ORBIT EQUIVALENCE OF MINIMAL ACTIONS ON THE CANTOR SET

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ONGOING PROJECT WITH CHRISTIAN SKAU.

LONG TERM GOAL: CLASSIFY UP TO O.E MINIMAL FREE ACTIONS OF $\mathbb{Z}^n$ ON THE CANTOR SET X.

REM: MEASURABLE CASE: DYE, KRIEGER
ORNSTEIN - WEISS
CONNES-FELDMAN-WEISS.

BOREL CASE: WEISS
DOUGHERTY-JACKSON-KECHRIS

FIRST CASE: MINIMAL ACTIONS OF $\mathbb{Z}$ KNOWN. [GPS]

GOAL: ANY MINIMAL $\mathbb{Z}^2$-ACTION O.E $\mathbb{Z}$-ACTION.

EXP: (S^1, R_α) IRRATIONAL ROTATION

DISCONNECT S^1 ALONG THE α-ORBIT OF 1 PT.

GET A CANTOR SET X

(X, R_α) C.M.S. DENJOY SYST.

- α, β RATION - INDEP.

DISCONNECT S^1 ALONG THE β-ORBIT OF EACH ELT OF THE α-ORBIT OF A PT.

GET CANTOR SET X

R_α, R_β GENERATE MIN. $\mathbb{Z}^2$-ACTION. φ

DOUBLE DENJOY SYST.

EQUIV. REL. INDUCED BY φ:

$$x \mathcal{R}_\varphi y \iff \exists\, n, m \in \mathbb{Z} \text{ st } y = R_\alpha^n R_\beta^m x$$

G COUNTABLE GRP ACTING FREELY ON X.

$$\mathcal{R}_G = \{ (x, gx) ; g \in G, x \in X \}$$

TOPOLOGIZE $\mathcal{R}_G$ VIA BIJECTION $X \times G \to \mathcal{R}_G$

$$(x, g) \mapsto (x, gx)$$

THEN $\mathcal{R}_G$ HAUSDORFF
LOC. COMPACT GROUPOID
σ-COMPACT

MOREOVER $r : \mathcal{R}_G \to X$ RANGE

$s : \mathcal{R}_G \to X$ SOURCE

LOCAL HOMEO.

$\mathcal{R}_G$ <u>ÉTALE EQUIV. REL</u>.

<u>PROP</u>: $(\mathcal{R}, \tau)$ ÉTALE EQ. REL ON X CANTOR

THEN $\mathcal{R} = \mathcal{R}_G$ FOR SOME COUNTABLE GRP OF HOMEO.

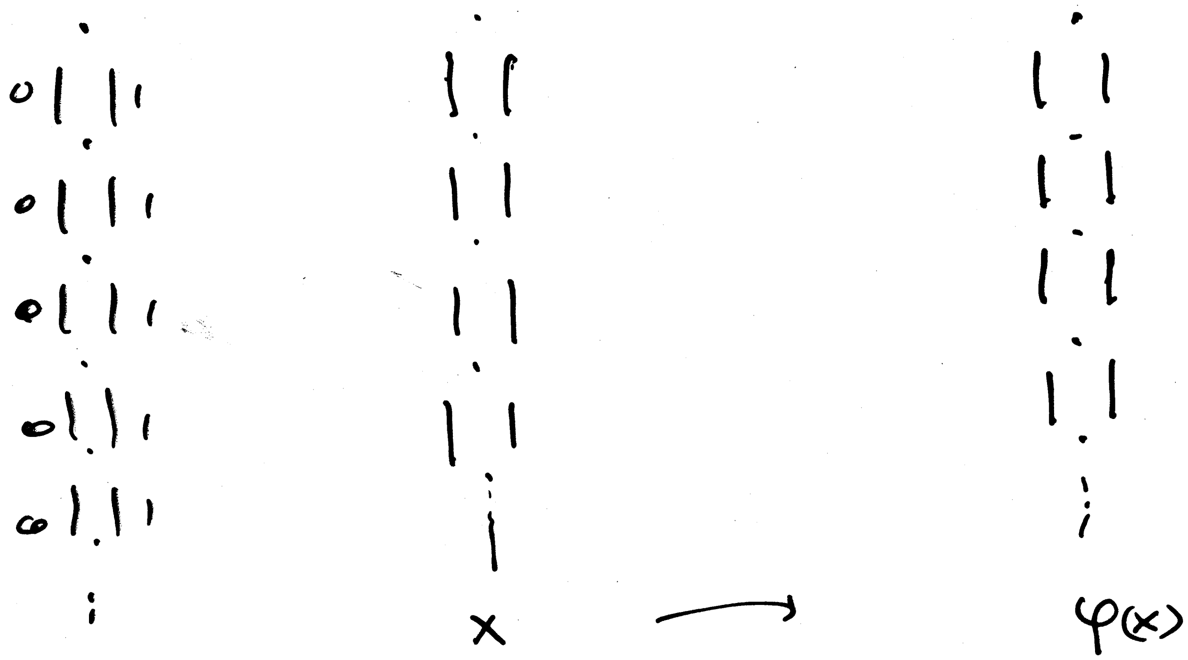
Exp. $X = \prod \{0,1\}$ $\quad \varphi = +(1,0,0,\dots)$ WITH CARRYING OVER

(X, φ) CMS. ODOMETER.

