Propagation of chaos and return to equilibrium for Kac's random walks

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Plan

I. From microscopic to macroscopic evolutions

II. Probabilistic foundation of kinetic theory

III. The main results

IV. The functional framework

V. Sketch of the proof of the abstract stability result

VI. Entropic chaos and relaxation rate

VII. Statistical stability and the BBGKY hierarchy

The problem at hand

- How to derive rigorously macroscopic evolution equations in terms of the microscopic laws?
- $\rightarrow$ Foundation of continuum mechanics (Hilbert 6-th pb)
- Statistical mechanics and kinetic theory for large number of particles as an intermediate step
- $\rightarrow$ Foundation of kinetic theory?

The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics. [...] Thus Boltzmann's work on the principles of mechanics suggests ***the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua****.*

The (Maxwell)-Boltzmann equation (1867-1872)

$$\underbrace{\partial_t f}_{\text{time change}} + \underbrace{v \cdot \nabla_x f}_{\text{space change}} = \underbrace{Q(f,f)}_{\text{collision operator}} \qquad \text{on } f(t,x,v) \geq 0$$

- Transport term $v \cdot \nabla_x$: straight line along velocity v
- Collision operator $Q(f,f)$:

$$Q(f,f)(v) = \int_{v_* \in \mathbb{R}^3} \int_{\sigma \in \mathbb{S}^2} \Big[\underbrace{f(v')f(v'_*)}_{(v',v'_*)\to(v,v_*)} - \underbrace{f(v)f(v_*)}_{(v,v_*)\to\ldots} \Big] B(v - v_*, \sigma)$$

Velocity collision rule (2 free parameters $\to \sigma \in \mathbb{S}^2$):

$$v' := \frac{v + v_*}{2} + \sigma \frac{|v - v_*|}{2}, \qquad v'_* := \frac{v + v_*}{2} - \sigma \frac{|v - v_*|}{2}$$

Structure of the Boltzmann equation

Symmetries: $\int_v Q(f,f)\varphi(v) = \frac{1}{4}\int_{v,v_*,\omega}[f'f'_* - ff_*]B(\varphi+\varphi_* - \varphi' - \varphi'_*)$

Conservation laws:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^{2d}} Q(f,f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v\,\mathrm{d}x = 0$$

H-Theorem:

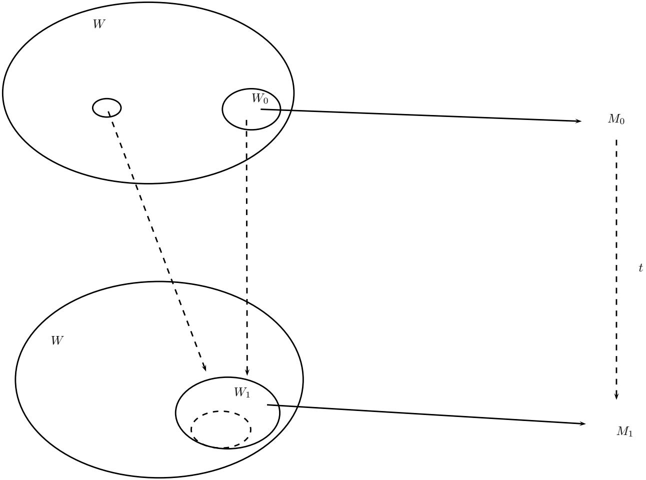
$$\frac{\mathrm{d}}{\mathrm{d}t}H(f) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} f \log f \,\mathrm{d}v\,\mathrm{d}x = -\int_{\mathbb{R}^{2d}} Q(f,f) \log f \,\mathrm{d}x\,\mathrm{d}v \leq 0$$

with cancellation only at $M_f = \frac{\rho}{(2\pi T)^{d/2}} e^{-|v-u|^2/2T}$ (Maxwellian)

Time-irreversible equation and mathematical basis for 2-d law of thermodynamic: natural question in the many-particle limit

Molecular chaos (I)

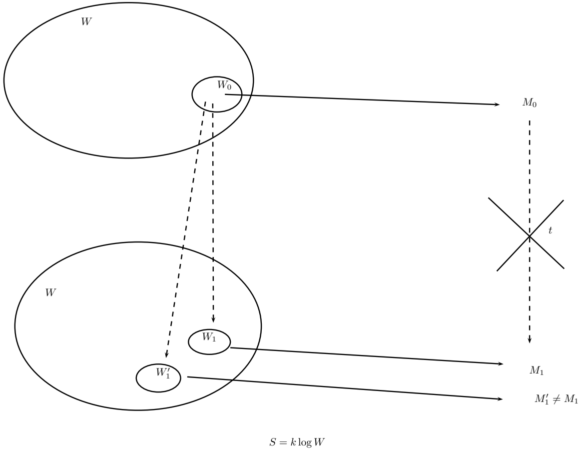
Irreversibility according to Boltzmann in terms of a “factorization” of a dynamics



$S = k \log W$

Molecular chaos (II)

Implicitly nontrivial “factorization” assumption of the dynamics:
$X_0 \in W(M_0) \Rightarrow T_t(X_0) \in W(F_t(M_0))$ T_t, F_t micro./macro. semigroups



Forbidden for macroscopic evolution laws (closed equation)

Molecular chaos (III)

How to justify this "factorization":

- Boltzmann's idea of molecular chaos ("Stosszahlansatz")
- Roughly speaking: for certain initial data (low correlations), the low correlations are mostly preserved with times and the Poincaré recurrence time is "sent to ∞" as $N \to +\infty$
- At least the time scale of such spurious "reversible fluctuations" remains out of the range of observations

"Proving" the Boltzmann equation (I)

Cercignani 1972:

The apparently paradoxical connection between the reversible nature of the basic equations of classical mechanics and the irreversible features of the gross description of large systems of classical particles satisfying those equations, came under strong focus with the celebrated H-theorem of Boltzmann and the related controversies between Boltzmann on one side and Loschmidt and Zermelo on the other.
[. . .]
In particular, it is not clear whether an averaging is taking place during the duration and over the region of a molecular collision. This averaging is related to another controversial point, i.e., **whether irreversibility can appear only through the intervention of a stochastic or random model or can be a consequence of the progressive weakening of the property of continuous dependence on initial conditions**.

"Proving" the Boltzmann equation (II)

- **Second viewpoint**: best result so far Lanford 1973: convergence for very short time (less than mean free time)
- Conceptually based on expansion of the solution in terms of the initial data but hard, deep and technical: see recent preprint Gallagher-Saint-Raymond-Texier
- At now, not adapted for the study of the long-time behavior
- **First viewpoint**: related to the question of the probabilistic foundation of kinetic theory (Kac 1956): randomness in the evolution itself and probabilistic methods

Plan

Kac's program (I)

- Goal: derive the spatially homogeneous Boltzmann Eq. and H-theorem from a many-particle Markov jump process
- The process is studied through its master equation (the equation on the law of the process)

Remarks: (1) This amounts intuitively to consider the spatial variable as a hidden variable inducing "loss of memory" and randomness on the velocity variable by ergodicity. → Open and interesting question...
(2) Limit different from the Boltzmann-Grad limit: mean-field limit

Kac's program (II)

"This formulation led to the well-known paradoxes which were fully discussed in the classical article of P. and T. Ehrenfest. These writers made it clear (a) that the "Stosszahlansatz" cannot be strictly derivable from purely dynamic considerations and (b) that the "Stosszahlansatz" has to be interpreted probabilistically. [...]The "master equation" approach which we have chosen seems to us to follow closely the intentions of Boltzmann."

Interpretation not clear! Cf. Cercignani 1972, Lanford 1973

But it raises a nice question:

If we have to introduce stochasticity, at least...
Can we keep it under control all along the process of many-particle limit and relate it to the dissipativity of the limit equation?

