$\mathbb{Q}$-lattices: quantum statistical mechanics & Galois Theory

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Quantum statistical mechanics of $\mathbb{Q}$-lattices

Based on:

- Alain Connes and M.M., *Quantum statistical mechanics of* $\mathbb{Q}$*-lattices*, math.NT/0404128 (*)

- Alain Connes, Niranjan Ramachandran, and M.M., *KMS states and complex multiplication*, preprint 2004(?)

- A. Connes & M.M. "$\mathbb{Q}$-lattices: quantum statistical mechanics & Galois Theory" (brief overview of (*))

Unified setting for:

- Modular Hecke algebra (Connes–Moscovici)
- Spectral realization of zeros of $\zeta(s)$ (Connes)
- Bost–Connes system with arithmetic spontaneous symmetry breaking
- NC boundary of modular curves (Manin-M)

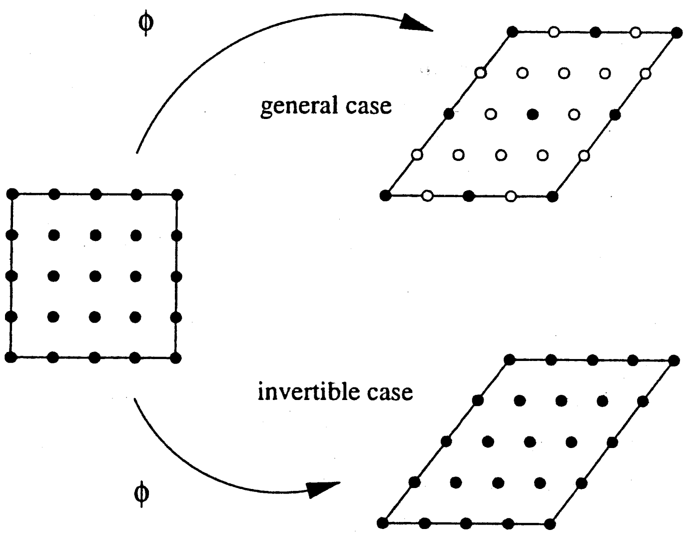
⇝ NCG & Class Field Theory

<u>$\mathbb{Q}$-lattices</u> (Λ, ϕ) $\mathbb{Q}$-lattice in $\mathbb{R}^n$

lattice $\Lambda \subset \mathbb{R}^n$ + labels of torsion points

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

group homomorphism (invertible $\mathbb{Q}$-lat is isom)



<u>Commensurability</u>

$$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$$

iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and

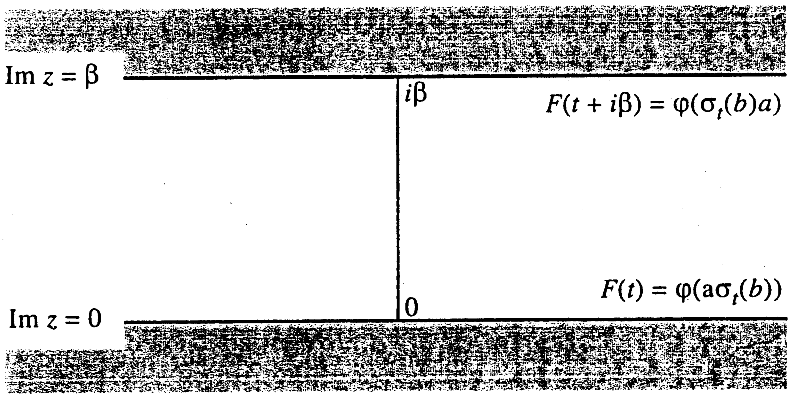
$$\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$$

<u>$\mathbb{Q}$-lattices</u> / <u>Commensurability</u> $\Rightarrow$ NC space

KMS states $\quad \varphi \in \text{KMS}_\beta \ (0 < \beta < \infty)$

$\forall a, b \in \mathcal{A} \ \exists$ holom function $F_{a,b}(z)$ on strip: $\forall t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \qquad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a)$$