FOR EXAMPLE, IF $x = (1,1,1,0,\dots)$, THEN

$\varphi(x) = (0,0,0,1,\dots)$

BRATTELI- VERSHIK MODEL.



X_B PATH SPACE

- $\mathcal{R}_\varphi \subset X \times X$. EQ. REL. INDUCED BY THE ACTION OF φ.

$$x \,\mathcal{R}_\varphi\, y \iff \exists\, n \in \mathbb{Z};\ y = \varphi^n(x).$$

- $\mathcal{R} \subset X \times X$ TAIL EQUIVALENCE.

$$x \,\mathcal{R}\, y \iff \exists\, n \text{ st } x_k = y_k,\ \forall k \geq n.$$

X AND Y COFINAL.

REM: ① $\mathcal{R} \subset \mathcal{R}_\varphi$.

x not cofinal to $(0,\dots,0,\dots)$; $(1,1,\dots,1,\dots)$, THEN $[x]_{\mathcal{R}} = [x]_{\mathcal{R}_\varphi}$

$$x = (0,\dots,0,\dots),\quad [x]_{\mathcal{R}} = \{\varphi^n(x),\ n \geq 0\}$$

IF $y = (1,\dots,1,\dots)$, $[y]_{\mathcal{R}} = \{\varphi^{-n}(y);\ n \geq 0\}$

$$[(0,\dots,0,\dots)]_{\mathcal{R}_\varphi} = [(1,1,\dots,1,\dots)]_{\mathcal{R}} \cup [\varphi(1,\dots,1,\dots)]_{\mathcal{R}}$$

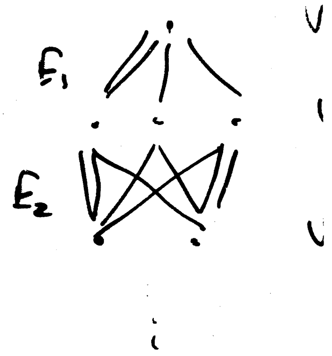
IN FACT, $\mathcal{R}_\varphi = \mathcal{R} \vee \{((1,\dots,1,\dots),\varphi(1,\dots,1,\dots))\}$.

② FOR $n \geq 1$, $\mathcal{R}_n$ EQU. REL DEF. BY:

$$x \,\mathcal{R}_n\, y \iff x_k = y_k,\ \forall k \geq n.$$

- $\mathcal{R}_n$ FINITE EQ. REL.
- $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{R}_3 \subset \cdots$ AND $\mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n$.

IN GENERAL: (X, φ) C.M.S.



[HPS]. $\exists$ SIMPLE BRATTELI DIAG. $B = (V, E)$

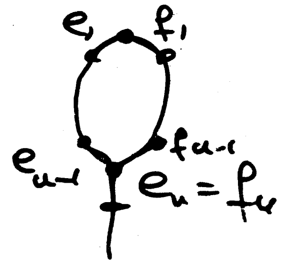
PARTIAL ORDER ON E

WITH UNIQUE MIN AND MAX PATH ST.

X_B PATH SPACE CANTOR SET

φ_V VERSHIK MAP. MINIMAL HOMEO.

(X_B, φ_V) CONJ. TO (X, φ).



REM: ① $\forall n \geq 1$. $e, f \in X_B$ $e \mathcal{R}_n f$ iff.

$\mathcal{R}_n$ FINITE EQU. REL.

- $\mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n$ TAIL EQUIV.
- $\mathcal{R} \subset \mathcal{R}_{\varphi_B}$
- IF x NOT COFINAL TO MAX, MIN, $[x]_{\mathcal{R}} = [x]_{\mathcal{R}_{\varphi_B}}$

REM: $(\mathcal{R}_n, \tau_n)$ COMPACT, WHERE τ_n RELATIVE TOPOL. $\mathcal{R}_n \subset X \times X$.

$$(\mathcal{R}, \tau) = \varinjlim (\mathcal{R}_n, \tau_n)$$

INDUCTIVE LIMIT TOPOL.

ÉTALE EQUIV. REL. AF-RELATION

CONVERSELY, ANY AF-RELATION IS GIVEN BY A BRATTELI DIAGRAM, $AF(V, E)$.

