{A})$ interpolating the spectral projections p_δ and $\mathbf{1}-q_\delta$ of x.

Since the projection p found above also interpolates the spectral projections of the cut-down elements x_γ when $\gamma>\delta$, it follows that $\alpha_r(x_\gamma)\le\delta<\gamma$. As δ can be chosen arbitrarily close to $\alpha_r(x)$ this implies that $\inf\{\gamma>0\mid\alpha_r(x_\gamma)<\gamma\}\le\alpha_r(x)$.

To prove the reverse inequality we note that if for some γ we have $\alpha_r(x_\gamma) < \beta < \gamma$ for some β, then $p_\beta \leq p \leq \mathbf{1} - q_\beta$ by Theorem 3, with p a projection in $M(\mathcal{A})$. Consequently $x - x_\beta = f(x) = (2p - \mathbf{1})|f(x)|$, where $f(t) = \operatorname{sign} t\ (|t| - \beta)_+$, and the self-adjoint element $(2p - \mathbf{1})(|f(x)| + \varepsilon\mathbf{1})$ is invertible for every $\varepsilon > 0$ and affiliated with $\mathcal{A}$. Since $\|x_\beta\| \leq \beta$ it follows that $\alpha_r(x) \leq \beta + \varepsilon$, whence in the limit $\alpha_r(x) \leq \gamma$. □

Remarks

The problems encountered when trying to remove zero from the spectrum of certain differential (Dirac) operators are well documented in the literature, see e.g. [LL, LP98, LP02, LLP99, Lot96]. We hope that our result can be of some use in this context.

Clearly we do not need the whole multiplier algebra to formulate the results in Theorem 9. What is required is a unital C^*−algebra $\mathcal{B}$ such that $f(x) \in \mathcal{B}$ for each f in $C(\mathbb{R} \cup \pm\infty)$ (the two-point compactification of $\mathbb{R}$), but also such that $f(x + b) \in \mathcal{B}$ if $b \in \mathcal{B}_{\mathrm{sa}}$.

On the other hand we see from the proof of Theorem 9 that if $y = x - a$ is an invertible bounded perturbation of x (so that $a \in M(\mathcal{A})_{\mathrm{sa}}$) then the invertible element $z = x_\delta + d_\delta(x) y^{-1} d_\delta(x)$ is an $\mathcal{A}$−perturbation of x_δ, so that we can assert that x_δ is invertible in the corona algebra $M(\mathcal{A})/\mathcal{A}$.

3 Some applications

3.1 Distance to the Symmetries

The formula for the distance between an element x in a C^*−algebra $\mathcal{A}$ and the group $\mathcal{U}(\mathcal{A})$ of unitaries was proved in [Ror88, Theorem 2.7], see also [Ped87, Theorem 10], in complete analogy with the formula for the von Neumann algebra case found by C.L. Olsen in [Ols89]. The same formula, but with $\alpha_q(\cdot)$ replacing $\alpha(\cdot)$, describes the distance to the set $\mathcal{E}(\mathcal{A})$ of extreme partial isometries in $\mathcal{A}$ by [BP97, Theorem 3.1]. We show below that the exact same formula – but now with $\alpha_r(\cdot)$ replacing $\alpha(\cdot)$ – describes the distance between a self-adjoint element x and the set $\mathcal{S}(\mathcal{A})$ of symmetries in $\mathcal{A}$. As for unitaries in C^*−algebras one can not in general hope to find an approximant to x in $\mathcal{S}(\mathcal{A})$, Example 5 provides a case in point, but in special cases they exist, cf. Corollary 11 and Proposition 12.

Proposition 10. *Let x be a non-invertible self-adjoint element in a unital C^*−algebra $\mathcal{A}$ and let $\mathcal{S}(\mathcal{A})$ denote the set of symmetries in $\mathcal{A}$. Then $[-\alpha_r(x), \alpha_r(x)] \subset \mathrm{sp}(ux)$ for every u in $\mathcal{S}(\mathcal{A})$. Moreover,*

$$\operatorname{dist}\{x, \mathcal{S}(\mathcal{A})\} = \max\{\|x\| - 1,\ \alpha_r(x) + 1\}.$$

Proof. If $u \in \mathcal{S}(\mathcal{A})$ and $\lambda \notin \mathrm{sp}(ux)$ for some real number λ, then $\lambda \neq 0$ and $\lambda\mathbf{1} - ux \in \mathcal{A}^{-1}$, whence $\lambda u - x \in \mathcal{A}_{\mathrm{sa}}^{-1}$. Therefore $|\lambda| = \|\lambda u\| \geq \alpha_r(x)$, proving the first statement.

