Commutators, paraproducts and BMO in non-homogeneous martingale harmonic analysis

Sergei Treil

Department of Mathematics
Brown University

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1. Main objects
   - Martingale difference decomposition
   - Martingale transforms, paraproducts and dyadic shifts
   - Square function, H^1 and BMO
2. Bounds on paraproducts and commutators
   - Paraproducts
   - Extended paraproducts and commutators
   - Basis properties of martingale difference spaces
3. From dyadic to classical
   - Random dyadic lattices
   - Averaging of dyadic shifts
   - H^1 vs. dyadic H^1

Haar system:

- For an interval I let h_I be the Haar function

$$h_I := |I|^{-1/2}(\mathbf{1}_{I_+} - \mathbf{1}_{I_-}),$$

where $I_\pm$ are the right and left halves of I respectively.

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$\{h_I\}_{I\in\mathcal{D}}$ is and orthonormal basis in $L^2(\mathbb{R})$

- It is also an unconditional basis in $L^p(\mathbb{R})$, $1 < p < \infty$:

$$f = \sum_{I\in\mathcal{D}} \langle f, h_I\rangle h_I$$

and the series converges unconditionally.

Martingale difference decomposition:

- $\mathbb{E}_I F := \left(|I|^{-1} \int_I f dx\right) \mathbf{1}_I$
- $\Delta_I := -\mathbb{E}_I + \sum_{J \in \text{child}(I)} \mathbb{E}_J$
- So $f = \sum_{I \in \mathcal{D}} \Delta_I f$;
- Note that on $\mathbb{R}$ we have $\Delta_I f = \langle f, h_I \rangle h_I$;

Advantage of MDD notation: the same notation in $\mathbb{R}^n$, where there are $2^n - 1$ Haar functions for each cube.

Can use arbitrary measure.

Martingale transforms and martingale multipliers

$\mathbb{R}^N$, Radon measure μ: $\mathbb{E}_Q = \mathbb{E}^\mu_Q$, $\Delta_Q = \Delta^\mu_Q$ are with respect to μ.

- $D_Q = D^p_Q := \Delta_Q L^p$ — martingale difference space;
- Martingale multiplier T_α, $\alpha = \{\alpha_Q\}_{Q\in\mathcal{D}}$,

$$T_\alpha f := \sum_{Q\in\mathcal{D}} \alpha_Q \Delta_Q f.$$

- Martingale transform T is a diagonal operator in the basis $\{D_Q : Q \in \mathcal{D}\}$:

$$Tf = \sum_{Q\in\mathcal{D}} T_Q(\Delta_Q f), \qquad T_Q : D_Q \to D_Q.$$

- For doubling μ, T is bounded in L^p, $1 < p < \infty$ iff T_Q are uniformly bounded: not true in general case for $p \neq 2$.

Paraproducts

- Let b locally integrable.
- Let M_b be the multiplication operator, $M_b f = bf$.
- Decompose M_b in the basis $\{D_Q : Q \in \mathcal{D}\}$ (recall $D_Q = \Delta_Q L^2$),

$$
\begin{aligned}
Tf &= \sum_{Q,R\in\mathcal{D}} \Delta_Q M_b \Delta_R f \\
&= \sum_{Q\subsetneq R} \dots + \sum_{Q=R} \dots + \sum_{R\subsetneq Q} \dots = \pi_b f + \Lambda_b f + \pi_b^* f
\end{aligned}
$$

- $\pi_b f = \sum_{Q\in\mathcal{D}} (\Delta_Q b)(\mathbb{E}_Q f)$ — paraproduct
- Λ_b is a martingale transform, commutes with all martingale multipliers.
- $\pi_b^* f = \sum_{Q\in\mathcal{D}} \mathbb{E}_Q(b\Delta_Q f) = \sum_{Q\in\mathcal{D}} \mathbb{E}_Q\left((\Delta_Q b)(\Delta_Q f)\right)$

Paraproducts: another point of view

- Decompose

$$bf = \sum_{Q,R\in\mathcal{D}} (\Delta_Q b)(\Delta_R f)$$
$$= \sum_{Q\subsetneq R} \dots + \sum_{R\subsetneq Q} \dots + \sum_{Q=R} \dots = \pi_b f + \Lambda_b^0 f + \pi_b^{(*)} f$$

