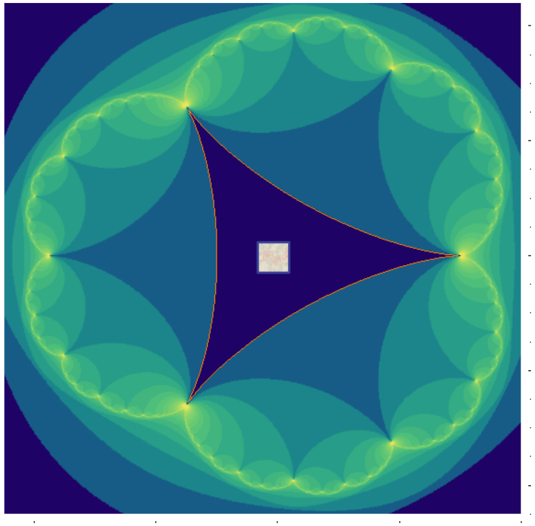
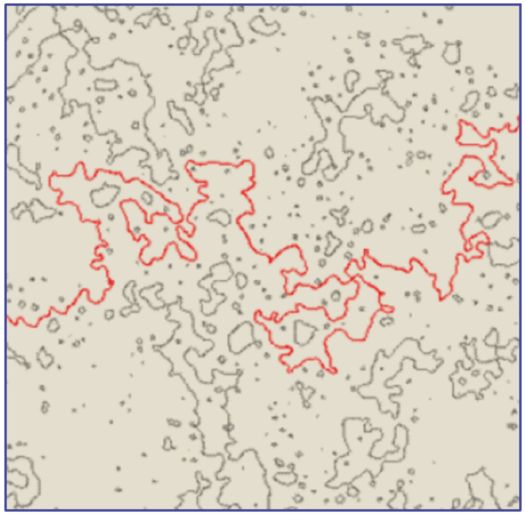
Eigenvalues of random normal matrices





Random normal matrix model:

probability on $n \times n$ normal matrices

$$M^*M = MM^*$$

Partition function:

$$Z_n = \int e^{-n \ \mathrm{Trace} Q(M)} dM,$$

i.e.

$$\mathrm{Prob}(M) \sim e^{-n \ \mathrm{Trace} Q(M)} \quad \text{(infinitesimaly)}.$$

Here

$$Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$$

is a given function ("potential")

Eigenvalues:

$$\{z_j\}_{j=1}^{n}$$

(n electrons in external field nQ).

As $n \to \infty$,

$$\frac{1}{n}\sum \delta_{\lambda_j} \to \hat{\sigma}.$$

Equilibrium measure $\hat{\sigma}$ minimizes Q-energy of a unit electric charge,

$$\mathcal{E}_Q[\sigma] = \int\int \log\frac{1}{|z-w|}d\sigma(z)d\sigma(w) + \int Q d\sigma.$$

Hele-Shaw time

$$t \mapsto \hat{\sigma}_t, \qquad (t > 0),$$

where $\hat{\sigma}_t$ is equilibrium measure of mass t,

$$\hat{\sigma}_t \to \min\{\mathcal{E}_Q[\sigma] : \sigma > 0, \|\sigma\| = t\}.$$

Existence, uniqueness if

- $Q(z) \gg \log|z|$ as $z \to \infty$
- Q is lower semi-continuous

(e.g., Q continuous on a closed set and $+\infty$ elsewhere)

Droplets

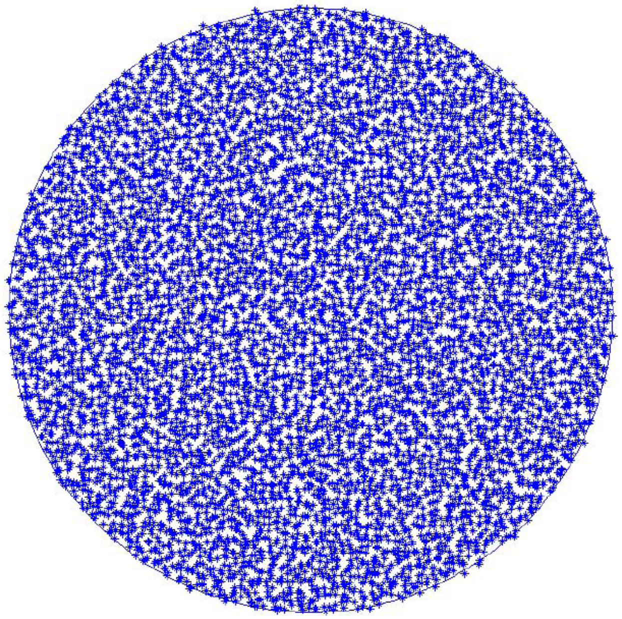
$$\boxed{S_t[Q] := \text{supp } \hat{\sigma}_t}$$