The propagation of chaos

- If $f_t^N = f_t^{\otimes N}$ tensorized on $t \in [0, T]$, then f_t satisfies the limit nonlinear Boltzmann equation on $t \in [0, T]$
- Tensorization property does not propagate in time (interactions)
- But the weaker property of chaoticity can be expected to propagate in time, in the correct scaling limit
- $(f^N)_{N \geq 1}$ symmetric probabilities on E^N is said f-chaotic if

$$f^N \sim f^{\otimes N} \quad \text{when} \quad N \to \infty$$

  (weak convergence of marginals)
- Many-particle limit reduced to the propagation of chaos

f_0-chaoticity of $(f_0^N)_{N \geq 1}$ implies f_t-chaoticity of $(f_t^N)_{N \geq 1}$

The notions of chaos and how to measure them (I)

- $f^N \in P_{\text{sym}}(E^N)$ is f-chaotic, $f \in P(E)$, if for any $\ell \in \mathbb{N}^*$ and any $\varphi \in C_b(E)^{\otimes \ell}$ there holds

$$\lim_{N\to\infty} \left\langle f^N, \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle = \left\langle f^{\otimes \ell}, \varphi \right\rangle$$

  which amounts to the weak convergence of any marginals
- Strong and weak topologies on $P(E)$
- Canonical distance M^1 for the strong topology
- But many distances for the weak topology
- In this talk Monge-Kantorovich-Rubinstein distance

$$W_1(\mu, \nu) = \sup_{\|\varphi\|_{Lip} \le 1} \int_E \varphi(\,\mathrm{d}\mu - \,\mathrm{d}\nu).$$

The notions of chaos and how to measure them (II)

- Finite-dimensional chaos: $K_\ell > 0$ and $\varepsilon(N) \to 0$, $N \to \infty$ s.t.

$$W_1\left(\Pi_\ell f^N, f^{\otimes \ell}\right) \leq K_\ell\, \varepsilon(N)$$

- Infinite-dimensional chaos:

$$\frac{W_1\left(f^N, f^{\otimes N}\right)}{N} \leq \varepsilon(N)$$

- (Infinite-dimensional) entropic chaos:

$$\frac{1}{N} H\left(f^N\right) \xrightarrow{N\to\infty} H(f), \quad H(f) := \int f \log f$$

- Other metrics by duality, e.g.

$$\left|\left\langle \Pi_\ell f^N - f^{\otimes \ell}, \varphi \right\rangle\right| \leq K_\ell\, \varepsilon(N), \quad \forall\, \varphi \in \mathcal{F}^{\otimes \ell} \subset C_b(E)^{\otimes \ell}$$

Kac's walk in the simpler case (I)

- Markov process on a continuous phase space with transition operator P
- Continuous in time: exponential random time

$$\mathbb{P}(T \geq t) = e^{-at} \quad \text{and} \quad \mathbb{E}(T) = \frac{1}{a} \quad \text{so that}$$

$$\mathbb{P}(n \text{ jumps before } t) = \frac{(at)^n}{n!} e^{-at} \quad \text{and}$$

$$f(t) = \sum_{n \geq 0} \mathbb{P}(n \text{ jumps before } t) P^n f(0)$$

$$= \sum_{n \geq 0} \frac{(at)^n}{n!} e^{-at} P^n f(0) = e^{ta(P - \mathrm{Id})} f(0)$$

- Differentiating the latter equation in time we obtain the Master equation (Kolmogorov forward equation)

$$\partial_t f = a(P - \mathrm{Id}) f$$

Kac's walk in the simpler case (II)

- Simplify collisions: one-dimensional velocities
- Trivial with momentum and energy conservation: drop momentum conservation
- Draw pairs (i, j) uniformly, with exponential time and perform

$$v_i' = v_i \cos\theta + v_j \sin\theta, \quad v_j' = -v_i \sin\theta + v_j \cos\theta$$

- Energy is preserved, normalize it as $(\sum_{i=1}^N v_i^2)/N = 1$
- In order to maintain $O(1)$ collisions happening per unit of time, scale the random exponential time so that $\mathbb{E}(T) = 1/N$

$$\frac{\partial f^N}{\partial t} = N(P - \mathrm{Id})f^N = L^N f^N$$

- Jump process on $\mathbb{S}^{N-1}(\sqrt{N})$ (Kac's walk):

$$\frac{\partial f^N}{\partial t} = \frac{2}{(N-1)} \sum_{i<j} \int_0^{2\pi} \left(f^N(\dots, v_i'(\theta), \dots, v_j'(\theta), \dots) - f^N \right) \frac{\mathrm{d}\theta}{2\pi}$$

Kac's main theorem

Kac's main propagation of chaos theorem

For the model above, $(f_t^N)_{N\geq 1}$ propagates chaos: if at time $t = 0$

$$f_k^N(t=0) \overset{N\to\infty}{\rightharpoonup} f_1^{\otimes k}(t=0), \quad \forall k \geq 1$$

then $\forall t > 0$

$$f_k^N(t) \overset{N\to\infty}{\rightharpoonup} f_1^{\otimes k}(t), \quad \forall k \geq 1,$$

where $f_k^N(t)$ marginals of the solution to the master equation

Remarks:

- Not quantitative
- On finite time
- Proof based on combinatorial series representation of the solution, available only for collision rates independent of the relative velocity...

Open problem 1

- McKean 1967 extends Kac's argument to the real geometry of collision but for cutoff Maxwell molecules ($a = \text{cst}$)
- But main short-range physical interaction: hard spheres for which $a = |v_i - v_j|$ (see later)
- Seemingly technical issue in fact related to difficulties for dealing with jump process whose jump times law depends on the velocity variables: "*The above proof suffers from the defect that it works only if the restriction on time is independent of the initial distribution. It is therefore inapplicable to the physically significant case of hard spheres because in this case our simple estimates yield a time restriction which depends on the initial distribution.*"
- Propagation of chaos for the hard spheres collision process? Incomplete attempt Grünbaum 1971 (important inspiration) and first proof Sznitman 1984 but with no rate

Open problem 2

- Spirit: going beyond “exact” combinatorics in order to deal with realistic collision processes
- Other paradigmatic long-range interaction: propagation of chaos for the true Maxwell molecules collision process?
- Difficulty: the particle system can undergo infinite number of collisions in a finite time interval, no “tree” representation of solutions available
- Physical example of long-range interactions, mathematical kind of fractional derivative operator and Lévy walk

Open problem 3

- Ergodic property of the Markov process under consideration $\rightarrow$ infinite number of Liapunov functions, including the L^2 norm and Boltzmann's entropy
- In contrast with it, the limit equation admits only (in general) the Boltzmann entropy as a Liapunov function.
- Kac then heuristically conjectures $H(f_t^N)/N \rightarrow H(f_t)$ along time, which would recover Boltzmann's H-theorem from the monotonicity of $H(f^N)/N$ for the Markov process.
- "*If the above steps could be made rigorous we would have a thoroughly satisfactory justification of Boltzmann's H-theorem*"
- In our notation the question is can one prove propagation of entropic chaos along time?

Open problem 4 (I)

- Relate long-time behavior of the many-particle system and of the limit nonlinear PDE
- First step proposed by Kac: L^2 spectral gap of the process
  "*Surprisingly enough this seems quite difficult and we have not succeeded in finding a proof. Even for the simplified model we have been considering, the question remains unsettled although we are able to give a reasonably explicit solution of the master equation.*"
- Recent works Carlen-Carvalho-Loss 2003, see also Diaconis-Saloff-Coste 2000, Janvresse 2001, Maslen 2003, Carlen-Geronimo-Loss 2011. . . who fully solved this question

Open problem 4 (II)

- But no hope of passing to the limit $N \to \infty$ in this spectral gap estimate, even if the spectral gap is independent of N. The L^2 norm is catastrophic in infinite dimension:

$$\|f^{\otimes N}\|_{L^2} \sim C^N \quad \text{geometric growth}$$

- Therefore following quite closely the intention of Kac, we reframe the question in a setting which "tensorizes correctly in the limit $N \to \infty$"
- In our notation: can one prove relaxation times *independent of the number of particles* on
  - normalized Wasserstein distance $\frac{W(f^N, \gamma^N)}{N}$?
  - normalized relative entropy $\frac{H(f^N|\gamma^N)}{N}$?

where γ^N denotes the N-particle invariant measure?