We see from above that

$$\|x - u\| = \|\mathbf{1} - ux\| \geq \rho(\mathbf{1} - ux) \geq 1 + \alpha_r(x);$$

and evidently $\|x - u\| \geq \|x\| - 1$, so that we have the inequality

$$\mathrm{dist}\,\{x, \mathcal{S}(\mathcal{A})\} \geq \max\{\|x\| - 1,\ \alpha_r(x) + 1\}.$$

To prove the reverse inequality we take $\delta > \alpha_r(x)$ and find a projection p in $\mathcal{A}$ with $p_\delta \leq p \leq \mathbf{1} - q_\delta$ using Theorem 3. Then with $u = 2p - \mathbf{1}$ we have

$$\begin{aligned}\|x - u\| &= \|(x - u)(p_\delta + q_\delta + (\mathbf{1} - p_\delta - q_\delta))\| \\ &= \max\{\|(x - u)p_\delta\|,\ \|(x - u)q_\delta\|,\ \|(x - u)(\mathbf{1} - p_\delta - q_\delta)\|\} \\ &\leq \max\{\|x_+\| - 1,\ 1 - \delta,\ \|x_-\| - 1,\ 1 + \delta\}.\end{aligned}$$

Since δ can be chosen arbitrarily near $\alpha_r(x)$ the result follows. □

Corollary 11. *If $\alpha_r(x) < \|x\| - 2$ there is a symmetry u in $\mathcal{A}$ such that*

$$\|x - u\| = \mathrm{dist}\,\{x, \mathcal{S}(A)\} = \|x\| - 1$$

Proof. By assumption we can find δ such that $\alpha_r(x) < \delta < \|x\| - 2$. Choosing the symmetry u as in Proposition 10 this means that $\|x - u\| = \|x\| - 1$, as desired. □

Proposition 12. *Let x be self-adjoint and invertible in a unital C^*-algebra $\mathcal{A}$ with polar decomposition $x = u|x|$. Then with $m(x)$ as in Definition 2 we have*

$$\mathrm{dist}\,\{x,\ \mathcal{S}(\mathcal{A})\} = \max\{\|x\| - 1,\ 1 - m(x)\} = \|x - u\|.$$

Proof. If $w \in \mathcal{S}(\mathcal{A})$ then evidently $\|x - w\| \geq \|x\| - 1$. Moreover, for each unit vector ξ we have $\|x - w\| \geq \|w(\xi)\| - \|x(\xi)\| = 1 - \|x(\xi)\|$, proving the inequality

$$\mathrm{dist}\,\{x,\ \mathcal{S}(\mathcal{A})\} \geq \max\{\|x\| - 1,\ 1 - m(x)\}.$$

On the other hand, with $x = u|x|$ we have by spectral theory that

$$\|x - u\| = \sup\{|t - \mathrm{sign}\,t| \mid t \in \mathrm{sp}(x)\} = \max\{\|x\| - 1,\ 1 - m(x)\},$$

as desired. □

3.2 The $\lambda-$Function

For every element x in the unit ball $\mathcal{A}^1$ of a unital C^*-algebra $\mathcal{A}$ the number $\lambda(x)$ is defined as the supremum of all λ in $[0,1]$ such that $x=\lambda u+(1-\lambda)y$ for some extreme point u in $\mathcal{A}^1$ and some arbitrary y in $\mathcal{A}^1$. This $\lambda-$*function* on $\mathcal{A}^1$ was completely determined in [BP97, Theorem 3.7] in terms of the numbers $\alpha_q(x)$ and $m_q(x)$. In particular it was shown that $\mathcal{A}$ has the $\lambda-$*property* ($\lambda(x)>0$ for every x in $\mathcal{A}^1$) if and only if $\lambda(x)\geq 1/2$ for every x, which happens precisely when $\mathcal{A}$ is extremally rich, cf. Section 1.3.

The simpler cases of unitaries or isometries were solved earlier in [Ped91, Theorems 5.1, 5.4 & 8.1] with formulae resembling the case above. The relevant function here is the *unitary* $\lambda-$*function*, $\lambda_u(x)$, defined on A^1 as the supremum of all λ in $[0,1]$ such that $x=\lambda u+(1-\lambda)y$ for some unitary u in A and y in A^1. We found that $\lambda(x)>0$ for every x in A^1 precisely when A has stable rank one, in which case actually $\lambda_u(x)\geq 1/2$.

We now define the *real* $\lambda-$*function* $\lambda_r(x)$ on $\mathcal{A}^1_{\rm sa}$ to be the supremum of all λ in $[0,1]$ such that $x=\lambda u+(1-\lambda)y$ for some symmetry u in $\mathcal{S}(\mathcal{A})$ and y in $\mathcal{A}^1_{\rm sa}$. As we shall see, the form of this function is completely analogous to the classical $\lambda-$function, with $\alpha_r(\cdot)$ and $m(\cdot)$ replacing $\alpha_q(\cdot)$ and $m_q(\cdot)$; and a C^*-algebra $\mathcal{A}$ has the *real* $\lambda-$*property* ($\lambda_r(x)>0$ for every x in $\mathcal{A}^1_{\rm sa}$) if and only if $\lambda_r(x)\geq 1/2$ for every x, which happens precisely when $\mathcal{A}$ has real rank zero.