Theorem. If Q is smooth in some nbh of S_t, then

$$d\hat{\sigma}_t = \frac{1}{2\pi}\Delta Q \cdot 1_{S_t} \cdot dA$$

Example:

$$Q(z) = |z|^2$$

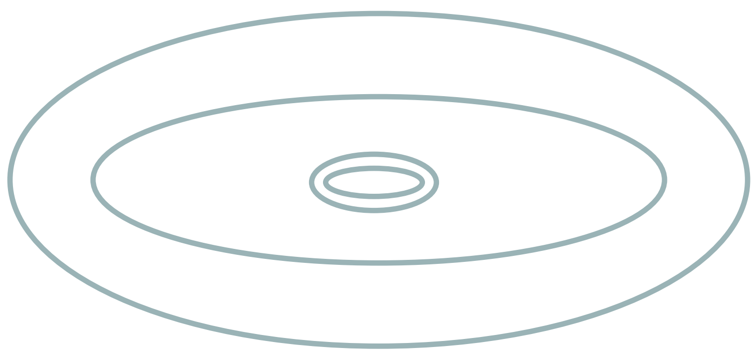


Algebraic (Abelian) potentials

$$Q(z) = |z|^2 - H(z)$$

$h := \frac{\partial H}{\partial z}$ is a rational function
("quadrature function" of Q)

Example:
$h(z) = az + b$ with $|a| < 1$



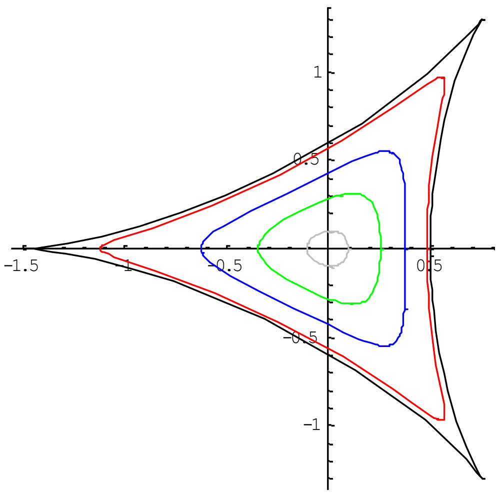
concentric ellipses

(if $|a| \geq 1$ then no equilibria)

Local droplets

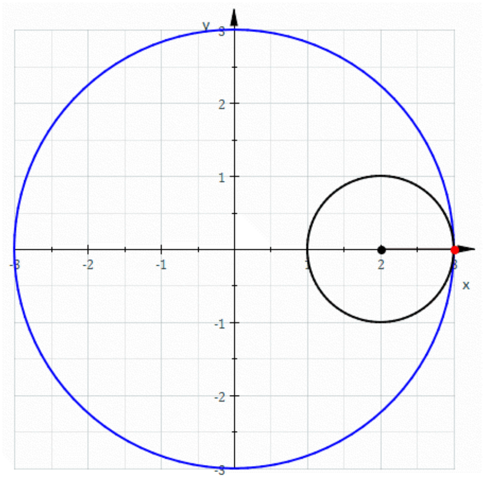
$$h(z) = z^2$$

[Wiegmann-Zabrodin]