Ground states $\quad (\beta = \infty,\ T = 0)$

At $T > 0$ simplex $\text{KMS}_\beta \ \rightsquigarrow$ extremal $\mathcal{E}_\beta$
(Points on NC space $\mathcal{A}$)

At $T = 0$: $\text{KMS}_\infty =$ weak limits of KMS_β

$$\varphi_\infty(a) = \lim_{\beta \to \infty} \varphi_\beta(a)$$

Quantum Statistical Mechanics

$\mathcal{A} =$ algebra of observables (C^*-algebra)

State: $\varphi : \mathcal{A} \to \mathbb{C}$ linear

$$\varphi(1) = 1, \quad \varphi(a^* a) \geq 0$$

Time evolution $\sigma_t \in \mathrm{Aut}(\mathcal{A})$

(rep on Hilbert space $\mathcal{H}$)

$$\text{Hamiltonian} \quad H = \frac{d}{dt} \sigma_t |_{t=0}$$

Equilibrium state (inverse temperature $\beta = 1/kT$)

$$\frac{1}{Z(\beta)} \mathrm{Tr}\left(a\, e^{-\beta H}\right) \qquad Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right)$$

Symmetries $\rightsquigarrow$ action on $\mathcal{E}_\beta$

- Automorphisms $G \subset \mathrm{Aut}(\mathcal{A})$, $g\sigma_t = \sigma_t g$

Mod Inner: $u =$ unitary $\sigma_t(u) = u$

$$a \mapsto uau^*$$

$\rho : \mathcal{A} \to \mathcal{A}$ *-hom.

- Endomorphisms $\rho\sigma_t = \sigma_t\rho$ $e = \rho(1)$

For $\varphi(e) \neq 0$

$$\rho^*(\varphi) = \frac{1}{\varphi(e)}\varphi \circ \rho$$

Mod Inner: $u =$ isometry $\sigma_t(u) = \lambda^{it}u$

Action: (on $\mathcal{E}_\infty$: warming up/cooling down)

$$W_\beta(\varphi)(a) = \frac{\mathrm{Tr}(\pi_\varphi(a)\, e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$$

The Bost–Connes system

J.B. Bost, A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (1995)

$r \in \mathbb{Q}/\mathbb{Z} \rightsquigarrow$ "phase operators" $e(r)$

Fock space $e(r)|n\rangle = \alpha(\zeta_r^n)|n\rangle$

$\zeta_{a/b} = \zeta_b^a$ abstract roots of 1 embedding $\alpha : \mathbb{Q}^{cycl} \hookrightarrow \mathbb{C}$

Optical phase: $|\theta_{m,N}\rangle = e\left(\frac{m}{N+1}\right) \cdot v_N$ superposition of occupation states $v_N = \frac{1}{(N+1)^{1/2}} \sum_{n=0}^{N} |n\rangle$

$n \in \mathbb{N}^\times = \mathbb{Z}_{>0} \rightsquigarrow$ changes of scale μ_n

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

1-dimensional $\mathbb{Q}$-lattices

$$(\Lambda, \phi) = (\lambda \mathbb{Z}, \lambda \rho) \quad \lambda > 0$$

$$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$$

Up to scaling λ: algebra $C(\hat{\mathbb{Z}}) \simeq C^*(\mathbb{Q}/\mathbb{Z})$

Commensurability Action of $\mathbb{N}^\times$

$$\alpha_n(f)(\rho) = f(n^{-1}\rho)$$

(up to scale)
1-dimensional $\mathbb{Q}$-lattices / Commensurability

$$\Rightarrow \text{ NC space } C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times$$

Bost – Connes

Hecke algebra $\mathcal{A} = \mathcal{A}_1$

$(\Gamma_0, \Gamma) = (P_{\mathbb{Z}}, P_{\mathbb{Q}})$ $ax + b$ group

$f : \Gamma_0 \backslash \Gamma / \Gamma_0 \to \mathbb{C}$ (or to $\mathbb{Q}$)

$$(f_1 * f_2)(\gamma) = \sum_{\Gamma_0 \backslash \Gamma} f_1(\gamma\gamma_1^{-1}) f_2(\gamma_1)$$