FACTS: ① [HPS]. $AF(V, E)$ MINIMAL $\iff (V, E)$ SIMPLE

② [E-K] $AF(V, E) \cong AF(V', E')$

(i.e. $\exists$ $f: X_B \to X_{B'}$ HOMEO ORBIT MAP.

$f \times f : AF(V, E)) \longrightarrow AF(V', E')$ HOMEO.

iff

$D(V, E) \cong D(V', E')$
ORDER ISOM.

COME BACK TO (X_B, φ_B).

$$\mathcal{R}_{\varphi_B} = \mathcal{R} \cup \{ (e_{MAX}, \underset{\varphi_B(e_{MAX})}{\overset{\shortparallel}{e_{MIN}}}) \}$$

THM [GPS]. $\mathcal{R}_{\varphi_B}$ OE $\mathcal{R}$

HENCE $\mathcal{R}_{\varphi_B}$ OE AF-RELATION.

STRATEGY FOR PROVING OE.

STEP 1: CLASSIFY UP TO O.E. AF-RELATIONS

STEP 2: (X, φ) MIN. FREE G-ACTION ON X, SHOW $\mathcal{R}_\varphi$ O.E. AF-RELATION.

STEP 1 IS KNOWN.:

INVARIANT: $(X, \mathcal{R})$ ÉTALE EQ. REL ON X CANTOR

$$D_m(X,\mathcal{R}) = C(X,\mathbb{Z}) \Big/ \{ f \,;\, \int f \, d\mu = 0, \ \forall \mu \in M_1(X,\mathcal{R}) \}$$

WHERE $M_1(X,\mathcal{R})$ $\mathcal{R}$-INV. PROB. MEAS. ON X.

POSITIVE CONE $D_m^+ = \{ [f] \,;\, f \geq 0 \}$.

THM [GPS] FOR MINIMAL AF-EQUIV REL.

$(D_m(X,\mathcal{R}), D_m^+, [1])$ COMPLETE INV FOR O.E.

THE $\mathbb{Z}$-CASE:

THM [GPS] (X, φ) CMS.

THEN $\mathcal{R}_\varphi$ O.E. AF-REL.

COR: $(D_m(X,\mathcal{R}_\varphi), D_m^+, [1])$ COMPLETE INV FOR O.E.

REM:

① [PUTNAM] $C^*(X, \varphi)$ SIMPLE UNITAL, RR0 AT-ALG.

② [GPS]. (X, φ), (X, ψ) CMS.

$C^*(X, \varphi) \cong C^*(X, \psi)$ IFF (X, φ) SOE (X, ψ)

③ [H. LIN]. (X, φ) O.E (X, ψ)

IFF

$C^*(X, \varphi)$ TRACIALLY EQUIV TO $C^*(X, \psi)$.

④ (X_i, φ_i) φ_i MIN HOMEO; X_i COMPACT CONNECT

φ_1 NOT FLIP CONJ. φ_2, BUT $C^*(X_1, \varphi_1) \cong C^*(X_2, \varphi$

⑤ [H. LIN - C. PHILLIPS]

STUDY OF $C^*(X, \varphi)$ φ MIN HOMEO

X COMPACT METRIC

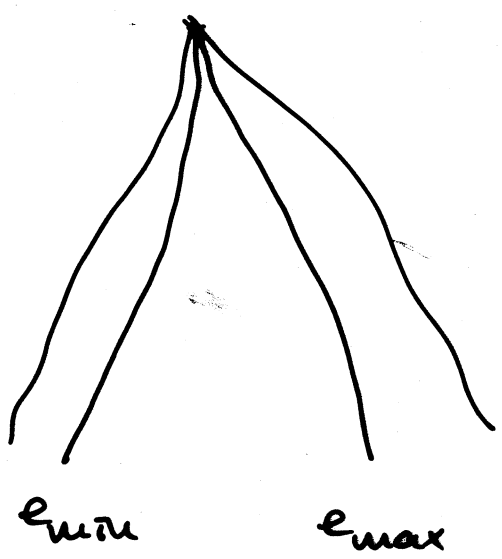
FINITE COVER. DIM.

STEP 2 FOR MIN. FREE ACTIONS φ OF $\mathbb{Z}^2$.

STEP 2.1

SHOW THAT "SMALL" EXTENSIONS OF A MIN. AF-REL. ARE O.E TO THE ORIGINAL ACTION.

RECALL: (X_B, φ_B) B-V. MODEL.



$\varphi_B(e_{max}) = e_{min}$.

$\mathcal{R}_{\varphi_B} = \mathcal{R} \vee \{(e_{min}, e_{max})\}$.