- $\Lambda_b^0 f = \sum_{R\in\mathcal{D}} (\mathbb{E}_R b)(\Delta_R f)$ — martingale multiplier, commutes with all martingale transforms.
- $\pi_b^{(*)} f = \sum_{Q\in\mathcal{D}} (\Delta_Q b)(\Delta_Q f)$
(recall that $\pi_b^* f = \sum_{Q\in\mathcal{D}} \mathbb{E}_Q \Big((\Delta_Q b)(\Delta_Q f) \Big)$).
- For classical Haar system (Lebesgue measure) on $\mathbb{R}$, $\pi_b^* = \pi_b^{(*)}$ (because $h_I^2 \equiv$ Const on I).

Paraproducts and commutators

- $M_b = \pi_b + \pi_b^* + \Lambda_b$, where Λ_b is a martingale transform, and so commutes with all martingale multipliers.
- Therefore, if π_b is bounded, then $[M_b, T] := M_b T - T M_b$ is bounded for any martingale multiplier $T = T_\alpha$
- $M_b = \pi_b + \pi_b^{(*)} + \Lambda_b^0$ where Λ_b^0 is a martingale multiplier, and so commutes with all martingale transforms.
- Therefore, if π_b and $\pi_b^{(*)}$ are bounded, then $[M_b, T] := M_b T - T M_b$ is bounded for any martingale transform $T = \mathrm{diag}\{T_Q : Q \in \mathcal{D}\}$.
- In fact, if $\pi_b^{(*)}$ is bounded, then π_b is bounded
- $\pi_b^{(*)} - \pi_b^* = \Lambda_b - \Lambda_b^0 = \sum_{Q\in\mathcal{D}} \Delta_Q \left((\Delta_Q b)(\Delta_Q f) \right)$ — one can split this term between π_b and π_b^*.

Paraproducts and Calderón–Zygmund operators

For CZO paraproducts catch some hidden oscillation ($T(1)$ theorem).

- If $Q \cap R = \varnothing$ then $\langle T\Delta_Q f, \Delta_R g\rangle$ is easy to estimate using only smoothness of the kernel.
- If $R \subset Q$ one needs to estimate $\langle T\mathbf{1}_{Q'}, \Delta_R g\rangle$; here Q' is the child of Q, $Q' \supset R$.
- if $T1 = 0$ this is equivalent the estimating $\langle T\mathbf{1}_{\mathbb{R}^N \setminus Q'}, \Delta_R g\rangle$, which can be done standard way
- If $T1 \neq 0$ one needs to replace T by $T - \pi_b$, $b = T1$
- Condition $b \in$ BMO implies that π_b is bounded
- Case $Q \subset R$ is treated symmetrically. Condition $T^*1 \in$ BMO is used.

- Let $\mathrm{child}(Q) = \mathrm{child}_1(Q)$ denote the children of Q, and let $\mathrm{child}_n(Q)$ be the grandchildren of order n.

Definition (Dyadic (Haar) shift)

with parameters m and n is an operator $T = \sum_{Q\in\mathcal{D}} A_I$, where

$$A_I : \oplus_{R\in\mathrm{child}_m(Q)} D_R \to \oplus_{R\in\mathrm{child}_n(Q)} D_R$$

where A_I can be represented as an integral operator with kernel $a_Q(x,y)$, $\|a_Q\|_\infty \le |Q|^{-1}$.
Complexity of T is $r = \max\{m,n\}$.

- Dyadic shift is not a martingale transform.
- But T can be decomposed $T = T_1 + T_2 + \ldots + T_r$, $T_k = \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}:\,\ell(Q)=2^{k+rj}} A_Q$;
- Each T_k can be treated as a martingale transform if one goes r steps at a time.

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Complexity of T is $r = \max\{m,n\}$.

- Bound $\|a_Q\|_\infty \le |Q|^{-1}$ means simply that after "renormalization" bilinear form of the operator A_Q is bounded on $L^1 \times L^1$.

Square function and Hardy space H^1_{d}

- Square function: $Sf(x) = \left(\sum_{Q \in \mathcal{D}} |\Delta_Q f(x)|^2 \right)^{1/2}$.