Plan

The Markov process (I)

- Start from Markov process $(\mathcal{V}_t^N)$ on $(\mathbb{R}^d)^N$
- *Scale time $t \to t/N$ in order that the number of interactions is of order $\mathcal{O}(1)$ on finite time interval*
- Denote by f_t^N the law of $\mathcal{V}_t^N$ and S_t^N the associated semigroup
- Master equation on $f_t^N = S_t^N f_0$ in dual form

$$\partial_t \langle f_t^N, \varphi \rangle = \langle f_t^N, G^N \varphi \rangle$$

$$(G^N \varphi)(V) = \frac{1}{N} \sum_{i,j=1}^{N} \Gamma\left(|v_i - v_j|\right) \int_{\mathbb{S}^{d-1}} b(\cos\theta_{ij}) \left[\varphi_{ij}^* - \varphi\right] \, \mathrm{d}\sigma$$

where $\varphi_{ij}^* = \varphi(V_{ij}^*)$ and $\varphi = \varphi(V) \in C_b(\mathbb{R}^{Nd})$
(see next slide for V_{ij}^*)

The Markov process (II) (short-range interaction)

(i) for any $i \neq j$, draw a random time $T_{\Gamma(|v_i - v_j|)}$ of collision (exponential law of parameter $\Gamma(|v_i - v_j|)$), then choose the collision time T_1 and the colliding couple (v_{i_0}, v_{j_0}) s.t.

$$T_1 = T_{\Gamma(|v_{i_0} - v_{j_0}|)} := \min_{1 \leq i \neq j \leq N} T_{\Gamma(|v_i - v_j|)};$$

(ii) then draw $\sigma \in \mathbb{S}^{d-1}$ according to the law $b(\cos\theta_{ij})$, where $\cos\theta_{ij} = \sigma \cdot (v_j - v_i)/|v_j - v_i|$;

(iii) the new state after collision at time T_1 becomes

$$V_{ij}^* = (v_1, \ldots, v_i^*, \ldots, v_j^*, \ldots, v_N),$$

$$v_i^* = \frac{v_i + v_j}{2} + \frac{|v_i - v_j|\,\sigma}{2}, \quad v_j^* = \frac{v_i + v_j}{2} - \frac{|v_i - v_j|\,\sigma}{2}.$$

The Markov process (III)

- Process invariant under velocity permutations and preserves momentum and energy at any jump

$$\sum_{j=1}^{N} v_j^* = \sum_{j=1}^{N} v_j \quad \text{and} \quad |V^*|^2 = \sum_{j=1}^{N} |v_j^*|^2 = \sum_{j=1}^{N} |v_j|^2 = |V|^2$$

- Hence process on $\mathbb{R}^{dN}$ but can be restricted to the manifold

$$\mathcal{S}^N := \left\{ \sum_{j=1}^{N} |v_j|^2 = \mathcal{E}, \quad \sum_{j=1}^{N} v_j = 0 \right\}$$

- At the level of the law, for $\phi : \mathbb{R} \to \mathbb{R}_+$

$$\int_{\mathbb{R}^{dN}} \phi\left(\sum_{j=1}^{N} v_j\right) \mathrm{d}f_t^N(V) = \int_{\mathbb{R}^{dN}} \phi\left(\sum_{j=1}^{N} v_j\right) \mathrm{d}f_0^N(V),$$

$$\int_{\mathbb{R}^{dN}} \phi(|V|^2)\,\mathrm{d}f_t^N(V) = \int_{\mathbb{R}^{dN}} \phi(|V|^2)\,\mathrm{d}f_0^N(V)$$

The Markov process (IV) (expected limit PDE)

- (Expected) limit nonlinear semigroup $S_t^{NL}(f_0) := f_t$ for any $f_0 \in P_2(\mathbb{R}^d)$ (probabilities with bounded second moment)

$$\partial_t f = Q(f,f) \quad \text{with} \quad B(v - v_*, \sigma) = \Gamma(|v - v_*|) b(\cos\theta)$$

$$Q(f,f)(v) := \int_{v_* \in \mathbb{R}^d} \int_{\sigma \in \mathbb{S}^{d-1}} \Big(f(v'_*) f(v') - f(v) f(v_*) \Big) B(v - v_*, \sigma)$$

(nonlinear spatially homogeneous Boltzmann equation)

- Conservation of momentum and energy for $t \geq 0$

$$\int_{\mathbb{R}^d} v \, \mathrm{d}f_t(v) = \int_{\mathbb{R}^d} v \, \mathrm{d}f_0(v), \;\; \int_{\mathbb{R}^d} |v|^2 \, \mathrm{d}f_t(v) = \int_{\mathbb{R}^d} |v|^2 \, \mathrm{d}f_0(v)$$

The collision operator Q

- Kernel $B := \Gamma(|v - v_*|)\, b(\cos\theta)$: physical information about molecular interaction (different from fluid mechanics model)
- Hard spheres: $\Gamma(Z) = Z$ and $b = 1$ in dimension 3
- Long-range interactions (inverse-power laws): $\Gamma(Z) = Z^\beta$, $\beta \in (-d, 1)$ and $b(\cos\theta) \sim C\,\theta^{-(d-1)-\alpha}$, $\theta \sim 0$, $\alpha \in (0, 2)$
- Intuition: $\Gamma \sim$ polynomial growth or decay of the coefficents in a PDE, and order of singularity of $b \sim$ (fractional) order of differentiation (cf. Lévy processes)
- Our theorems cover (in $d = 3$): $\gamma = 1$ and $\alpha = d - 1$ (hard spheres) and $\gamma = 0$ and $\alpha = 1/2$ (Maxwell molecules)

What we prove (I)

In short we answer the four open problems formulated above.

- Propagation of chaos with quantitative rates for hard spheres and Maxwell molecules without cutoff
- Most importantly estimates uniform in time:
  $\Rightarrow$ however "top-down" instead of "bottom-up" as was suggesting Kac (see later)
- Infinite-dimensional chaos (in Wasserstein or entropic form)
- Estimates of relaxation times independent of the number of particles (in Wasserstein or entropic form)
- New method based on perturbative intuition: (1) consistency estimates and (2) stability estimates on the limit PDE
  [*No compactness argt. or expansion in terms of initial data*]

What we prove (II)

- Uniform in time finite-dimensional chaos

$$\sup_{t\geq 0} W_1\left(\Pi_\ell f_t^N, f_t^{\otimes \ell}\right) \leq K_\ell\, \varepsilon(N)$$

with ε poly. (Maxwell mol.) or power of logarithm (HS)

- Infinite-dimensional chaos

$$\sup_{t\geq 0} \frac{W_1\left(f_t^N, f_t^{\otimes N}\right)}{N} \leq K\, \varepsilon(N)$$

and entropic chaos

$$\forall\, t \geq 0, \quad \frac{H\left(f_t^N\right)}{N} \xrightarrow{N\to\infty} H(f)$$

- Estimates on relaxation times indep. of N

$$\forall\, N \geq 1, \quad \frac{W_1\left(f_t^N, \gamma^N\right)}{N} \leq \beta(t) \ \text{ with } \beta(t) \xrightarrow[t\to+\infty]{} 0$$

(also in entropic form for Maxwell molecules)
Rk: Rate $\beta(t)$ not optimal

A flavor of the key stability estimate

$f_0 \in P(\mathbb{R}^d)$ with cpct support, $f_0^N \in P(\mathcal{S}^N)$ and f_0-chaotic:

(i) Hard spheres: for any $\ell \in \mathbb{N}^*$ and any $N \geq 2\ell$:

$$\sup_{t \geq 0} \sup_{\|\varphi\|_{Lip(\mathbb{R}^d)^{\otimes \ell}} \leq 1} \left\langle \Pi_\ell \left[S_t^N \left(f_0^{\otimes N} \right) \right] - S_t^{NL}(f_0)^{\otimes \ell}, \varphi \right\rangle \leq \ell \, \varepsilon(N)$$

with $\varepsilon(N) \to 0$ as a power of logarithm

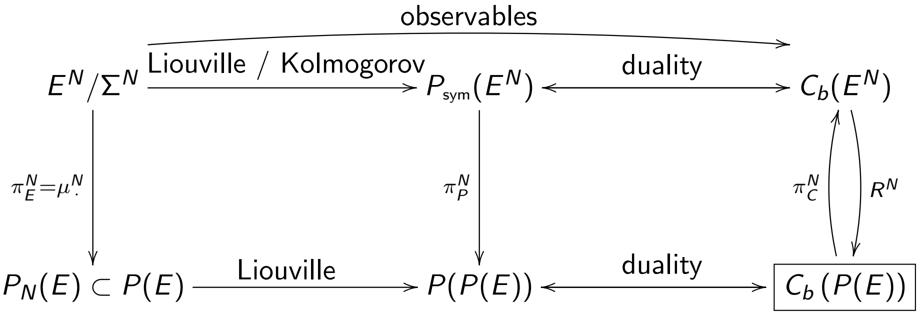
(ii) Maxwell molecules: $\forall \; \ell \in \mathbb{N}^*$, $N \geq 2\ell$, $\eta << 1$:

$$\sup_{t \in (0,\infty)} \sup_{\|\varphi\|_{\mathcal{F}^{\otimes \ell}} \leq 1} \left\langle \Pi_\ell \left[S_t^N \left(f_0^{\otimes N} \right) \right] - S_t^{NL}(f_0)^{\otimes \ell}, \varphi \right\rangle \leq \ell^2 \, \frac{C_\eta}{N^{\frac{1}{2(d+4)} - \eta}}$$

$$\mathcal{F} := \left\{ \varphi : \mathbb{R}^d \to \mathbb{R}; \; \|\varphi\|_{\mathcal{F}} := \int_{\mathbb{R}^d} (1 + |\xi|^4) \, |\hat{\varphi}(\xi)| \, d\xi < \infty \right\},$$

Plan

Functional diagram



- $E = \mathbb{R}^d$ (or Polish space)
- Σ^N N-permutation group
- $P_{\text{sym}}(E^N)$ symmetric probabilities

Inspiration: Grünbaum 1971, some intersection with ideas in Kolokoltsov 2010

Maps of the diagram

- μ_V^N denotes the *empirical measure*:

$$\mu_V^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{v_i}, \quad V = (v_1, \dots, v_N)$$

- $P_N(E) = \{\mu_V^N,\ V \in E^N\} \subset P(E)$
- $\forall\, V \in E^N/\mathfrak{S}^N, \quad \pi_E^N(V) := \mu_V^N$
- $\forall\, \Phi \in C_b\left(P(E)\right),\ \forall\, V \in E^N,\ \left(\pi_C^N \Phi\right)(V) := \Phi\left(\mu_V^N\right)$
- $\forall\, \phi \in C_b(E^N),\ \forall\, f \in P(E), \quad R^N[\phi](f) := \left\langle f^{\otimes N}, \phi \right\rangle$
- $\forall\, \Phi \in C_b(P(E)),\ \forall\, f^N \in P_{\text{sym}}(E^N),\ \left\langle \pi_P^N f^N, \Phi \right\rangle = \left\langle f^N, \pi_C^N \Phi \right\rangle$

Evolution N-particle semigroups

- Process $(\mathcal{V}_t^N)$ on E^N = trajectories: stochastic ODEs (Markov process), or deterministic ODEs (Hamiltonian flow).
  Flow commutes with permutations (part. indistinguishable)
- Corresponding linear semigroup S_t^N on $P_{\text{sym}}(E^N)$:

$$\partial_t f^N = A^N f^N, \qquad f^N \in P_{\text{sym}}(E^N),$$

  Forward Kolmogorov equation or Liouville equation
- *Dual* linear semigroup T_t^N of S_t^N:

$$\forall\, f^N \in P(E^N),\ \phi \in C_b(E^N), \quad \left\langle f^N, T_t^N(\phi) \right\rangle := \left\langle S_t^N(f^N), \phi \right\rangle$$

  Semigroup of the observables: $\partial_t \phi = G^N(\phi), \quad \phi \in C_b(E^N).$

Evolution limit semigroups

- (Nonlinear) semigroup S_t^{NL} on $P(E)$ solution to

$$\partial_t f_t = Q(f_t), \quad f_0 = f.$$

- Pullback linear semigroup T_t^∞ on $C_b(P(E))$:

$$\forall\, f \in P(E),\ \Phi \in C_b(P(E)), \quad T_t^\infty[\Phi](f) := \Phi\left(S_t^{NL}(f)\right)$$

  solution to the *linear* evolution equation on $C_b(P(E))$:

$$\partial_t \Phi = G^\infty(\Phi) \quad \text{with generator} \quad G^\infty.$$

- T_t^∞ can be interpreted physically as the semigroup of the evolution of observables of the nonlinear Boltzmann equation.

Interpretation of the pullback semigroup T_t^∞ (I)

- Given a nonlinear ODE $V' = F(V)$ on $\mathbb{R}^d$, one can define (at least formally) the linear Liouville transport PDE

$$\partial_t \rho + \nabla_v \cdot (F\,\rho) = 0,$$

  where $\rho_t(V) = V_t^*(\rho_0) = \rho_0 \circ V_{-t}$
- Dual viewpoint of observables: for ϕ_0 function defined on $\mathbb{R}^d$, evolution $\phi_t(v) = \phi_0(V_t(v)) = (V_t)_*(\phi_0) = \phi_0 \circ V_t$ solution to the linear PDE

$$\partial_t \phi - F \cdot \nabla_v \phi = 0,$$

- Duality relation:

$$\langle \phi_t, \rho_0 \rangle = \langle \phi_0, \rho_t \rangle$$

Interpretation of the pullback semigroup T_t^∞ (II)

- Go "one level above": replace $\mathbb{R}^d$ by $\mathcal{B} = P(E)$:
  The infinite dimensional "ODE" $V' = Q(V)$ on $\mathcal{B}$ yields first the abstract transport equation

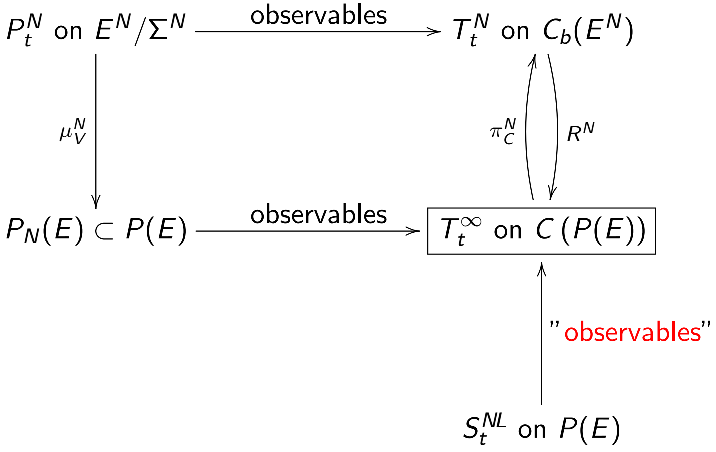
  $$\partial_t \pi + \nabla \cdot (Q(v)\, \pi) = 0, \qquad \pi \in P(\mathcal{B})$$

  and second the abstract dual observable equation

  $$\partial_t \Phi - Q(v) \cdot \nabla \Phi = 0, \qquad \Phi \in C_b(\mathcal{B}).$$

- Provide intuition and formal formula for the generator "$G^\infty \Phi = Q(v) \cdot \nabla \Phi$": but requires to develop abstract differential calculus on $C_b(H)$ to give sense to this heuristic.
- Note that for a *dissipative equation*, no reversed "characteristics" and observable viewpoint more natural

Diagram of connection between the two dynamics



The metric issue (I)

- Fundamental space of connection $C_b(P(E))$
- At the topological level there are two canonical choices:
  1. strong (total variation)
  2. weak topology.
- Two different sets: $C_b(P(E), w) \subset C_b(P(E), TV)$
- $\|\Phi\|_{L^\infty(P(E))}$ *does not* depend on the choice of topology on $P(E)$, and induces a Banach topology on the space $C_b(P(E))$.