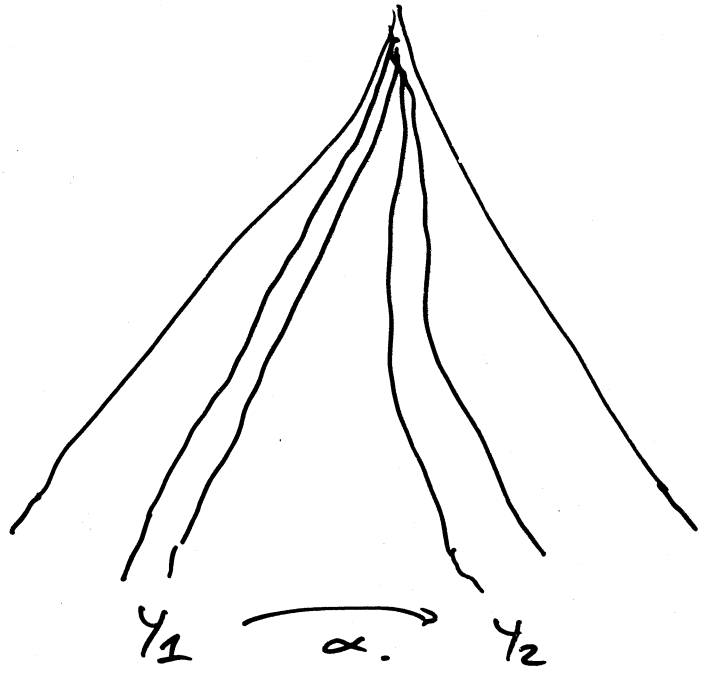
OE. $\mathcal{R}$.

THM: THE ABSORPTION THM.

$(X, \mathcal{R})$ MINIMAL AF-REL.

Y_1, Y_2 CLOSED

$\alpha : Y_1 \to Y_2$ HOMEO. ST.



1) $\mathcal{R} \cap (Y_1 \times Y_2) = \emptyset$

2) Y_i THIN.

$(\mu(Y_i) = 0 \ \ \forall \mu \in M(X, \mathcal{R}))$.

3) $\mathcal{R} \cap (Y_i \times Y_i)$ ETALE REL. ON Y_i

4) $y_1 \mathcal{R} y_1'$ IFF $\alpha(y_1) \mathcal{R} \alpha(y_1')$.

THEN $\mathcal{R} \vee \text{GRAPH}(\alpha)$ OE. $\mathcal{R}$.

(IN PARTICULAR $\mathcal{R} \vee \text{GRAPH}(\alpha)$ IS AF).

STEP. 2.2

(X,φ) FREE MIN. ACTION OF $\mathbb{Z}^2$ ON X.

CONSTRUCT A SEQUENCE $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots$ OF COMPACT OPEN SUBEQUIV. OF $\mathcal{R}_\varphi$ ST.

① $\mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n$ MINIMAL AF

② Y_1, Y_2 CLOSED, $\alpha : Y_1 \to Y_2$ HOMEO AS IN THE ABSORPTION THM.

THEN $\mathcal{R}_\varphi = \mathcal{R} \vee \mathrm{GRAPH}(\alpha)$.

REM: - A. FORREST LARGE AF SUBREL OF $\mathcal{R}_\varphi$

- IMPLICIT CONSTR.

BELLISARD - BENEDETTI - GAMBAUDO

BENEDETTI - GAMBAUDO.

COCYCLE AND POSITIVE COCYCLES.

(X, φ) FREE ACTION OF $\mathbb{Z}^2$ ON X.

- $\theta : X \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}$. (ONE-) COCYCLE.

  θ CONT. AND $\theta(x, m+n) = \theta(x, m) + \theta(\varphi^m(x), n)$

REM: $\theta : \mathcal{R}_\varphi \to \mathbb{Z}$ HOMOM.

$\theta((x,y)(y,z)) = \theta(x,z) = \theta(x,y) + \theta(y,z)$.

- A COCYCLE θ IS A COBOUNDARY IF

  $\theta(x, n) = h(x) - h(\varphi^{-n}(x)) \quad h \in C(X, \mathbb{Z})$.

- $H^1(X, \varphi) \; (= H^1(\mathcal{R}_\varphi)) = Z^1(X, \varphi) / B^1(X, \varphi)$.

REM: $\theta \in Z^1(X, \varphi)$.

$\ker \theta = \{ (x,y) \in \mathcal{R}_\varphi ; \; \theta(x,y) = 0 \} \subset \mathcal{R}_\varphi$.

Exp 1: $\varphi : X \circlearrowleft$ HOMEO. ; $R = R_\varphi$

$\theta \in Z^1(X,\varphi)$; $\theta : X \times \mathbb{Z} \to \mathbb{Z}$.

SET $f(x) = \theta(x,1)$. $\quad \theta(x,2) = \theta(x,1) + \theta(\varphi(x),1)$
$= f(x) + f(\varphi(x))$.

SO. $\theta \longleftrightarrow f$.