$f^*(\gamma) := \overline{f(\gamma^{-1})}$

Regular rep $\ell^2(\Gamma_0 \backslash \Gamma) \Rightarrow$ von Neumann alg
$\rightsquigarrow$ time evolution

$\gamma \in \Gamma/\Gamma_0$: $L(\gamma) = \#\Gamma_0\gamma$ and $R(\gamma) = L(\gamma^{-1})$

$$\sigma_t(f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)}\right)^{-it} f(\gamma)$$

$\sigma_t(\mu_n) = n^{it}\mu_n \qquad \sigma_t(e(r)) = e(r)$

σ_t st. state

$\varphi_1(f) = \langle \pi(f)\varepsilon_e, \varepsilon_e \rangle$ KMS$_1$

↑ cyclic sep vect. for reg. rep

Representations   Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^\times)$

$\alpha \in \hat{\mathbb{Z}}^* = \mathrm{GL}_1(\hat{\mathbb{Z}}) \Leftrightarrow$ embedding $\alpha : \mathbb{Q}^{cycl} \hookrightarrow \mathbb{C}$

$$\pi_\alpha(e(r))\, \epsilon_k = \alpha(\zeta_r^k)\, \epsilon_k$$

$$\pi_\alpha(\mu_n)\, \epsilon_k = \epsilon_{nk}$$

Hamiltonian $H\,\epsilon_k = \log k \;\; \epsilon_k$

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_{k=1}^{\infty} k^{-\beta} = \zeta(\beta)$$

Partition function = Riemann zeta function

Structure of KMS states (Bost–Connes)

- $\beta \leq 1 \Rightarrow$ unique KMS$_\beta$ state
- $\beta > 1 \Rightarrow \mathcal{E}_\beta \cong \hat{\mathbb{Z}}^*$ (free transitive action)

$$\varphi_{\beta,\alpha}(x) = \frac{1}{\zeta(\beta)} \operatorname{Tr}\left(\pi_\alpha(x)\, e^{-\beta H}\right)$$

- $\beta = \infty$ Galois action $\theta : \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \overset{\cong}{\to} \hat{\mathbb{Z}}^*$

$$\gamma\,\varphi(x) = \varphi(\theta(\gamma)\,x)$$

$\forall \varphi \in \mathcal{E}_\infty \quad \forall \gamma \in \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$
and $\forall x \in \mathcal{A}_\mathbb{Q}$ (arithmetic subalgebra)
$\varphi(\mathcal{A}_\mathbb{Q}) \subset \mathbb{Q}^{cycl}$

NCG and class field theory

$\mathbb{K}$ = number field $[\mathbb{K}:\mathbb{Q}] = n$, an algebraic closure $\bar{\mathbb{K}} \rightsquigarrow$ group of symmetries Gal$(\bar{\mathbb{K}}/\mathbb{K})$

Max abelian extension $\mathbb{K}^{ab}$:

$$\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}) = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})^{ab}$$

Kronecker–Weber: $\mathbb{K} = \mathbb{Q}$

$$\mathbb{Q}^{ab} = \mathbb{Q}^{cycl} \quad \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^*$$

Hilbert 12th problem (explicit class field theory)

Generators of $\mathbb{K}^{ab}$ + action of Gal$(\mathbb{K}^{ab}/\mathbb{K})$

Solved for: $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-d})$ (imaginary quadratic)

Question: Can NCG say something new?
(at least for real quadratic $\mathbb{Q}(\sqrt{d})$)

Bost–Connes system $\rightsquigarrow$ Kronecker–Weber

<u>adèlic description:</u> $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ and $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$

$$\mathcal{E}_\infty \simeq \mathrm{GL}_1(\mathbb{Q}) \backslash \mathrm{GL}_1(\mathbb{A}) / \mathbb{R}^*_+$$