The metric issue (II)

- The transformations π_C^N and R^N satisfy:

$$\left\|\pi_C^N \Phi\right\|_{L^\infty(E^N)} \leq \|\Phi\|_{L^\infty(P(E))} \text{ and } \|R^N[\phi]\|_{L^\infty(P(E))} \leq \|\phi\|_{L^\infty(E^N)}.$$

- π_C^N is well defined from $C_b(P(E), w)$ to $C_b(E^N)$, but it does not map $C_b(P(E), TV)$ into $C_b(E^N)$ since $V \in E^N \mapsto \mu_V^N \in (P(E), TV)$ is not continuous
- R^N is well defined from $C_b(E^N)$ to $C_b(P(E), w)$, and therefore also from $C_b(E^N)$ to $C_b(P(E), TV)$:
  For any $\phi \in C_b(E^N)$ and for any sequence $f_k \to f$ weakly, we have $f_k^{\otimes N} \to f^{\otimes N}$ weakly, and then $R^N[\phi](f_k) \to R^N[\phi](f)$.

The metric issue (III)

- Different metric structures inducing the weak topology not crucial at the level of $C_b(P(E), w)$. However any differential structure strongly depends on this choice.
- Define weak metrics on $P(E)$ by restricting from larger spaces with foliation by moments constraints (dictacted by conservation laws of the dynamics, and/or relaxation of moments)
- Ex. 1: Dual-Hölder (or Zolotarev's) distances

$$[\varphi]_s := \sup_{x,y\in E} \frac{|\varphi(y) - \varphi(x)|}{dist_E(x,y)^s}, s \in (0,1], \qquad [\varphi]_{Lip} := [\varphi]_1.$$

and

$$\forall\, f, g \in P_1(E), \qquad [g-f]_s^* := \sup_{\varphi\in C_0^{0,s}(E)} \frac{\langle g-f, \varphi\rangle}{[\varphi]_s}.$$

The metric issue (IV)

- Ex. 2: Wasserstein distances

$$\forall f, g \in P_q(E), \quad W_q(f,g)^q := \inf_{\pi \in \Pi(f,g)} \int_{E\times E} \mathrm{dist}_E(x,y)^q \,\mathrm{d}\pi(x,y)$$

- Ex. 3: Fourier-based norms

$$\forall f \in \mathcal{T}P\dots, \quad |f|_s := \sup_{\xi \in \mathbb{R}^d} \frac{|\hat{f}(\xi)|}{|\xi|^s}, \quad s \in \dots$$

- Ex. 4: Negative Sobolev norms for $s \in (d/2, \dots)$:

$$\forall f \in \mathcal{T}P\dots, \quad \|f\|_{\dot{H}^{-s}(\mathbb{R}^d)} := \left\| \frac{\hat{f}(\xi)}{|\xi|^s} \right\|_{L^2}$$

Differential calculus on $C(P(\mathbb{R}^d))$ (I)

- Less than one derivative: $C^{0,\theta}_{\Lambda}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)$ for some metric spaces $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$, some weight function $\Lambda : \tilde{\mathcal{G}}_1 \mapsto \mathbb{R}^*_+$ and some $\theta \in (0,1]$, defined by

$$\forall\, f_1,\ f_2 \in \tilde{\mathcal{G}}_1 \qquad dist_{\tilde{\mathcal{G}}_2}(\mathcal{S}(f_1), \mathcal{S}(f_2)) \le C\, \Lambda(f_1, f_2)\, dist_{\tilde{\mathcal{G}}_1}(f_1, f_2)^{\theta},$$

  with $\Lambda(f_1, f_2) := \max\{\Lambda(f_1), \Lambda(f_2)\}$.
- $C^{1,\theta}_{\Lambda}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2))$ defined as the space of continuous functions from $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$ admitting a second order expansion with a weighted $(1+\theta)$-power control on the second order term
- Nice usual composition rules. . .

Differential calculus on $C(P(\mathbb{R}^d))$ (II)

- Well suited to deal with the different objects we have (1-particle semigroup, polynomial, generators, ...) once restricting to correct "leaf" by fixing moments of measures
- Main novelty is the use of this differential calculus to state differential stability conditions on the limiting semigroup
- Roughly speaking the latters measure how this limiting semigroup handles fluctuations around chaoticity
- Corner stone of our analysis, in particular for uniform in time results, and the hardest part to prove on the limit PDE
- Under appropriate assumptions, allows to make rigorous the heuristic derivation of $(G^\infty \Phi)(f) := \langle D\Phi[f], Q(f)\rangle$

Plan

I. From microscopic to macroscopic evolutions

II. Probabilistic foundation of kinetic theory

III. The main results

IV. The functional framework

V. Sketch of the proof of the abstract stability result

VI. Entropic chaos and relaxation rate

VII. Statistical stability and the BBGKY hierarchy

Heuristic

- Consider a discretization problem in numerical analysis:

$$\partial_t f = \Delta f \quad \longrightarrow \ldots \longrightarrow \quad \partial_t f^N = \Delta_N f^N$$

where $G^N = \Delta_N$ is the discretized version of $G^\infty = \Delta$

- Well-known: convergence = consistency + stability
- Consistency: $\|(\Delta - \Delta_N)g\|_{C^0} \leq CN^{-1}\|g\|_{C^3}\ldots$ loss of derivatives
- Stability: propagation of C^3 regularity for the limit equation
- How to quantify the convergence? Cf. Trotter-Kato...

$$T_t^N - T_t^\infty = \int_0^t T_{t-s}^N [G^N - G^\infty] T_s^\infty \, \mathrm{d}s$$

- "Follow" this heuristic in $C(P(E))\ldots$

Assumptions (I)

(A1) On the N-particle system. Support and moment bounds...

(A2) Existence of the generator of the pullback semigroup.
For a distance $\text{dist}_{\mathcal{G}_1}$ on $\mathcal{P}_{\mathcal{G}_1}(E)$ (with some constraints):
continuity of S_t^{NL} and C^{1+0} in time, and Q Hölder...

(A3) Convergence of the generators.

$$\left\| \left(M^N_{m_1} \right)^{-1} \left(G^N \, \pi^N_E - \pi^N_E \, G^\infty \right) \, \Phi \right\|_{L^\infty(\mathbb{E}_N)} \leq \varepsilon(N) \sup_{\mathbf{r} \in \mathcal{R}} [\Phi]_{C^{1,\eta}_{\Lambda_1}(P_{\mathcal{G}_1,\mathbf{r}})}$$

where the constraints $\mathbf{r} \in \mathcal{R}$ fix mass and energy (reflecting in $\mathbb{E}_N$) and $M^N_m = \frac{1}{N} \sum_{i=1}^N m(v_i)$ and $\Lambda_1(f) = \langle f, m_1 \rangle$

Assumptions (II)

(A4) Differential stability of the limiting semigroup.

$$\int_0^T \left([S_t^{NL}]_{C^{1,\eta}_{\Lambda_2}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2})} + [S_t^{NL}]^2_{C^{0,(1+\eta)/2}_{\Lambda_2}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2})} \right) dt \leq C_T^\infty$$

where $\eta \in (0,1)$ is the same as in **(A3)**, $\Lambda_2 = \Lambda_1^{1/2}$ and $\mathcal{G}_2 \supset \mathcal{G}_1$ are some normed spaces

(A5) Continuity stability of the limiting semigroup.
For $P_{\mathcal{G}_3}$ (with weight and constraints) and any $T > 0$ there exists a concave function $\Theta_T : \mathbb{R}_+ \to \mathbb{R}_+$ s.t.

$$\forall\, f_1, f_2 \in P_{\mathcal{G}_3}$$

$$\sup_{[0,T)} \mathrm{dist}_{\mathcal{G}_3} \left(S_t^{NL}(f_1), S_t^{NL}(f_2) \right) \leq \Theta_T \left(\mathrm{dist}_{\mathcal{G}_3}(f_1, f_2) \right)$$