Proposition 13. *The real $\lambda-$function on the self-adjoint part of the unit ball $\mathcal{A}^1_{\rm sa}$ of a unital C^*-algebra $\mathcal{A}$ is given by the following formulae:*

$$\lambda_r(x)=\tfrac{1}{2}(1+m(x)) \qquad \text{if} \quad x\in\mathcal{A}^{-1}_{\rm sa} \tag{3}$$
$$\lambda_r(x)=\tfrac{1}{2}(1-\alpha_r(x)) \qquad \text{if} \quad x\notin\mathcal{A}^{-1}_{\rm sa}. \tag{4}$$

Proof. If $x\in\mathcal{A}^{-1}_{\rm sa}$ with polar decomposition $x=u|x|$ then with $\lambda=(1+m(x))/2$ we define the element $y=(1-\lambda)^{-1}(x-\lambda u)$ in $\mathcal{A}_{\rm sa}$. Using Definition 2 it follows by easy computations in spectral theory that $\|y\|\leq 1$. Thus $x=\lambda u+(1-\lambda y)$ in $\mathcal{A}^1_{\rm sa}$, whence $\lambda_r(x)\geq(1+m(x))/2$.

Conversely, if $x=\lambda w+(1-\lambda)z$ for some w in $\mathcal{S}(\mathcal{A})$ and z in $\mathcal{A}^1_{\rm sa}$ then by Proposition 12

$$1-m(x)\leq\|w-x\|=\|(1-\lambda)(w-z)\|\leq 2(1-\lambda),$$

whence $\lambda\leq(1+m(x))/2$, as desired.

If $x\notin\mathcal{A}^{-1}_{\rm sa}$ and $x=\lambda w+(1-\lambda)z$ for some w in $\mathcal{S}(\mathcal{A})$ and z in $\mathcal{A}^1_{\rm sa}$ then by Proposition 10

$$1+\alpha_r(x)\leq\|w-x\|=\|(1-\lambda)(w-z)\|\leq 2(1-\lambda),$$

whence $\lambda\leq(1-\alpha_r(x))/2$, which is therefore an upper bound for $\lambda_r(x)$.

On the other hand, if $\alpha_r(x)\neq 1$ and $\alpha_r(x)<\delta<1$ we can by Theorem 3 find a projection p in $\mathcal{A}$ with $p_\delta\leq p\leq\mathbf{1}-q_\delta$. With $u=2p-\mathbf{1}$ and $\lambda=(1-\delta)/2$

we claim that the element $y = (1-\lambda)^{-1}(x - \lambda u)$ has norm at most one. To prove this we compute

$$\|yp_\delta\| = \|(1-\lambda)^{-1}(x-\lambda)p_\delta\| \leq 1.$$

Similarly $\|yq_\delta\| \leq 1$. Finally,

$$\|y(\mathbf{1} - p_\delta - q_\delta)\| \leq \|(1-\lambda)^{-1}(\delta + \lambda)\| \leq 1.$$

Consequently $\|y\| \leq 1$. Since $x = \lambda u + (1-\lambda)y$ by construction, we see that $\lambda_r(x) \geq \lambda$, whence in the limit as $\delta \to \alpha_r(x)$ we obtain the desired estimate $\lambda_r(x) \geq (1 - \alpha_r(x))/2$. □

Remark 14. We see from the formulae in Proposition 13 that if $\lambda_r(x) > 0$ for every x in $\mathcal{A}^1_{\mathrm{sa}}$ then $\alpha_r(x) < 1$ for every x. But if $\alpha_r(x) > 0$ for some element x in $\mathcal{A}^1_{\mathrm{sa}}$ then by Corollary 4 we have a non-zero element x_α in $\mathcal{A}_{\mathrm{sa}}$ with $\|x_\alpha\| = \alpha_r(x_\alpha)$. Thus the element $y = \|x_\alpha\|^{-1}x_\alpha$ will violate the real $\lambda-$condition ($\lambda_r(y) = 0$). The only way to avoid this situation is to demand that $\alpha_r(x) = 0$ for all x, so that $\mathcal{A}$ has real rank zero. In this case, of course, $\lambda_r(x) \geq 1/2$ for every x in $\mathcal{A}^1_{\mathrm{sa}}$.

3.3 Projectionless C*–Algebras

It is well known that there are C^*-algebras, even simple ones, that contain no non-trivial projections. Such algebras may be regarded as opposite to the real rank zero C^*-algebras. Theorem 3 allows us to reformulate this property in terms of the distance from the invertible self-adjoint elements in the algebra.