Number field $\mathbb{K}$ with $[\mathbb{K} : \mathbb{Q}] = n$
$\mathbb{K}^* \hookrightarrow \mathrm{GL}_n(\mathbb{Q})$

$\Rightarrow$ Seek extension of Bost–Connes system to $\mathrm{GL}_n(\mathbb{A})$

GL_2 case $\rightsquigarrow$ Modular curves, modular forms

(A.Connes, M.M. *Quantum Statistical Mechanics of $\mathbb{Q}$-lattices*)

2-dimensional $\mathbb{Q}$-lattices

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}\tau), \lambda\rho)$$

$\lambda \in \mathbb{C}^*$, $\tau \in \mathbb{H}$, $\rho \in M_2(\hat{\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2)$

Up to scale $\lambda \in \mathbb{C}^*$ and isomorphism

$$M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \quad \text{mod } \Gamma = SL(2, \mathbb{Z})$$

Commensurability action of $GL_2^+(\mathbb{Q})$ (partially defined)

$\Rightarrow$ NC space

Functions on

$\mathcal{U} = \{(g, \rho, z) \in GL_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H} | g\rho \in M_2(\hat{\mathbb{Z}})\}$

invariant under $\Gamma \times \Gamma$ action $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 z)$

2-dim $\mathbb{Q}$-lattices

$\tilde{\mathcal{U}} = \{ (g, \rho, \alpha) \in GL_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R}) : g\rho \in M_2(\hat{\mathbb{Z}})$

$\tilde{\mathcal{U}}/\mathbb{C}^*$ up to scaling

R_2 : functions on $\tilde{\mathcal{U}}$ convolution prod groupoid, but $\mathbb{C}^*$ action has nontrivial stabilizers $R_2/\mathbb{C}^*$ for ($\phi=0$, 1 with nontrivial autom.)

Hecke algebra $\mathcal{A} = \mathcal{A}_2$

convolution product

$$(f_1 * f_2)(g, \rho, z) = \sum_{s \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q}) : s\rho \in M_2(\hat{\mathbb{Z}})} f_1(gs^{-1}, s\rho, s(z)) f_2(s, \rho, z)$$

$$f^*(g, \rho, z) = \overline{f(g^{-1}, g\rho, g(z))}$$

Time evolution:

$$\sigma_t(f)(g, \rho, z) = \det(g)^{it} f(g, \rho, z)$$

Hilbert spaces: for $\rho \in M_2(\hat{\mathbb{Z}})$

$$G_\rho = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) \, : \, g\rho \in M_2(\hat{\mathbb{Z}})\}$$

$$\mathcal{H}_\rho = \ell^2(\Gamma \backslash G_\rho)$$

Representations:

$L = (\Lambda, \phi) = (\rho, z)$ gives representation on $\mathcal{H}_\rho$:

$$(\pi_L(f)\xi)(g) = \sum_{s \in \Gamma \backslash G_\rho} f(gs^{-1}, s\rho, s(z))\, \xi(s)$$

$L = (\Lambda, \phi)$ invertible $\Rightarrow \mathcal{H}_\rho \cong \ell^2(\Gamma \backslash M_2^+(\mathbb{Z}))$

Hamiltonian

$$H\, \epsilon_m = \log \det(m)\ \epsilon_m$$

$\Rightarrow$ positive energy

Partition function $\quad \sigma(k) = \sum_{d|k} d$

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} \det(m)^{-\beta}$$

$$= \sum_{k=1}^{\infty} \sigma(k)\, k^{-\beta} = \zeta(\beta)\zeta(\beta - 1)$$

Structure of KMS states $\quad GL_2$-system

- $\beta \leq 1$ No KMS states

- $1 < \beta < 2$ Unique ?