$\theta \in B^1(X,\varphi)$, $\theta(x,1) = f(x) = h(x) - h(\varphi^{-1}(x))$

HENCE,
$$H^1(X,\varphi) \cong C(X,\mathbb{Z}) / \{ f\circ\varphi^{-1} - f \} = K^0(X,\varphi).$$

Exp. 2: φ $\mathbb{Z}^2$- ACTION. ; $R = R_\varphi$.

NOTATION: $\alpha \in$ HOMEO (X). $\partial_\alpha : C(X,\mathbb{Z}) \circlearrowleft$

$\partial_\alpha f = f - f \circ \alpha^{-1}$

$\theta \in Z^1(X,\varphi)$. ; a, b GENERATORS OF $\mathbb{Z}^2$

$\theta^a(x) = \theta(x,a)$, $\theta^b(x) = \theta(x,b)$

THEN $\partial_b(\theta^a) = \partial_a(\theta^b)$

CONVERSELY: $f, g \in C(X,\mathbb{Z})$ ST. $\partial_b f = \partial_a g$

THEN THERE IS A UNIQUE COCYCLE θ ST.

$\theta^a = f$ AND $\theta^b = g$

SPECIAL CASE:

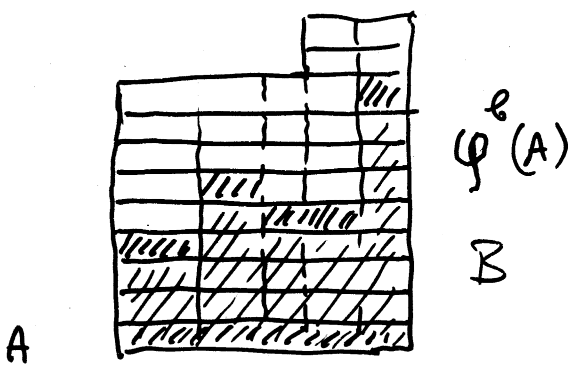
A, B CLOPEN WITH

1) $A \cap \varphi^a(B) = \emptyset$

2) $A \cup \varphi^a(B) = B \cup \varphi^b(A)$

THEN (χ_A, χ_B) COCYCLE.

EXP:



TOWER FOR φ^a

POSITIVE COCYCLES

EXP: (X, φ) CMS. $\varphi : X \circlearrowleft$

$\theta \in Z^1(X, \varphi) \longleftrightarrow f(x) = \theta(x, 1)$.

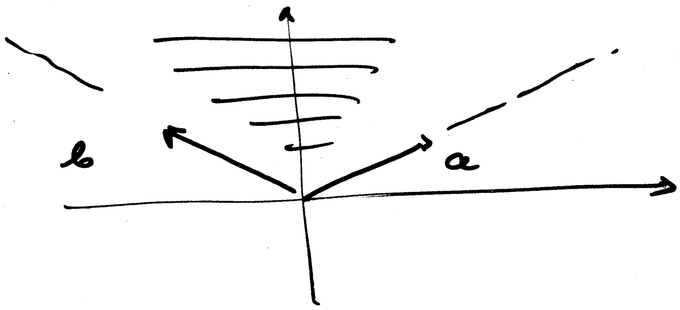
THEN: $f \geq 0$ $\qquad \theta|_{X \times \mathbb{N}} \geq 0$

$f \neq 0$ $\qquad \theta|_{X \times \mathbb{N}}$ PROPER MAP.

- φ FREE $\mathbb{Z}^2$- ACTION.

IF a, b GEN. OF $\mathbb{Z}^2$, SET

$C(a,b) =$



$C(a,b) = \{ ka + lb \; ; \; k, l \geq 0 \}$.

DEF: $\theta \in Z^1(X, \varphi)$.

θ POSITIVE WRT $C(a,b)$ IF $\theta|_{X \times C(a,b)} \geq 0$.

θ STRICTLY POSITIVE WRT $C(a,b)$ IF

$\theta : X \times C(a,b) \longrightarrow \{0, 1, 2, ..\}$ PROPER.

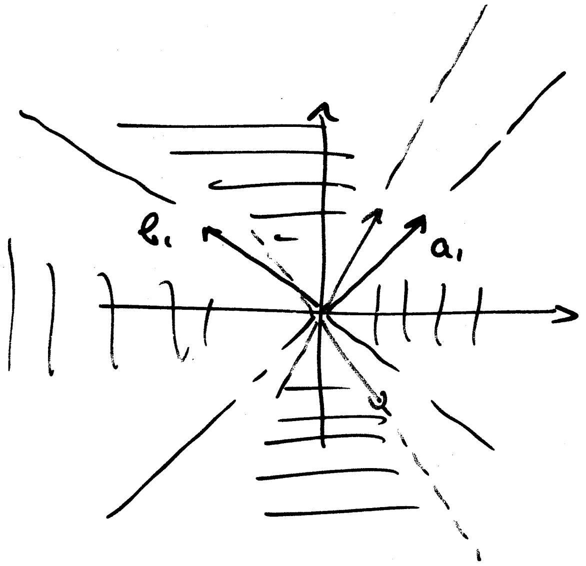
THM: $C_1, --, C_k$ CONES IN $\mathbb{Z}^2$ ST.

$$\bigcup_{i=1}^{k} C_i \cup (-C_i) = \mathbb{Z}^2$$

θ_i STRICT. ≥ 0 WRT. C_i

THEN $\bigcap_{i=1}^{k} \ker \theta_i$ COMPACT ETALE EQ. REL.