Statement

Theorem

For any $N, \ell \in \mathbb{N}^*$, with $N \geq 2\ell$, and $\varphi \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3)^{\otimes \ell}$ (where $\mathcal{F}_i$ are the dual of $\mathcal{G}_i$)

$$\sup_{[0,T)} \left| \left\langle \left(S_t^N(f_0^N) - \left(S_t^{NL}(f_0) \right)^{\otimes N} \right), \varphi \right\rangle \right|$$
$$\leq C \left[\ell^2 \frac{\|\varphi\|_\infty}{N} + C_{T,m_1}^N \, C_T^\infty \, \varepsilon_2(N) \, \ell^2 \, \|\varphi\|_{\mathcal{F}_2^2 \otimes (L^\infty)^{\ell-2}} \right.$$
$$\left. + \ell \, \|\varphi\|_{\mathcal{F}_3 \otimes (L^\infty)^{\ell-1}} \, \Theta_T \left(\mathcal{W}_{1,P_{\mathcal{G}_3}} \left(\pi_P^N f_0^N, \delta_{f_0} \right) \right) \right],$$

where $\mathcal{W}_{1,P_{\mathcal{G}_3}}$ is the Monge-Kantorovich distance in $P(P_{\mathcal{G}_3}(E))$:

$$\mathcal{W}_{1,P_{\mathcal{G}_3}} \left(\pi_P^N f_0^N, \delta_{f_0} \right) = \int_{E^N} \mathrm{dist}_{\mathcal{G}_3}(\mu_V^N, f_0) \, \mathrm{d} f_0^N(V)$$

Comments on the statement

- Assume furthermore $(f_0^N)_{N\geq 1}$ is f_0-chaotic, i.e. $\mathcal{W}_{\theta_3, P_{\mathcal{G}_3}}(\pi_P^N f_0^N, \delta_{f_0}) \to 0$, then f_t^N is f_t-chaotic in the quantified way above (chaos propagation)
- Treatment of the N-particles system as a perturbation (in a very degenerated sense) of the limiting problem, minimize assumptions on the many-particle systems in order to avoid complications of many dimensions dynamics.
- In the applications worst decay rate in the right-hand side is always the last one, which deals with the chaoticity of the initial data (law of large number in probability space)
- Fluctuations estimates explicit in terms of the constant in the assumptions, therefore if these constants are uniform in times, so are the chaos propagation estimates

Scheme of the proof (I)

For $\varphi \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3)^{\otimes \ell}$, break up the term to be estimated into three parts:

$$\left| \left\langle \left(S_t^N(f_0^N) - (S_t^\infty(f_0))^{\otimes N} \right), \varphi \otimes 1^{\otimes N-\ell} \right\rangle \right| \leq$$

$$\leq \left| \left\langle S_t^N(f_0^N), \varphi \otimes 1^{\otimes N-\ell} \right\rangle - \left\langle S_t^N(f_0^N), R_\varphi^\ell \circ \mu_V^N \right\rangle \right| \qquad (=: \mathcal{T}_1)$$

$$+ \left| \left\langle f_0^N, T_t^N(R_\varphi^\ell \circ \mu_V^N) \right\rangle - \left\langle f_0^N, (T_t^\infty R_\varphi^\ell) \circ \mu_V^N) \right\rangle \right| \qquad (=: \mathcal{T}_2)$$

$$+ \left| \left\langle f_0^N, (T_t^\infty R_\varphi^\ell) \circ \mu_V^N) \right\rangle - \left\langle (S_t^\infty(f_0))^{\otimes \ell}, \varphi \right\rangle \right| \qquad (=: \mathcal{T}_3)$$

Scheme of the proof (II)

We deal separately with each part step by step:

- $\mathcal{T}_1$ controlled by a classical purely combinatorial arguments [In some sense price to pay for using the injection π_E^N based on empirical measures]: $\pi_C^N \circ R_\varphi^N \sim \varphi$, $N \to \infty$
- $\mathcal{T}_2$ controlled thanks to the consistency estimate **(A3)** on the generators, the differential stability assumption **(A4)** on the limiting semigroup and the moments propagation **(A1)**
- $\mathcal{T}_3$ controlled in terms of the chaoticity of the initial data, which is propagated thanks to the weak stability assumption **(A5)** on the limiting semigroup (and support controls in **(A1)**)

Step 1: Estimate of the first term $\mathcal{T}_1$

Let us prove that for any $t \geq 0$ and any $N \geq 2\ell$ there holds

$$\mathcal{T}_1 := \left| \left\langle S_t^N(f_0^N), \varphi \otimes 1^{\otimes N-\ell} \right\rangle - \left\langle S_t^N(f_0^N), R_\varphi^\ell \circ \mu_V^N \right\rangle \right| \leq \frac{2\,\ell^2\,\|\varphi\|_{L^\infty(E^\ell)}}{N}.$$

Since $S_t^N(f_0^N)$ is a symmetric probability measure, consequence of:

Lemma

$$\forall\, N \geq 2\ell, \qquad \left| \left(\varphi \otimes \mathbf{1}^{\otimes N-\ell} \right)_{sym} - \pi_N R_\varphi^\ell \right| \leq \frac{2\,\ell^2\,\|\varphi\|_{L^\infty(E^\ell)}}{N}$$

and for any symmetric measure $f^N \in P(E^N)$ we have

$$\left| R_\varphi^\ell(\mu_V^N)\rangle - \langle f^N, \varphi \rangle \right| \leq \frac{2\,\ell^2\,\|\varphi\|_{L^\infty(E^\ell)}}{N}$$

Step 2: Estimate of the second term $\mathcal{T}_2$ (I)

Let us prove that for any $t \in [0, T)$ and any $N \geq 2\ell$ there holds

$$\begin{aligned} \mathcal{T}_2 &:= \left| \left\langle f_0^N, T_t^N \left(R_\varphi^\ell \circ \mu_V^N \right) \right\rangle - \left\langle f_0^N, \left(T_t^\infty R_\varphi^\ell \right) \circ \mu_V^N \right\rangle \right| \\ &\leq C_{T,m_2}^N \, C_T^\infty \, \|\varphi\|_{\infty, \mathcal{F}_2^2 \otimes (L^\infty)^{\ell-2}} \, \ell^2 \, \varepsilon(N). \end{aligned}$$

We start from the following identity (cf. Trotter-Kato)

$$T_t^N \pi_N - \pi_N T_t^\infty = - \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \left(T_{t-s}^N \, \pi_N \, T_s^\infty \right) \mathrm{d}s$$

$$= \int_0^t T_{t-s}^N \left[G^N \pi_N - \pi_N G^\infty \right] T_s^\infty \, \mathrm{d}s$$

Step 2: Estimate of the second term $\mathcal{T}_2$ (II)

From assumptions **(A1)** and **(A3)**, we have for any $t \in [0, T)$

$$\left| \left\langle f_0^N, T_t^N \left(R_\varphi^\ell \circ \mu_V^N \right) \right\rangle - \left\langle f_0^N, \left(T_t^\infty R_\varphi^\ell \right) \circ \mu_V^N \right\rangle \right|$$

$$\leq \int_0^T \left| \left\langle M_{m_1}^N \, S_{t-s}^N \left(f_0^N \right), \left(M_{m_1}^N \right)^{-1} \left[G^N \pi_N - \pi_N G^\infty \right] \left(T_s^\infty R_\varphi^\ell \right) \right\rangle \right| ds$$

$$\leq \left(\sup_{0 \leq t < T} \left\langle f_t^N, M_{m_1}^N \right\rangle \right) \left(\int_0^T \left\| \left(M_{m_1}^N \right)^{-1} \left[G^N \pi_N - \pi_N G^\infty \right] \left(T_s^\infty R_\varphi^\ell \right) \right\|_\infty ds \right)$$

$$\leq \varepsilon(N) \, C_{T,m}^N \int_0^T \left[T_s^\infty R_\varphi^\ell \right]_{C_{\Lambda_1}^{1,\theta}(P_{\mathcal{G}_1})} ds$$

with

$$\left[T_s^\infty \left(R_\varphi^\ell \right) \right]_{C_{\Lambda_2^2}^{1,\theta}(P_{\mathcal{G}_2})} \leq \left[S_t^{NL} \right]_{C_{\Lambda_2}^{1,\theta}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2})} \left\| R_\varphi^\ell \right\|_{C^{1,\theta}(P_{\mathcal{G}_2})}$$