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- Linearized (vector) square function ($\vec{S}f(x) \in \ell^2$): $\vec{S}f(x) = \{\Delta_Q(x) : Q \ni x,\, \ell(Q) = 2^{-k}\}_{k\in\mathbb{Z}}$.
- $|Sf(x)| = \|\vec{S}f(x)\|_{\ell^2}$.

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- $|Sf(x)| = \|\vec{S}f(x)\|_{\ell^2}$.
- Let $1 < p < \infty$. Then $f \in L^p$ iff $Sf \in L^p$, and $\|Sf\|_p \asymp \|f\|_p$ (Paley, Burkholder).
- Define H^1_{d} (dyadic Hardy space): $f \in H^1_{\mathrm{d}}$ iff $Sf \in L^1$.
- H^1_{d} can be treated as a subspace of $L^1(\ell^2)$, consisting of $f = \{f_k\}_{k\in\mathbb{Z}}$, $\int_{\mathbb{R}^N} \left(\sum_{k\in\mathbb{Z}} |f_k(x)|^2\right)^{1/2} d\mu(x) < \infty$ such that
  1. f_k is constant on children of $Q \in \mathcal{D}$, $\ell(Q) = 2^{-k}$;
  2. $\int_Q f_k d\mu(x) = 0$ for all $Q \in \mathcal{D}$, $\ell(Q) = 2^{-k}$.

Maximal function characterization of H^1_{d}

- $f \in H^1_{\mathrm{d}}$ iff $Mf \in L^1$, where

$$Mf(x) = \sup_{Q \ni x} |\mathbb{E}_Q f(x)|$$

- M also can be interpreted as a linear (but vector-valued) operator ($\vec{M}f(x) \in \ell^\infty$),

$$\vec{M}f = \{\mathbb{E}_Q(x) : Q \ni x,\, \ell(Q) = 2^{-k}\}_{k \in \mathbb{Z}}$$

$\mathrm{BMO}_{\mathrm{d}}$ as dual of H^1_{d}

Definition

$f \in \mathrm{BMO}^p_{\mathrm{d}}$ iff

1. $\forall Q_0 \in \mathcal{D} \int_{Q_0} \left| \sum_{Q \subset Q_0} |\Delta_Q f(x)|^2 \right|^{p/2} d\mu(x) \leq C\mu(Q_0)$
2. $\sup_{Q \in \mathcal{D}} \|\Delta_Q f\|_\infty < \infty$.

Condition 2 follows from 1 for doubling measures μ.

- All $\mathrm{BMO}^p_{\mathrm{d}}$, $1 \leq p < \infty$ coincide (not a trivial fact). Will write simply $\mathrm{BMO}_{\mathrm{d}}$.
- For $p = 2$ condition 1 can be rewritten as

$$\sum_{Q \subset Q_0} \|\Delta_Q f\|_2^2 \leq C\mu(Q_0)$$

- $\mathrm{BMO}_{\mathrm{d}} = (H^1_{\mathrm{d}})^*$

BMO_d (continued)

- Recall: $f \in \mathrm{BMO}_\mathrm{d}$ iff
  1. $\forall Q_0 \in \mathcal{D}\ \int_{Q_0} \left|\sum_{Q \subset Q_0} |\Delta_Q f(x)|^2\right|^{p/2} d\mu(x) \le C\mu(Q_0)$
  2. $\sup_{Q\in\mathcal{D}} \|\Delta_Q f\|_\infty < \infty$.

  Any $p \in [1, \infty)$ works.
- For $p \in (1, \infty)$ this equivalent to

$$\mu(Q)^{-1} \int_Q |f - \mathbb{E}_{\widetilde{Q}} f|^p \le C\mu(Q) \qquad \forall Q \in \mathcal{D}$$

  where $\widetilde{Q}$ is the parent of Q.
- $p = 1$ also works.

Why $\mathrm{BMO_d} = (H^1_\mathrm{d})^*$

- Easier to see from the square function definition.
- Inclusion $(H^1_\mathrm{d})^* \subset \mathrm{BMO_d}$:
  - Martingale difference decomposition of H^1_d can be identified with a subspace of $L^1(\ell^2)$;
  - so $(H^1_\mathrm{d})^*$ is a quotient space of $L^\infty(\ell^2)$
  - Averaging (going from arbitrary $f \in L^\infty(\ell^2)$ to a martingale difference) does not preserve $L^\infty(\ell^2)$ norm, but it preserves (local) $L^p(\ell^2)$ norm for $p \in (1, \infty)$.
    Especially easy for $p = 2$.
- Inclusion $\mathrm{BMO_d} \subset (H^1_\mathrm{d})^*$: need to show that that any function $\varphi \in \mathrm{BMO_d}$ defines a bounded linear functional on H^1_d.
  Can be done analyzing level sets of square function.