REM:



$\equiv$ $R^1 \cup (R^1)^{-1}$

$|||$ $R^2 \cup (R^2)^{-1}$

DEF: φ FREE MIN ACTION OF $\mathbb{Z}^2$

φ HAS <u>ARBIT. SMALL STRICTLY ≥ 0 COCYCLES</u>.

IF FOR EVERY PAIR OF GEN. $a, b \in \mathbb{Z}^2$

FOR $n \geq 1$,

$\exists$ $\theta \in Z^1(R_\varphi)$ 1) STRICTLY ≥ 0 WRT $C(a,b)$

2) $\theta^{n(a+b)} \leq 1$.

THM: DENJOY (α, β) HAVE <u>ASSP</u>. COCYCLE

TWO ODOMETERS.

PRODUCT OF 2 C.M.S

REM: ① $\exists\ \varphi$ $\mathbb{Z}^2$ - ACTION OF X

MINIMAL, BUT NOT FREE

ST. $H^1(X, \varphi) = \mathbb{Z}^2$

② IN THE ODOMETER CASE,

$$0 \to \mathbb{Z} \to H^1(X, \varphi) \to \mathbb{Z}[\tfrac{1}{2}] \to 0$$

③ IN THE DENJOY EXAMPLE

$$H^1(X, \varphi) \cong \mathbb{Z}^3$$

MORE GEN. FOR EVERY $\mathbb{Z}^2$-ACTION COMING FROM THE CUT AND PROJ. METH.

$$H^1(X, \varphi) \cong \mathbb{Z}^k \quad , k \geq 3.$$

<u>THM</u> [GPS]. (X, φ) FREE MIN. $\mathbb{Z}^2$-ACTION ON CANTOR SET X

SUPPOSE FOR EVERY $a, b \in \mathbb{Z}^2$ SYST OF GEN

THERE EXIST ASSP COCYCLES ON $C(a,b)$

THEN $\mathcal{R}_\varphi$ OE. AF-EQUIV. REL.

<u>COR</u>: $(X_1, \mathcal{R}_1)$, $(X_2, \mathcal{R}_2)$ MIN. ACTIONS ON CANTOR

ST. $\mathcal{R}_i$ EITHER AF, $\mathcal{R}_{\mathbb{Z}}$, $\mathcal{R}_{\mathbb{Z}^2}$ WITH ASSP COCY.

THEN. $\mathcal{R}_1$ OE $\mathcal{R}_2$.

IFF

$$\left(D_m(X_1, \mathcal{R}_1), D_m^+, [1]\right) \cong \left(D_m(X_2, \mathcal{R}_2), D_m^+, [1]\right)$$

ORDER ISOM.

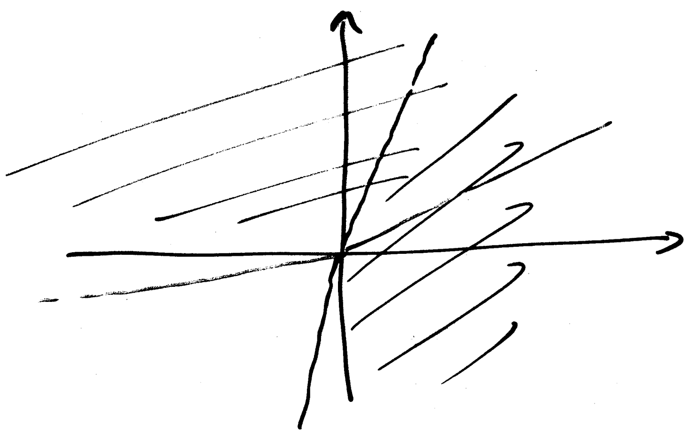
REMARKS ON THE PF:

① CONSTRUCTION OF $\mathcal{R}_1$.