Proposition 15. *In a unital C^*-algebra $\mathcal{A}$ the following conditions are equivalent:*

(i) $\mathcal{A}$ has no non-trivial projections.
(ii)$\alpha_r(x) = \min\{\|x_+\|,\ \|x_-\|\}$ for every element x in $\mathcal{A}_{\mathrm{sa}}$.
(iii)$\mathcal{A}^{-1}_{\mathrm{sa}} \subset -\mathcal{A}_+ \cup \mathcal{A}_+$.

Proof. (i) $\Longrightarrow$ (ii) If we can find δ such that $\alpha_r(x) < \delta < \min\{\|x_+\|,\ \|x_-\|\}$ for some x in $\mathcal{A}_{\mathrm{sa}}$, then the spectral projections p_δ and q_δ are both non-zero (in $\mathcal{A}^{**}$). Applying Theorem 3 we obtain a projection p in $\mathcal{A}$ such that $p_\delta \leq p \leq \mathbf{1} - q_\delta$, which means that p is non-trivial.

(ii) $\Longrightarrow$ (iii) We always have $-\mathcal{A}_+ \cup \mathcal{A}_+ \subset (\mathcal{A}^{-1}_{\mathrm{sa}})^=$ and $\alpha_r(x) \leq \min\{\|x_+\|,\ \|x_-\|\}$. If now $x \in (\mathcal{A}^{-1}_{\mathrm{sa}})^= \setminus (-\mathcal{A}_+ \cup \mathcal{A}_+)$ then $\alpha_r(x) = 0$, but $\min\{\|x_+\|,\ \|x_-\|\} > 0$.

(iii) $\Longrightarrow$ (i) If p is a non-trivial projection in $\mathcal{A}$ then $2p - \mathbf{1} \in \mathcal{A}^{-1}_{\mathrm{sa}} \setminus (-\mathcal{A}_+ \cup \mathcal{A}_+)$. □

3.4 The Unitary Case Revisited

It is well worth noticing that the method from Theorem 3 also can be used to give a short and transparent proof of the result mentioned in 1.1 about unitary polar decomposition of arbitrary elements in a C^*-algebra $\mathcal{A}$. With some more work it will even give the corresponding result in [BP95, Theorem 2.2] for quasi-invertible elements, but we shall here only show the former.

If $x \in \mathcal{A}$ with polar decomposition $x = v|x|$ we define the operator matrix

$$a = \begin{pmatrix} vc_\delta(|x|) & d_\delta(|x^*|) \\ d_\delta(|x|) & -x^* \end{pmatrix}$$

with c_δ and d_δ as in Lemma 1. The observant reader will notice the similarity between a and the standard unitary dilation of a contraction x. Straightforward computations show that

$$a^*a = \begin{pmatrix} \delta^2\mathbf{1} & 0 \\ 0 & \delta^2\mathbf{1} \vee xx^* \end{pmatrix} \quad \text{and} \quad aa^* = \begin{pmatrix} \delta^2\mathbf{1} & 0 \\ 0 & \delta^2\mathbf{1} \vee x^*x \end{pmatrix},$$

so that $m(a) \geq \delta$ and $m(a^*) \geq \delta$. If therefore $\alpha(x) < \delta$, so that we can find y in $\mathcal{A}^{-1}$ with $\|x^* - y\| < \delta$, then the matrix

$$b = \begin{pmatrix} vc_\delta(|x|) & d_\delta(|x^*|) \\ d_\delta(|x|) & -y \end{pmatrix}$$

is invertible (in $\mathbb{M}_2(\mathcal{A})$), since both $m(b) > 0$ and $m(b^*) > 0$, cf. Definition 2. As in the proof of Theorem 3 this implies that also the element $z = vc_\delta(|x|) + d_\delta(|x^*|)y^{-1}d_\delta(|x|)$ is invertible (in $\mathcal{A}$). Since $ze_\delta = \delta ve_\delta$ and $f_\delta z = \delta f_\delta v$ by construction, it follows that if $z = u|z|$ is the polar decomposition of z then u is a unitary in $\mathcal{A}$ such that $ue_\delta = ve_\delta$ and $f_\delta u = f_\delta v$. We have reproved [Ror88, Theorem 2.2] and [Ped87, Theorem 5]

Theorem 16. *If $x = v|x|$ is the polar decomposition of an element in a unital C^*-algebra $\mathcal{A}$ then for each $\delta > \alpha(x)$ there is a unitary u in $\mathcal{A}$ such that $ue_\delta = ve_\delta$. Equivalently, for every continuous function f vanishing on $[0,\delta]$ we have $vf(|x|) = uf(|x|)$.* □

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