Why $p = 1$ works in the definition of BMO

Classical way: use John–Nirenberg inequality (the measure of the set where $|f(x)| > \lambda$ decays exponentially.

Why $p = 1$ works in the definition of BMO

A neat trick: As we discussed, the quantities

$$\sup_Q \left(\fint_Q |f - \mathbb{E}_{\widetilde{Q}} f|^2 \right)^{1/2} \quad \text{and} \quad \sup_Q \left(\fint_Q |f - \mathbb{E}_{\widetilde{Q}} f|^3 \right)^{1/3}$$

are equivalent for $f \in \mathrm{BMO}$.
Take $\varphi = (f - \mathbb{E}_{\widetilde{Q}} f)\mathbf{1}_Q$ where Q is a cube where $\sup_Q \left(\fint_Q |f - \mathbb{E}_{\widetilde{Q}} f|^2 \right)^{1/2}$ is almost attained. Then by Cauchy–Scwartz

$$\|\varphi\|_{Q,2}^2 = \fint_Q |\varphi|^2 = \fint_Q |\varphi|^{1/2} |\varphi|^{3/2} \leq \|\varphi\|_{Q,1}^{1/2} \|\varphi\|_{Q,3}^{3/2} \leq C \|\varphi\|_{Q,1}^{1/2} \|\varphi\|_{Q,2}^{3/2}$$

so $\|\varphi\|_{Q,2} \leq C \|\varphi\|_{Q,1}$.
The converse inequality is trivial.

The above trick works for multiparameter H^1 spaces.

A surprising observation

There exists $f \in \mathrm{BMO_d}(\mathbb{R})$ such that the decomposition $\sum_{I\in\mathcal{D}} \Delta_I f$ diverges a.e.

- For $k \geq 1$ let $I_k = [0, 2^k)$, and let $\Delta_{I_k} f := \mathbf{1}_{[0,2^{k-1})} - \mathbf{1}_{[2^{k-1},2^k)}$;
- Clearly for any $J \in \mathcal{D}$

$$\sum_{I_k \subset J} \|\Delta_{I_k} f\|_2^2 \leq 2|J|,$$

  but $\sum_k \Delta_{I_k} f(x) = +\infty$ for all x (the sum have finitely many 0s and -1s, and infinitely many 1s)
- f can be defined as $\sum_k (\Delta_{I_k} f - \mathbf{1}_{[0,\infty)})$ (can add arbitrary constant also)

Theorem

A paraproduct π_b is bounded in L^p, $1 < p < \infty$ if and only if $b \in BMO_{\mathrm{d}}$.

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Theorem

A paraproduct π_b is bounded in L^p if and only if it is uniformly bounded on characteristic functions $\mathbf{1}_Q$, $Q \in \mathcal{D}$,

$$\mu(Q)^{-1} \int_Q \left| \sum_{R \subset Q} \Delta_R b \right|^p d\mu$$

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- Trivial for $p = 2$, easy for $p < 2$ and hard for $p > 2$.
- Unlike $b \in \mathrm{BMO}_{\mathrm{d}}$, this condition depends on p
- In the homogeneous case it is just the condition $b \in \mathrm{BMO}_{\mathrm{d}}$

What about BMO?

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Theorem

An extended (adjoint) paraproduct $\pi_b^{(*)}$ in bounded in L^p, $1 < p < \infty$ if and only if $b \in BMO_{\mathrm{d}}$.

The condition is the same for all p.

- the condition $\sup_{I \in \mathcal{D}} \|\Delta_Q\|_\infty < \infty$ responsible for the boundedness of the difference

$$\pi_b^{(*)} f - \pi_b^* f = \sum_{Q \in \mathcal{D}} \Delta_Q \Big((\Delta_Q b)(\Delta_Q f) \Big)$$

- This theorem is much easier to prove than the result about paraproducts.