- $\beta > 2 \;\Rightarrow\quad \mathcal{E}_\beta \cong GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/\mathbb{C}^*$

$$\varphi_{\beta,L}(f) = \frac{1}{Z(\beta)} \sum_{m \in \Gamma\backslash M_2^+(\mathbb{Z})} f(1, m\rho, m(z)) \det(m)^{-\beta}$$

$L = (\rho, z)$

(Hard to show: all $\varphi \in \mathcal{E}_\beta$ of this form)

Two phase transitions

Symmetries $\quad 2 < \beta \leq \infty$

$\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q})\mathrm{GL}_2(\hat{\mathbb{Z}})$ acts on $\mathcal{A}_2$

- $\mathrm{GL}_2(\hat{\mathbb{Z}})$ by automorphisms (deck transformations of coverings of modular curves)

$$\theta_\gamma(f)(g, \rho, z) = f(g, \rho\gamma, z)$$

- $\mathrm{GL}_2^+(\mathbb{Q})$ by endomorphisms

$$\theta_m(f)(g, \rho, z) = \begin{cases} f(g, \rho\tilde{m}^{-1}, z) & \rho \in m\, M_2(\hat{\mathbb{Z}}) \\ 0 & \text{otherwise} \end{cases}$$

$\tilde{m} = \det(m) m^{-1}$

$\mathbb{Q}^* \hookrightarrow \mathrm{GL}_2(\mathbb{A}_f)$ acts by inner

$\Rightarrow \quad S = \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$ symmetries on $\mathcal{E}_\beta$

Action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathcal{E}_\infty$: warming/cooling

Arithmetic subalgebra

Look back at $\mathcal{A}_{\mathbb{Q}} = \mathcal{A}_{1,\mathbb{Q}}$ (Bost–Connes case):
homogeneous functions of weight zero on 1-dim $\mathbb{Q}$-lattices

$$\epsilon_{k,a}(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-k}$$

normalized by covolume $e_{k,a} := c^k \, \epsilon_{k,a}$
($c(\Lambda) \sim |\Lambda|$ covolume with $(2\pi\sqrt{-1})c(\mathbb{Z}) = 1$)

GL_2 case $\mathcal{A}_{\mathbb{Q}} = \mathcal{A}_{2,\mathbb{Q}}$: Eisenstein series

$$E_{2k,a}(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-2k}$$

and

$$X_a(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-2} - \sum_{y \in \Lambda}{}' \, y^{-2}$$

normalized to weight zero

$\Rightarrow$ Modular functions

Arithmetic Subalgebra

of unbounded multipliers

$f \in$ Continuous functions on

$$Z = \{(g,\rho,z) \in GL_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H} : g\rho \in M_2(\hat{\mathbb{Z}})\}$$
mod $\Gamma \times \Gamma$ action

$$f(g,\rho,z) = f_{(g,\rho)}(z) \qquad f_{(g,\rho)} \in C(\mathbb{H})$$

- Support in $\Gamma \backslash GL_2^+(\mathbb{Q})$ finite
- $f_{(g,\rho)} = f_{(g,p_N(\rho))}$ $\quad p_N : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$
  finite level $\longrightarrow$ invariant under congr. $\Gamma(N) \cap g^{-1}\Gamma g$
- cyclotomic condition $\qquad f_{(g,\rho)} \in F$

$$f_{(g,\alpha(u)m)} = \mathrm{cycl}(u)\, f_{(g,m)}$$

$\forall g \in GL_2^+(\mathbb{Q})$ diagonal and $\forall u \in \hat{\mathbb{Z}}^* \simeq \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$

w/ $\quad \alpha(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathrm{cycl}(u)$ = action on coeff. of q-expansion

Want all Galois conjugates to appear among coefficients (rule out scalars in $\mathbb{Q}^{ab}$ else no fabulous states: $\mathbb{C}$-linearity of states)

The modular field

Weierstrass $\wp$-function:

parameterization $w \mapsto (1, \wp(w;\tau,1), \frac{d}{dw}\wp(w;\tau,1))$
of elliptic curve $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ by $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$