hypotrochoids

deltoid

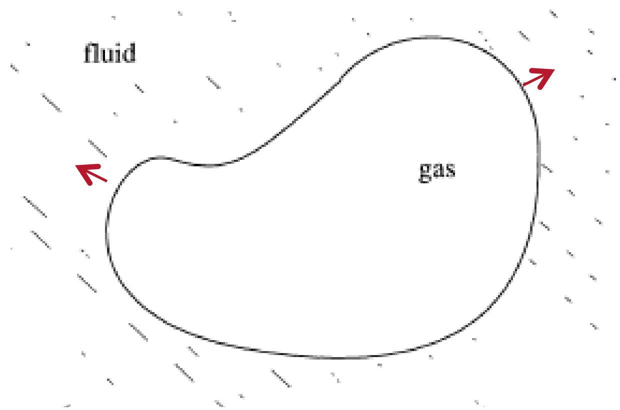


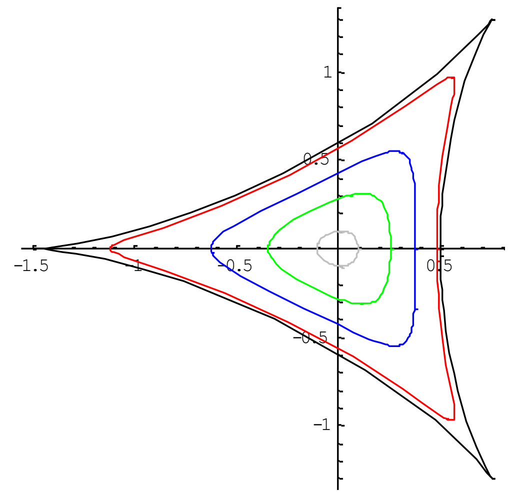
Definition: K is a local droplet if

$1_K \cdot dA$ is equilibrium measure of

$$Q_K := Q \cdot 1_K + \infty \cdot 1_{K^c}$$

Hele-Shaw flow





If $\{K_t\}_{0<t<T}$ is a monotone family of local Q-droplets, $t = A(K_t)$, then it's a generalized solution of Hele-Shaw equation

$$\text{normal velocity} = 2\pi\nabla G(\cdot, \infty).$$

First result

Theorem. If $h(z) = z^2$ (quadrature function of Q), then for all $t \leq T$ there is exactly one local droplet of area t; T is the area of deltoid.

Proof: show local droplets are connected; then use classical Hele-Shaw

Related statement in inverse problem: if two domains $\Omega_1, \Omega_2 \subset \hat{\mathbb{C}}$ satisfy

$$\Omega = \text{clos int } \Omega, \quad \infty \in \Omega, \quad 0 \notin \text{clos } \Omega$$

and have same moments

$$m_0 = A(\Omega^c), \quad m_j = \int_\Omega z^{-k} \, dA(z),$$

and if $m_j = 0$ for all $j \geq 4$, then $\Omega_1 = \Omega_2$

Schwarz function

Definition. Let Ω be a domain in $\hat{\mathbb{C}}$, $\infty \notin \partial\Omega$. Then

$$S : \mathrm{clos}(\Omega) \to \hat{\mathbb{C}}$$

is the Schwarz function of Ω is S is continuous, meromorphic in Ω, and

$$S(z) = \bar{z} \qquad \text{on} \quad \partial\Omega.$$

Meaning:
$\bar{S}$ is like reflection near the boundary and like (anti-)rational function away from $\partial\Omega$

S-function and local droplets

If K is a local droplet, then every complementary component has S-function,

$$S(z) = h(z) - C^K.$$

$C^K = \partial U^K$; U^K is logarithmic potential of 1_K, and C^K is Cauchy integral.

$$U^K + Q = \text{const} \quad \text{on } K$$

so if $Q(z) = |z|^2 - H(z)$, then $C^K + \bar{z} - h(z) = 0$ on ∂K

Dynamics of S-function in case $h(z) = z^2$

$$\bar{S} : \Omega \to \hat{\mathbb{C}}, \qquad D := \bar{S}^{-1}\Omega$$

Proper maps

$$\bar{S} : D \xrightarrow{2} \Omega, \quad \bar{S} \ : \mathrm{clos}\, \Omega \setminus D \xrightarrow{3} K$$

Two pieces of dynamics:

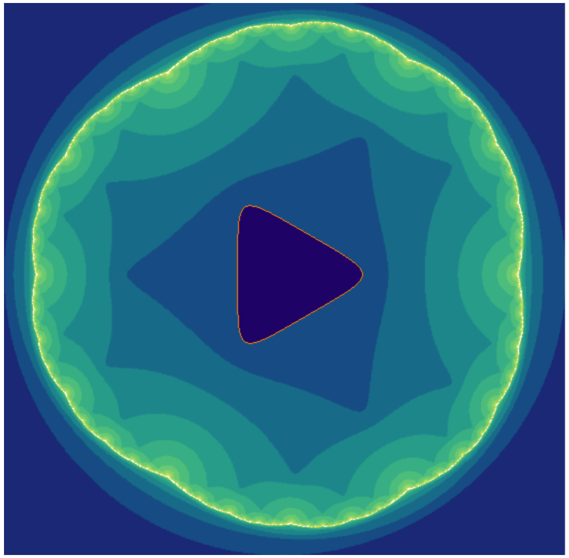
$$\{z : \ z, \ \bar{S}(z), \ \bar{S}^2(z), \ \ldots \ \in \Omega\}$$