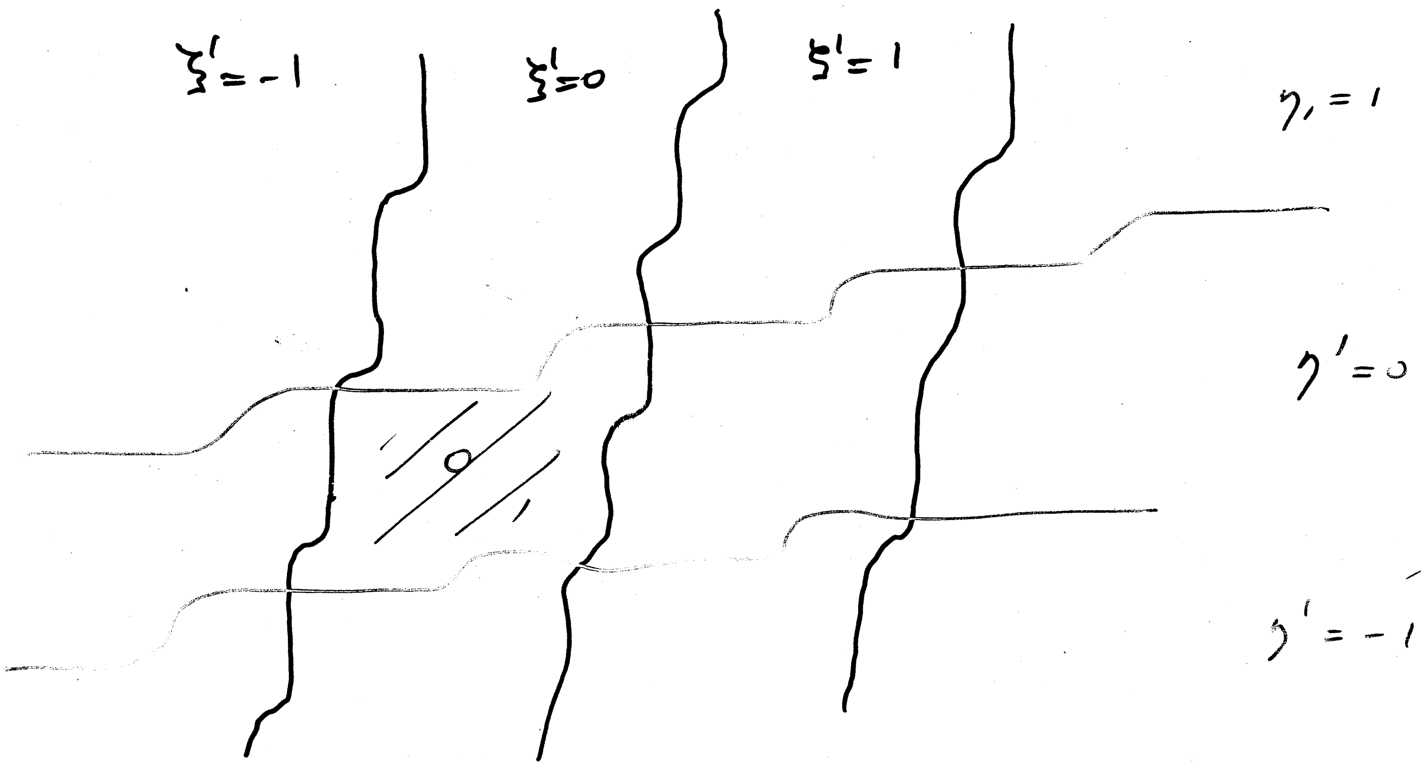
FIND PAIR OF SURJECTIVE SP. COCYCLES. ξ', η' ST.

ξ' SP.

η' SP



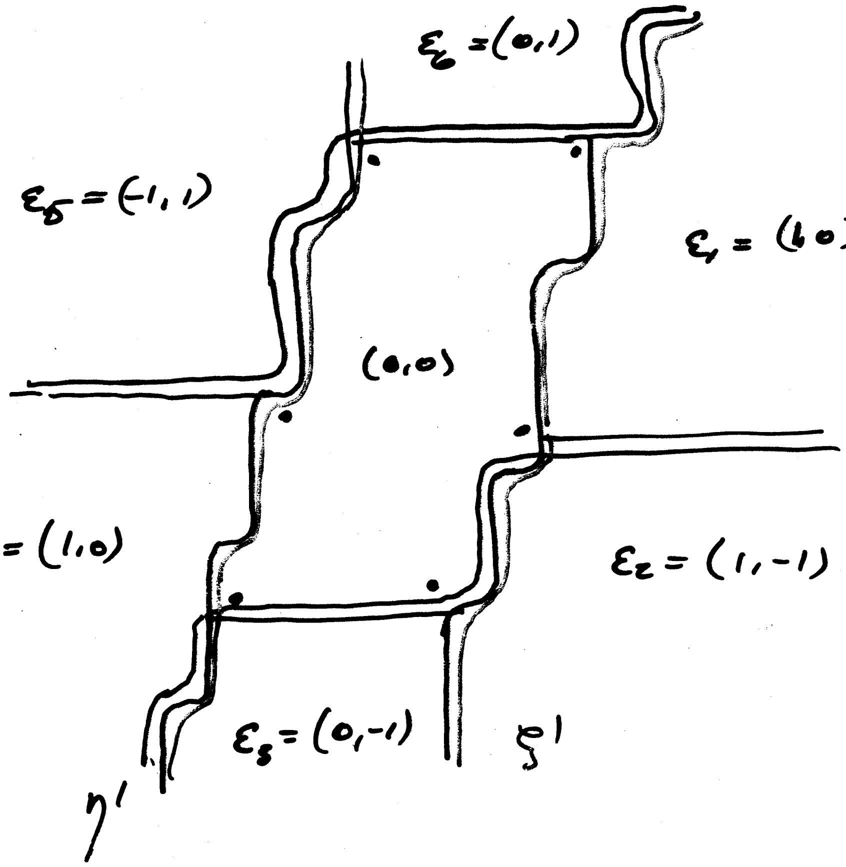
$\mathbb{Z}^2$-ORBIT OF $x \in X$.