Fricke functions: (homog weight zero) $v \in \mathbb{Q}^2/\mathbb{Z}^2$

$$f_v(z) = -2^7 3^5 \frac{g_2(z)g_3(z)}{\Delta(z)} \wp(\lambda_z(v); z, 1)$$

$\Delta(z) = g_2^3 - 27g_3^2$ discriminant, $\lambda_z(v) := v_1 z + v_2$

Generators of the modular field F

(Shimura) $\quad \mathrm{Aut}(F) \cong \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_f)$

$\tau \in \mathbb{H}$ generic $\Rightarrow$ evaluation $f \mapsto f(\tau)$
$\rightsquigarrow$ embedding $F \hookrightarrow \mathbb{C}$ image F_τ

$$\theta_\tau : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_f)$$

CM case (Connes – M – Ramachandran)

Intermediate case between BC & GL_2

1-dimensional $\mathbb{K}$-lattices

$\mathbb{K} = \mathbb{Q}(\tau)$ imaginary quadratic field ($\tau \in \mathbb{H}$ a generator)

$\Lambda \subset \mathbb{K}$
(free $\mathbb{Z}$-submodule $rk = [\mathbb{K} : \mathbb{Q}] = 2$)

$\phi : \mathbb{K}/\mathcal{O} \longrightarrow \mathbb{Q}\Lambda/\Lambda$

$\Rightarrow$ also 2-dim $\mathbb{Q}$-lattices

$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$ commensurable if underlying 2-dim $\mathbb{Q}$-lattices commensurable and identif. $\mathbb{Q}\Lambda_1 \cong \mathbb{Q}\Lambda_2$ is multipl. by some $x \in \mathbb{K}^*$.

Has properties in common w/ BC system & with specialization of GL_2-system at (non-generic) CM pts.

As for CM

- unique KMS between $0 < \beta \leq 1$ (?)
- free/trans. action of $GL_1(\mathbb{A}_{\mathbb{K}})/\mathbb{K}^*$ on KMS_β $\beta > 1$
- $Z_{\mathbb{K}}(\beta) = \sum_{x \in \mathcal{O}^\times/\mathcal{O}^*} n(x)^{-\beta}$

$n : \mathcal{O}^\times \to \mathbb{N}^\times$ norm

As for GL_2

- Symmetries autos + endos
  $1 \to GL_1(\hat{\mathcal{O}})/\mathcal{O}^* \longrightarrow GL_1(\mathbb{A}_{\mathbb{K},f})/\mathbb{K}^* \to Cl(\mathcal{O}) \to 1$
  autom — endom
- $\mathbb{K}^*$ acts by inner
- Rational subalg from modular functions
- Class field theory of $\mathbb{K} = \mathbb{Q}(\tau)$ im. from $Aut(F)$ (Shimura)

Galois action on $\mathcal{E}_\infty$

State $\varphi = \varphi_{\infty,L} \in \mathcal{E}_\infty$ $(L = (\rho, \tau)$ generic) $\Rightarrow$

$$\theta_\varphi : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_f)$$

$$\theta_\varphi(\gamma) = \rho^{-1}\,\theta_\tau(\gamma)\,\rho$$

$\varphi = \varphi_{\infty,L}$ with $L = (\rho, \tau)$ invertible $\mathbb{Q}$-lattice
generic

$\forall f \in \mathcal{A}_{2,\mathbb{Q}}$ and $\forall \gamma \in \mathrm{Gal}(F_\tau/\mathbb{Q})$

$$\gamma\,\varphi(f) = \varphi(\theta_\varphi(\gamma) f)$$

$\varphi(\mathcal{A}_{2,\mathbb{Q}}) \subset F_\tau$

Shimura varieties: $Sh(G,X) = G(\mathbb{Q})\backslash G(\mathbb{A}_f) \times X$

$$Sh(\mathrm{GL}_2, \mathbb{H}^\pm) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm$$

$$= \mathrm{GL}_2^+(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H} = \mathrm{GL}_2^+(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*$$

Components: $\pi_0(Sh(\mathrm{GL}_2, \mathbb{H}^\pm)) = Sh(\mathrm{GL}_1, \{\pm 1\})$