- Recall:

$$M_b = \pi_b + \pi_b^* + \Lambda_b = \pi_b + \pi_b^{(*)} + \Lambda_b^0,$$

  Λ_b^0 commutes with martingale transforms, Λ_b commutes with martingale multipliers.

- If $b \in \mathrm{BMO_d}$, then π_b and $\pi_b^{(*)}$ are bounded, so the commutator $[M_b, T] = M_b T - T M_b$ is bounded in L^p for any bounded martingale transform T.
- If T satisfies some *mixing properties*, then boundedness of $[M_b, T]$ (in some L^p) implies $b \in \mathrm{BMO_d}$.
  Generalizes classical result of S. Janson about commutators on d-adic martingales.
- In the general case, it is impossible to get that $b \in \mathrm{BMO_d}$ even if $[M_b, T]$ are uniformly bounded for *all* martingale multipliers T, $\|T\| \leq 1$.

Definition

A sequence of subspace E_k is called an unconditional basis if any vector $x \in X$ admits unique representation

$$x = \sum_k x_k, \qquad x_k \in E_k$$

and the sum converges unconditionally (i.e. independently of the ordering of indices).

Well known: martingale difference spaces $\Delta_Q L^p$, $1 < p < \infty$ form an unconditional basis in L^p.

Definition

An unconditional basis is called a *strong unconditional* basis if one can define an equivalent norm in X using a lattice norm on the sequence $\{\|x_k\|\}_k$.

Strong unconditional bases

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An unconditional basis is called a *strong unconditional* basis if one can define an equivalent norm in X using a lattice norm on the sequence $(\|x_k\|)_k$.

- Lattice (Banach lattice) norm means that if $\alpha = (\alpha_k)_k$ and $\beta = (\beta_k)_k$ satisfy $|\alpha_k| \leq |\beta_k|$ then $\|\alpha\| \leq \|\beta\|$
- Any unconditional basis in a Hilbert space is trivially a strong unconditional basis (ℓ^2 norm is the lattice norm giving an equivalent norm).
- Martingale difference spaces $\Delta_Q L^p$ form a strong unconditional basis if the underlying measure is doubling
- Unfortunately, for general measures that is not the case

Strong unconditional bases

- There exist measures such that the martingale difference spaces $\Delta_Q L^p$ do not form strong unconditional basis in L^p, $1 < p < \infty$.
- The construction of counterexamples is related to the counterexamples showing that the boundedness of the papraproduct π_b does not imply that $b \in$ BMO.

Random lattices

- $\xi_k = \xi_k(\omega)$, $\omega \in \Omega$, $k \in \mathbb{Z}$ — independent $0, 1$ Bernoulli, $\mathbb{P}(1) = \mathbb{P}(0) = 1/2$;
- $s_k(\omega) := \sum_{j:j<k} 2^j \xi_j(\omega)$; note that s_k are uniformly distributed on $[0, 2^k]$.
- $\mathcal{D}_\omega := \{2^k([0,1) + j) + s_k(\omega) : j, k \in \mathbb{Z}\}$

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If $\xi_0(\omega) = 0$

$$2I_0 + s_0 + \xi_0$$

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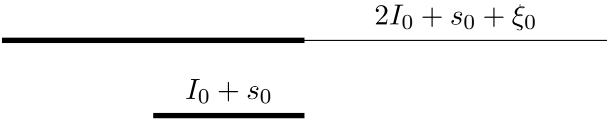

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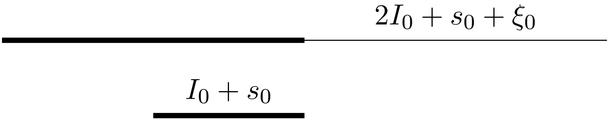
- Let $I_0 = [0, 1)$

If $\xi_0(\omega) = 1$



- An elementary interpretation: average over translations over $s \in [-R, R]$, and take limit as $R \to \infty$.

Definition (An elementary Haar shift)

$\mathbb{S}f = \sum_{I\in\mathcal{D}} \langle f, h_I\rangle (h_{I_+} - h_{I_-})$,
I_+ and I_- — right ald left halves of I.