Step 3: Estimate of the third term $\mathcal{T}_3$

Let us prove that for any $t \geq 0$, $N \geq \ell$

$$\mathcal{T}_3 := \left| \left\langle f_0^N, \left(T_t^\infty R_\varphi^\ell \right) \circ \mu_V^N \right\rangle - \left\langle \left(S_t^\infty(f_0) \right)^{\otimes \ell}, \varphi \right\rangle \right| \leq$$

$$\leq [R_\varphi]_{C^{0,1}} \; \Theta_T \left(\mathcal{W}_{1, P_{\mathcal{G}_3}} \left(\pi_P^N p_0^N, \delta_{f_0} \right) \right).$$

We compute using **(A5)** and support assumptions in **(A1)**:

$$\begin{aligned}
\mathcal{T}_3 &= \left| \left\langle f_0^N, R_\varphi^\ell \left(S_t^{NL}(\mu_V^N) \right) \right\rangle - \left\langle f_0^N, R_\varphi^\ell \left(S_t^{NL}(f_0) \right) \right\rangle \right| \\
&= \left| \left\langle f_0^N, R_\varphi^\ell \left(S_t^{NL}(\mu_V^N) \right) - R_\varphi^\ell \left(S_t^{NL}(f_0) \right) \right\rangle \right| \\
&\leq \left[R_\varphi \right]_{C^{0,1}(P_{\mathcal{G}_3})} \left\langle f_0^N, \, \mathrm{dist}_{\mathcal{G}_3} \left(S_t^{NL}(f_0), S_t^{NL}(\mu_V^N) \right) \right\rangle \\
&\leq \left[R_\varphi \right]_{C^{0,1}(P_{\mathcal{G}_3})} \left\langle f_0^N, \, \Theta_T \left(\mathrm{dist}_{\mathcal{G}_3}(f_0, \mu_V^N) \right) \right\rangle \\
&\leq \left[R_\varphi \right]_{C^{0,1}(P_{\mathcal{G}_3})} \Theta_T \left(\left\langle f_0^N, \, \mathrm{dist}_{\mathcal{G}_3}(f_0, \mu_V^N) \right\rangle \right. \\
&\leq \left[R_\varphi \right]_{C^{0,1}(P_{\mathcal{G}_3})} \Theta_T \left(\mathcal{W}_{1, P_{\mathcal{G}_3}} \left(\pi_P^N f_0^N, \delta_{f_0} \right) \right)
\end{aligned}$$

Application to the Boltzmann equation (I)

- In order to apply the abstract problem to a nonlinear PDE: establish the stability estimates **(A4)** (differentiability of the flow) and **(A5)** (Hölder stability of the flow)
  $\rightarrow$ choose correct metrics for each
- Metric chosen for **(A4)** impacts and constrained by consistency estimate **(A3)**
  $\rightarrow$ total variation metric for hard spheres
  $\rightarrow$ Fourier-based weak metric for Maxwell molecules
- **(A5)** was proved recently for hard spheres Fournier-CM 2009

Application to the Boltzmann equation (II)

On the differentiability estimate **(A4)**: Two solutions f_t and g_t and

$$h_t := \mathcal{D}_t^{NL}[f_0](g_0 - f_0)$$

the solution to the linearized equation around f_t:

$$\begin{cases} \partial_t f_t = Q(f_t, f_t), & f_{|t=0} = f_0 \\ \partial_t g_t = Q(g_t, g_t), & g_{|t=0} = g_0 \\ \partial_t h_t = 2\, Q(h_t, f_t), & h_{|t=0} = h_0 := g_0 - f_0. \end{cases}$$

Estimates on the expansion of the nonlinear limit semigroup in a scale of spaces

$$|f_t - g_t|_2 \le C_\eta \, e^{-(1-\eta)\,\lambda\, t} \, M(f_0 + g_0)^{\frac{1}{2}} \, |f_0 - g_0|_2^\eta ,$$

$$|h_t|_2 \le C_\eta \, e^{-(1-\eta)\,\lambda\, t} \, M(f_0 + g_0)^{\frac{1}{2}} \, |f_0 - g_0|_2^\eta ,$$

$$|\omega_t|_4 \le C \, e^{-(1-\eta)\,\lambda\, t} \, M(f_0 + g_0)^{\frac{1}{2}} \, \big|g_0 - f_0\big|_2^{1+\eta}$$

where $\omega_t := g_t - f_t - h_t$ and some (admissible) weight M

Application to the Boltzmann equation (III)

- Making these estimates uniform in time requires more work, and relies on the most recent a priori estimates on homogeneous Boltzmann equation:
  - appearance of exponential moments for hard spheres by Mischler-CM 2006
  - contraction in higher-order Fourier-based distance metric for Maxwell molecules by Carlen-Gabetta-Toscani, Carrillo et al.
  - interpolation with the estimates of exponential relaxation in P_2 for hard spheres CM 2006
- Question of the optimal rate: dictated by LLN/CLT at initial time, non-trivial pb in general in Banach setting (sampling of a distribution by empirical measures)

Plan

From finite-dimensional to infinite-dimensional chaos

- Recent result Hauray-Mischler: for any $f \in P(\mathbb{R}^d)$ and sequence $f^N \in P_{\text{sym}}(\mathbb{R}^d)$ we have

$$\frac{W_1\left(f^N, f^{\otimes N}\right)}{N} \leq C \left(W_1\left(\Pi_2\left[f^N\right], f^{\otimes 2}\right)^{\alpha_1} + \frac{1}{N^{\alpha_2}} \right)$$

  for some constructive constant C, α_1, $\alpha_2 > 0$.

- Idea of the proof: pass "through" a Hilbert negative Sobolev setting in order to make use of cancellations, and pass "through" generalized Wasserstein distance in $P(P(E))$, and estimate error (change of norm, combinatorial)
- Morally: the 2-particle correlation measure is enough to control the N-particle correlation measure once correctly scaled (extensivity)

Propagation of entropic chaos (I)

- Maxwell molecules (with or without cutoff) or hard spheres
- Initial data f with exponential moment bounds
- Sequence of N-particle initial data $(f_0^N)_{N\geq 1}$ constructed by conditioning to $\mathcal{S}^N$.
- Then if the initial data is entropy-chaotic in the sense

$$\frac{1}{N} H\left(f_0^N|\gamma^N\right) \xrightarrow{N\to+\infty} H\left(f_0|\gamma\right)$$

with

$$H\left(f_0^N\right) := \int_{\mathcal{S}^N} f_0^N \log \frac{f_0^N}{\gamma^N} \,\mathrm{d}V$$

the solution is also entropy chaotic for any later time:

$$\forall\, t \geq 0, \quad \frac{1}{N} H\left(f_t^N|\gamma^N\right) \xrightarrow{N\to+\infty} H\left(f_t|\gamma\right).$$

- Derivation of the H-theorem

Propagation of entropic chaos (II)

Sketch of the proof

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{N}H\left(f_t^N|\gamma^N\right) = -D^N\left(f_t^N|\gamma^N\right)$$

$$D^N\left(f^N\right) := \frac{1}{2N^2}\int_{\mathcal{S}^N}\sum_{i\neq j}\int_{\mathbb{S}^{d-1}}\left(f^N(r_{ij,\sigma}(V)) - f^N(V)\right)\log\frac{f^N(r_{ij,\sigma}(V))}{f^N(V)}\,B$$

Hence

$$\forall\, t\geq 0,\quad \frac{1}{N}H\left(f_t^N|\gamma^N\right) + \int_0^t D^N\left(f_s^N\right)\,\mathrm{d}s = \frac{1}{N}H\left(f_0^N|\gamma^N\right)$$

and at the limit

$$\forall\, t\geq 0,\quad H\left(f_t|\gamma\right) + \int_0^t D^\infty\left(f_s\right)\,\mathrm{d}s = H\left(f_0|\gamma\right)$$

Propagation of entropic chaos (III)