$Sh(\mathrm{GL}_1, \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A}_f) \times \{\pm 1\} = \mathbb{Q}_+^* \backslash \mathbb{A}_f^*$

Commutative pts of NC space $\mathrm{GL}_1(\mathbb{Q})\backslash \mathbb{A}_f$ Bost–Connes

$$\mathcal{A} \simeq C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^* \quad \text{Morita equiv}$$

Modular curves: $Sh(\mathrm{GL}_2, \mathbb{H}^\pm) =$ adèlic version of modular tower

$\varprojlim \Gamma\backslash\mathbb{H}$ congruence subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{Z}) \rightsquigarrow$
congr subgr in $\mathrm{SL}(2,\mathbb{Q}) \rightsquigarrow$ components $Sh(\mathrm{GL}_2, \mathbb{H}^\pm)$

NC Shimura varieties

Bost–Connes system:

$$Sh^{(nc)}(\{\pm 1\}, \mathrm{GL}_1) := \mathrm{GL}_1(\mathbb{Q})\backslash(\mathbb{A}_f \times \{\pm 1\})$$

$$= \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}^{\cdot}/\mathbb{R}^*_+$$

$\mathbb{A}^{\cdot} := \mathbb{A}_f \times \mathbb{R}^*$

Compactification:

$$\overline{Sh^{(nc)}}(\{\pm 1\}, \mathrm{GL}_1) = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}^*_+$$

Dual space (crossed product by time evolution σ_t)

$$\mathcal{L} = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A} \to \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}^*_+$$

$\mathbb{R}^*_+$-bundle

Spectral realization of zeros of ζ (Connes)

1-dim $\mathbb{Q}$-lattices (not up to scaling)

mod commensurability

GL$_2$-system:

$$Sh^{(nc)}(\mathbb{H}^{\pm}, \mathrm{GL}_2) := \mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f) \times \mathbb{H}^{\pm})$$

Compactification:

$$\overline{Sh^{(nc)}}(\mathbb{H}^{\pm}, \mathrm{GL}_2) := \mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbb{C}))$$

$$= \mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A})/\mathbb{C}^*$$

$\mathbb{P}^1(\mathbb{C}) = \mathbb{H}^{\pm} \cup \mathbb{P}^1(\mathbb{R})$

$\Rightarrow$ Adding NC boundary of modular curves

Dual system: $\mathbb{C}^*$-bundle $\mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A})$
Modular forms (instead of modular functions)
$\Rightarrow$ Modular Hecke algebra (Connes–Moscovici)

Compatibility:

$$\det \times \mathrm{sign} : Sh(\mathbb{H}^{\pm}, \mathrm{GL}_2) \to Sh(\{\pm 1\}, \mathrm{GL}_1)$$

passing to connected components π_0

Class field theory $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$

$\tau \in \mathbb{H}$ CM point $\mathbb{K} = \mathbb{Q}(\tau)$

Evaluation $F \to F_\tau \subset \mathbb{C}$ not embedding

$$F_\tau \simeq \mathbb{K}^{ab} \quad \{f(\tau),\, f \in F\} \text{ generators}$$

Galois action

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{K}^* & \longrightarrow & \mathrm{GL}_1(\mathbb{A}_{\mathbb{K},f}) & \xrightarrow{\simeq} & \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}) & \longrightarrow & 1 \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & \mathbb{Q}^* & \longrightarrow & \mathrm{GL}_2(\mathbb{A}_f) & \xrightarrow{\simeq} & \mathrm{Aut}(F) & \longrightarrow & 1. \end{array}$$

$\mathbb{A}_{\mathbb{K},f} = \mathbb{A}_f \otimes \mathbb{K}$

$\Rightarrow$ Specialization of GL_2-system at CM points
$\rightsquigarrow$ CFT for $\mathbb{K} = \mathbb{Q}(\tau)$

What about $\mathbb{Q}(\sqrt{d})$??

NC boundary of mod curves in $\overline{Sh^{(nc)}}(\mathbb{H}^\pm, \mathrm{GL}_2)$

(Manin's real multiplication program)