SINGULAR DISTRIBUTIONS AND SYMMETRY OF THE SPECTRUM

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We’ll discuss the "Fourier symmetry" of measures and distributions on the circle in relation with the size of their support.

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The talk is based on joint work with Gady Kozma (to appear in Annales de L'Institute Fourier)

Introduction

Notation:

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K denotes the support of S.
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In general it has a polynomial growth.

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In general it has a polynomial growth.

If $\hat{S}(n) = o(1)$,
then the Fourier series of the distribution S

$$\sum \hat{S}(n)e^{int}$$

converges to zero outside of the support.

Menshov:

There is a (non-trivial) singular, compactly supported measure μ on the circle with Fourier transform vanishing at infinity.

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Corollary:

A non-trivial trigonometric series

$$\sum_{n\in\mathbb{Z}} c(n)e^{int} \qquad (1)$$

may converge to zero almost everywhere.

Abel + Privalov.

An "analytic" series

$$\sum_{n\geq 0} c(n)e^{int} \qquad (2)$$

cannot converge to zero on a set of positive measure unless it is trivial.

One-side Frostmann theorem

Frostmann :

(i) Let $0 < \beta \leq 1$*.*
If a compact set K *supports a measure* μ *s.t.*

$$\sum |\hat{\mu}(n)|^2/|n|^{1-\beta} < \infty \qquad (3)$$

then $\dim K \geq \beta$*.*

(ii) If $\dim\ K > \beta$ *then* K *supports a probability measure* μ *satisfying (3)*.

Beurling:

If K *supports a distribution* S *satisfying (3) then it also supports a probability measure with this property.*

Theorem 1

If K supports a distribution S, s.t.

$$\sum_{n<0} |\hat{S}(n)|^2/|n|^{1-\beta} < \infty,$$

then $\dim K \geq \beta$.

Proof:

1. Let $\beta = 1$.
S is a distribution with "anti-analytic" part in L^2.
F-L type theorems in the disc (Dalberg, Berman).
$\dim K < 1$ implies $S = 0$.

2. Reduction of the general case.
Take a "Salem measure" ν, supported by $E, \dim E > 1 - \beta$
$\hat{\nu}(n) = O(1/|n|)^{(1-\beta)/2}$.

$S' := S * \nu$.
The anti-analytic part of S' belongs to L^2.
$\dim suppS' = 1$.
$\dim(K + E) \leq \dim K + \dim E$.
Minkowski dimension

Almost analytic singular pseudo-functions

Compare two-sides and one-side results.

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Theorem 2. (G.K.,A.O., Annals of Math.,2006)

There is a distribution S with the properties:
(i) $\hat{S}(n) = o(1)$;
(ii) $mK_S = 0$
(iii) $\sum_{n<0} |\hat{S}(n)|^2 < \infty$.

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Classical Riemannian theory:

"Uniqueness implies Fourier formulas for coefficients".

Du Bua-Reymond-Lebesgue- Vallee-Poussin- Privalov

Let K be a compact ,which is a uniqueness set. If a trigonometric series converges on ${}^cK$ to an integrable function f then it is the Fourier series of f.

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In a contrast:
Consider S from Th.2.
Then $\sum_{\mathbb{Z}} \hat{S}(n)e^{int} = 0(t \in^c K)$
Both "halves" converge pointwisely on ${}^cK$.

The anti -analytic part is an L^2- function.
It admits the "analytic" decomposition, which is unique, but not the Fourier series.

Critical size of the support

Theorem 3

If S *is a (non-trivial) distribution,*
s.t. $\hat{S} \in l^2(\mathbb{Z}^-)$
then $\Lambda_h(K) > 0$*,*
where
$h(t) := t \log 1/t$
and Λ_h *is the corresponding Hausdorf measure.*

Theorem 4.
There exists a (non-trivial) pseudo-function S, such that

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Take a Cantor set K on T of exact size,
Let μ be the natural probability measure on K,
u be the harmonic extension of this measure into the disc, v is the conjugate harmonic function.
Set: $F(z) := e^{(u+iv)}$.

F defines an "analytic distribution" on the boundary:

$$G := \sum_{n \geq 0} c(n) e^{int}$$

But pointwise limit is an an $L^{\infty}-$ function f.
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Theorems 3,4 characterize the critical size of exceptional sets for "non-classic" analytic decompositions.

Smoothness

A stronger version of Th.3 was proved in the cited paper:
There is a singular pseudo-function s.t. the amplitudes in negative part of the spectrum decrease faster then any power.

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Question
How the critical size of the support K depends on order of smoothness?

Non-symmetry for measures

Symmetry theorems for measures:

Rajchman Theorem.
$\hat{\mu}(n) = o(1)$ for $n > 0$ implies the same for $n > 0$.

Chrushev-Peller, Koosis-Pihorides:
$\sum_{n<0} |\hat{\mu}(n)|^2/|n| < \infty$
implies the same for $n > 0$.

However non-symmetry is also possible.

Theorem 5

Given $d > 0, p > 2/d$, there is a compact set K of dimension d which supports a measure ν s.t.

$$\hat{\nu} \in l^p(\mathbb{Z}^-), \quad \hat{\nu} \notin l^p(\mathbb{Z}^+) \quad (p > 2/d).$$

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Question
Let K supports a distribution

$$S : \hat{S}(n) = o(1) \qquad (n > 0).$$

Does it support a distribution with the two-side condition?

Arithmetics of compact sets

Classical examples:
When the Cantor set K_θ is a uniqueness set?
(Bari-Salem-Zygmund)

Piatetskii-Shapiro:
There is a compact set K which supports a distribution with Fourier transform vanishing at infinity, but does not support such a measure.

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Wiener Theorem on cyclic vectors:
1) $x = \{x_k\}$ is cyclic in $l^1(\mathbb{Z})$ iff $X(t) := \sum x_k e^{ikt}$ has no zeros;
2) x is cyclic in $l^2(\mathbb{Z})$ iff $X(t) \neq 0$ a.e.

Wiener conjecture:

x is cyclic in $l^p(\mathbb{Z})$ iff the set of zeros of $X(t)$ is "negligible".

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Theorem 6 (N.Lev, A.O., Annals of Math.- 2011).

Let $1 < p < 2$. Then there are two vectors $x, yinl^1(\mathbb{Z})$ s.t.
(i) Zero sets of X and Y are the same;
(ii) x is cyclic in $l^p(\mathbb{Z})$, while y is not.