Theorem (S. Petermichl, 2000)

Hilbert transform T can be represented as

$$T = C\int_1^2 \int_\Omega \mathbb{S}_{r\mathcal{D}(\omega)}\, d\mathbb{P}(\omega)\frac{dr}{r}.$$

- Other classical operators (Riesz transforms, Beurling–Ahlfors transform) can be represented as averages of dyadic shifts
- Beurling–Ahlfors transform can be even represented as average of dyadic multipliers (O. Dragičević–A. Volberg, 2003)
- Antisymmetric convolution operators on $\mathbb{R}$ with sufficiently smooth kernel can be represented as and average of dyadic shifts of fixed complexity (A. Vagharshakyan).

Recall that a *Calderón–Zygmund operator* (CZO) in $\mathbb{R}^N$, $d \leq N$, is a bounded in L^2 integral operator with kernel K satisfying the following growth and smoothness conditions

1. $|K(x,y)| \leq \dfrac{C_{\text{cz}}}{|x-y|^N}$ for all $x, y \in \mathbb{R}^N$, $x \neq y$.
2. There exists $\alpha > 0$ such that

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C_{\text{cz}} \frac{|x-x'|^\alpha}{|x-y|^{N+\alpha}}$$

for all $x, x', y \in \mathbb{R}^N$ such that $|x-x'| < |x-y|/2$.

General CZO as averages of the Haar shifts

- We do not know how to represent a general CZO as an average of Haar shifts of *fixed* complexity

Theorem (T. Hytonen, 2010)

A general Calderón–Zygmund operator in $\mathbb{R}^N$ *can be represented as*

$$T = C \int_\Omega \sum_{m,n \in \mathbb{Z}_+} 2^{-(m+n)\alpha/2}\, \mathbb{S}^\omega_{m,n}\, d\mathbb{P}(\omega) + C \int_\Omega \left(\pi_{b_1(\omega)} + \pi^*_{b_2(\omega)}\right) d\mathbb{P}(\omega);$$

Here $\mathbb{S}^\omega_{m,n}$ *are the dyadic shifts with parameters* m *and* n *corresponding to the lattice* $\mathcal{D}(\omega)$*, and* $\pi_{b_1(\omega)}$*,* $\pi_{b_2(\omega)}$ *are paraproducts.*

The measure μ is the Lebesgue measure in $\mathbb{R}^N$

- Let $\{\varphi_I\}_{I\in\mathcal{D}}$ be a reasonable wavelet system (say Meyer wavelets).
- Recall: $f \in H^1$ if $\left(\sum_{I\in\mathcal{D}} |\langle f, \varphi_I\rangle|^2 |I|^{-1}\mathbf{1}_I\right)^{1/2} \in L^1$

Theorem (B. Davis, Garnett–Jones, Pipher–Ward, Treil)

The following are equivalent

1. $f \in H^1$.
2. $\int_\Omega \|f\|_{H^1_{\mathcal{D}(\omega)}} d\mathbb{P}(\omega) < \infty$
3. $\left(\int_\Omega |S_{\mathcal{D}(\omega)} f(\,\cdot\,)|^2 d\mathbb{P}(\omega)\right)^{1/2} \in L^1$

- Treat dyadic square function $S_\mathcal{D}$ as a vector-valued linear operator.
- The average of $S_{\mathcal{D}(\omega)}$ is a a Hilbert-space-valued Calderón–Zygmund operator.

- Situation in the non-homogeneous case is far from clear.
- One of the most natural BMOs in the non-homogeneous case is X. Tolsa's RBMO (restricted BMO); the corresponding H^1 is the so-called H^1_{atb} (atomic block H^1)
- Averaging (non-homogeneous) dyadic H^1 over random lattices does not give H^1_{atb}
- What one gets averaging non-homogeneous dyadic H^1 over random lattices?
- What are the dyadic analogues of H^1_{atb} and RBMO?

Some open problems

- Multiparameter H^1–BMO theory: well understood in the homogeneous case, but very little is known in the non-homogeneous situation.
- Two weight theory for paraproducts and Haar shifts. L^2 case is trivial for paraproducts and is known for Haar shifts and their generalizations (Nazarov–Treil–Volberg).
  But in the L^p situation little is known. The statements in the same generality as in the L^2 case are false (a counterexample by F. Nazarov).
  But what if one assume size conditions like for the Haar shifts?

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