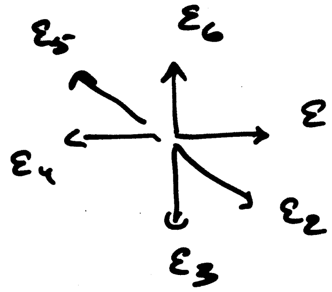
FACT: $C_{\xi_1} \cup (-C_{\xi_1}) \cup (C_{\eta_1}) \cup (-C_{\eta'}) = \mathbb{Z}^2$

$\Rightarrow$ Ker $\xi_1 \times \eta'$ COMPACT OPEN SUBEQU.

CAN ASSUME (AFTER SMALL MODIF BY COROUND.)



$= \xi' x \eta'$



SET $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, $E^* = \{\varepsilon_4, \varepsilon_5, \varepsilon_6\}$

<u>THEN</u> $x \in X$, $\xi' x \eta' (x, E) \subset E \cup \{(0,0)\}$
$\qquad\qquad\qquad\qquad\quad E^* \qquad E^*$

$E'(x) = \xi' x \eta' (x, E \cup E^*) - \{(0,0)\}$

<u>THEN</u> $\# E'(x) \leq 2$.

IF $\# E'(x) = 2$, THEN $E'(x) = \{\varepsilon_i, \varepsilon_{i-1}\}$ AND

$\xi' x \eta' (x, \varepsilon_i) = \varepsilon_i$, $\xi' x \eta' (x, \varepsilon_{i-1}) = \varepsilon_{i-1}$

THEY ARE THE "<u>CORNERS</u>"

IF $\# E'(x) = 1$, THEN $\xi' x \eta' (x, \varepsilon_i) = \varepsilon_i$

<u>BOUNDARIES</u>

② CAN CONSTRUCT SEQ OF SSP COCYCLES.

ξ^n, η^n ST. $\ker \xi^n \times \eta^n \supset \ker \xi^{n-1} \times \eta^{n-1}$

$$\underset{\mathcal{R}_n}{\parallel} \qquad \underset{\mathcal{R}_{n-1}}{\parallel}$$

$\mathcal{R} = \cup \mathcal{R}_n$ AF_REL., MIN.. $\mathcal{R} \subset \mathcal{R}_\varphi$

IF $B_i = \{ x \in X ; \ \xi^1 \times \eta^1 (x, \varepsilon_i) = \varepsilon_i \ ; \ \# E_n(x) = 1 \ \forall$

THEN $B = \bigcup_{i=1}^{3} B_i$, $B^* = \bigcup_{i=4}^{6} B_i$ BOUNDARIES

- B, B^* THIN, CLOSED
- $\varphi^{\varepsilon_i}(B_i) = B_{i+3}$ HENCE $\beta : B \to B^*$
- $\mathcal{R}_\varphi = \mathcal{R} \vee$ Graph β.

BUT $\mathcal{R}|_{B \times B}$ AND $\mathcal{R}|_{B^* \times B^*}$ NOT ÉTALE

③ DEFINE $\underset{\text{AF MIN}}{\mathcal{R}' \subset \mathcal{R}}$ ST $\mathcal{R}'|_{B \times B}$ ÉTALE

BY COLOURING.

THEN. $\mathcal{R}' \vee$ Graph $\beta = \tilde{\mathcal{R}}$ OE $\mathcal{R}'$

④ IF $C_i = \{ x \in X ; \ E^n(x) = \{ \varepsilon_i, \varepsilon_{i-1} \} \ \forall n \geq 1 \}$,

$\tilde{\mathcal{R}}|_{C_i \times C_i}$ = EQUALITY , HENCE ÉTALE

THEN $\mathcal{R}_\varphi$ FINITE EXT. OF $\widetilde{\mathcal{R}}$

(VIA $\gamma_3 : C_1 \to C_3$ $\gamma_3 = \varphi^{2_6}$, ...)

HENCE $\mathcal{R}_\varphi$ OE. AF-REL.