Sketch of the proof

Then prove that the many-particle relative entropy and entropy production functionals defined above are lower semi-continuous in $P(P(\mathbb{R}^d))$ in terms of f^N: if $(f^N)_{N\geq 1}$ is f-chaotic then (known)

$$\liminf_{N\to\infty} \frac{1}{N} H\left(f^N|\gamma^N\right) \geq H(f|\gamma) \geq 0$$

and ($\sim$new)

$$\liminf_{N\to\infty} D^N\left(f^N\right) \geq D^\infty(f) \geq 0$$

If we assume furthermore that

$$\frac{1}{N} H\left(f_0^N|\gamma^N\right) \to H\left(f_0|\gamma\right)$$

at initial time, then it implies the convergence of the functionals at time t and concludes the proof

Many-particle relaxation time (I)

Maxwell molecules or hard spheres and conditionned initial data.

$$\forall\, N \geq 1,\ \forall\, t \geq 0, \quad \frac{W_1\left(f_t^N, \gamma^N\right)}{N} \leq \beta(t)$$

for some $\beta(t) \to 0$ as $t \to \infty$, where γ gaussian equilibrium with energy $\mathcal{E}$ and γ^N uniform probability measure on $\mathcal{S}^N(\sqrt{N\mathcal{E}})$

Sketch of the proof: "interpolation between chaos and relaxation of the limit equation"

$$\frac{W_1\left(f_t^N, \gamma^N\right)}{N} \leq \frac{W_1\left(f_t^N, f_t^{\otimes N}\right)}{N} + \frac{W_1\left(f_t^{\otimes N}, \gamma^{\otimes N}\right)}{N} + \frac{W_1\left(\gamma^{\otimes N}, \gamma^N\right)}{N}$$

Many-particle relaxation time (II)

Sketch of the proof - bis

Then

$$\forall\, t \geq 0, \quad \frac{W_1\left(f_t^N, f^{\otimes N}\right)}{N} + \frac{W_1\left(\gamma^{\otimes N}, \gamma^N\right)}{N} \leq \alpha(N)$$

and

$$\frac{W_1\left(f_t^{\otimes N}, \gamma^{\otimes N}\right)}{N} \leq W_1\left(f_t, \gamma\right)$$

which implies by CM 2006

$$\frac{W_1\left(f_t^N, \gamma^N\right)}{N} \leq \alpha(N) + Ce^{-\lambda_1 t}$$

From the L^2 spectral gap estimate in Carlen-Geronimo-Loss and Carlen-Carvalho-Loss one can deduce

$$\forall\, N \geq 1,\ \forall\, t \geq 0, \quad \frac{W_1\left(f_t^N, \gamma^N\right)}{N} \leq C^N\, e^{-\lambda_2\, t}$$

Conclusion by optimization of $\max\{\alpha(N) + C_1 e^{-\lambda t}; C_2^N e^{-\lambda_2 t}\}$

Many-particle relaxation rate in the H-theorem (I)

In the case of Maxwell molecules, and assuming moreover that the Fisher information of the initial data f_0 is finite:

$$\int_{\mathbb{R}^d} \frac{|\nabla_v f_0|^2}{f_0} \, dv < +\infty,$$

the following estimate on the relaxation induced by the H-theorem *uniformly in the number of particles* also holds:

$$\forall\, N \geq 1, \quad \frac{1}{N} H\left(f_t^N | \gamma^N\right) \leq \beta(t)$$

for some polynomial function $\beta(t) \to 0$ as $t \to \infty$

Many-particle relaxation rate in the H-theorem (II)

Sketch of the proof

First prove propagation of the Fisher information

$$\forall\, t \geq 0, \quad \frac{I\left(f_t^N|\gamma^N\right)}{N} \leq \frac{I\left(f_0^N|\gamma^N\right)}{N}$$

with

$$I\left(f^N|\gamma^N\right) := \int_{\mathcal{S}^N} \frac{\left|\nabla f^N\right|^2}{f^N}\,\mathrm{d}V$$

then use the HWI interpolation inequality on the manifold $\mathcal{S}^N$

$$\begin{aligned}\frac{1}{N}\,H\left(f^N|\gamma^N\right) &\leq \frac{W_2\left(f^N,\gamma^N\right)}{\sqrt{N}}\sqrt{\frac{I\left(f^N|\gamma^N\right)}{N}} - \frac{K}{2N}\,W_2\left(f^N,\gamma^N\right)^2 \\ &\leq \frac{W_2\left(f^N,\gamma^N\right)}{\sqrt{N}}\sqrt{\frac{I\left(f^N|\gamma^N\right)}{N}}.\end{aligned}$$

Plan

I. From microscopic to macroscopic evolutions

II. Probabilistic foundation of kinetic theory

III. The main results

IV. The functional framework

V. Sketch of the proof of the abstract stability result

VI. Entropic chaos and relaxation rate

VII. Comments and perspectives

Statistical stability

- In Braun-Hepp-Dobrushin theory of mean-field limit Lipschitz estimate in W_1 on S_t^∞
- Here crucial point is higher than C^1 differentiability of the flow in terms of the initial data
- Measure distance in order to handle empirical measure, but possible to use strong spaces with mollification and interpolation (w/ Marahrens)
- E.g.: differentiability C^2 of S_t^∞ in terms of initial data $\Leftrightarrow$ propagation of regularity C^2 of the pullback semigroup $T_t^\infty \Leftrightarrow$ propagation of "negative" regularity C^{-2} for the statistical flow $(T_t^\infty)^*$ on $P(P(E))$
- Statistical stability: controls fluctuations in perturbation of T_t^∞ by T_t^N around chaos

Connection to the BBGKY hierarchy

Possible to reframe this theory in the framework of BBGKY hierarchy: correct assumption in order to prove quantitative estimates of stability on the BBGKY hierarchy (cf. Spohn 1981)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle f_\ell^N, \varphi \right\rangle = \left\langle f_{\ell+1}^N, G_{\ell+1}^N(\varphi) \right\rangle$$

$f_{t,\ell}^N \rightharpoonup \pi_{t,\ell}$ in $P(E^\ell) \Longrightarrow$ (Hewitt-Savage's theorem) $\pi \in P(P(E))$

$$\text{Statistical solutions} \quad \partial_t \pi = A^\infty(\pi) \;\; \text{on} \;\; P\left(P\left(\mathbb{R}^d\right)\right)$$

$$\text{Dual evolution} \quad \partial_t \Phi = G^\infty \Phi \;\; \text{on} \;\; C_b\left(P\left(\mathbb{R}^d\right)\right)$$

Theorem

Under the previous assumptions, these evolution problems are well-posed, they propagates chaos (Dirac mass structure) and the evolution of Φ is provided by the pullback semigroup T_t^∞ previously constructed

Going back to the probabilistic interpretation

- In negative Sobolev space distance and on finite time intervals optimal rate of CLT for Maxwell molecules

$$\sup_{t \in [0,T]} \operatorname{dist}\left(\Pi_\ell f_t^N, f^{\otimes \ell}\right) \leq \frac{K_{\ell,T}}{N^{1/2}}$$

- We prove quantitative LLN in $P(P(E))$, i.e. deviation estimates in $P(P(E))$
- Possible to obtain CLT in $P(P(E))$ ("gaussian" = solution to the linearized flow + Ornstein-Uhlenbeck noise)
- Large deviation?
- How uniform in time convergence? Stochastic trajectories departs from deterministic trajectories like their variance...
- For Brownian motion (diffusion): time-scale $O(\sqrt{N})$...
- Here at the level of the laws: ergodicity time-scale wins over time-scale of the effect of trajectories fluctuations

Related works and perspectives

- With Mischler and Wennberg: other applications to inelastic collisions, Fokker-Planck, jump + diffusion, Vlasov and McKean-Vlasov (with regular interaction potential)...
- In progress with Mischler: new Liapunov function for some inelastic collision operator plus diffusion by mean-field limit? [Cf. Original goal of Kac: recover new information on the limiting equation from the many-particle Markov process]
- Bodineau-Lebowitz-CM-Villani: "top-down" strategy inspiration for constructing new relative entropies for nonlinear diffusion with inhomogeneous Dirichlet conditions
- Hydrodynamical limit for the zero-range process with quantitative estimates w/ Marahrens