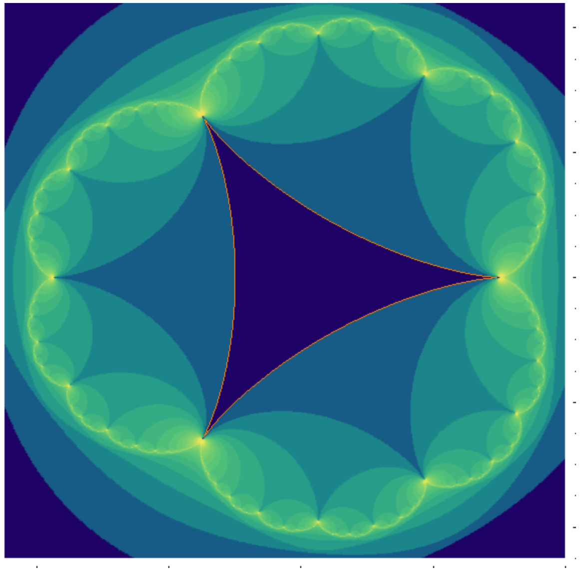
and

$$\bigcup_{n \geq 0} \bar{S}^{-n} K$$

The two pieces meet along "Julia set" J

Two pieces of the dynamics





Quasiconformal surgery

$\bar{S} : D \to \Omega$ extends to a 2-cover

$$\bar{R} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

such that

- $\bar{R}$ has attracting fixedpoints in each component of K
- R is topologically (and q.c. if no singularities) conjugate to a quadratic polynomial

Conclusion: K is connected

?-function as conformal welding (mating)

$$0 \qquad \frac{1}{2} \qquad 1$$

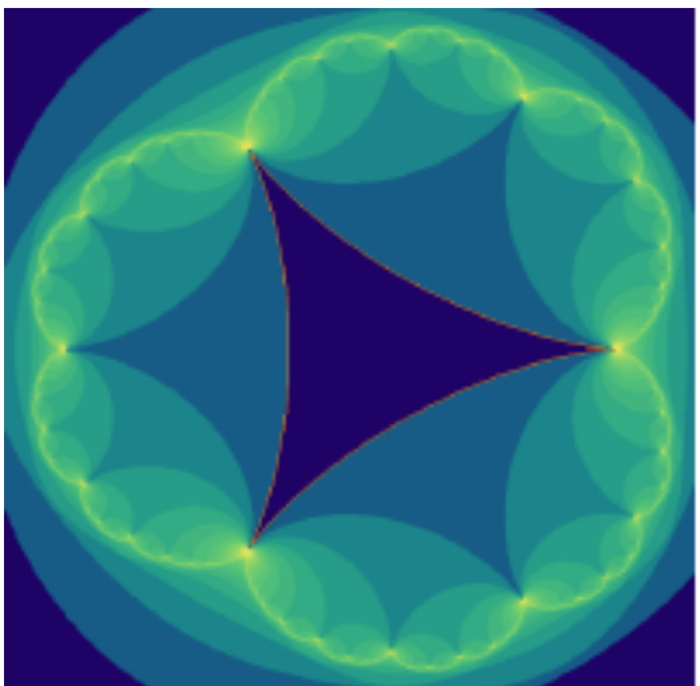
$$\frac{1}{4} \qquad \frac{3}{4}$$

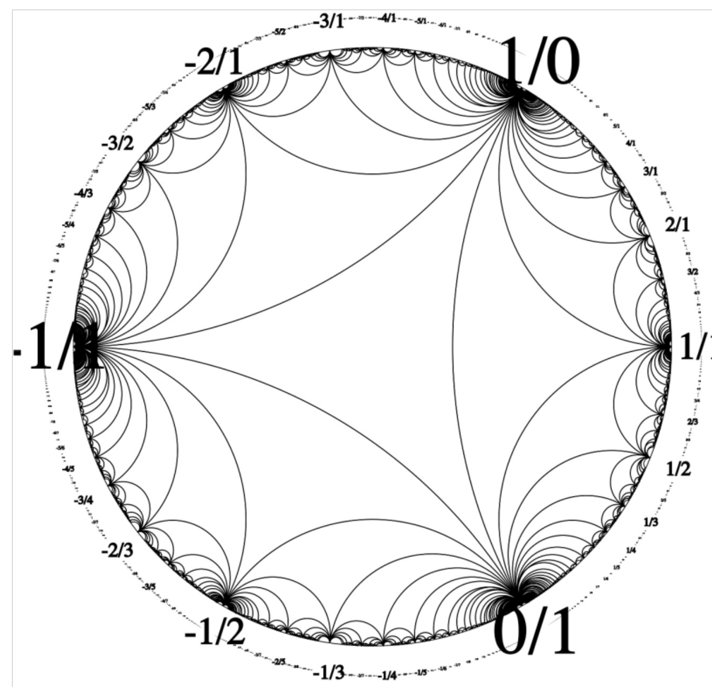
$$\frac{1}{8} \qquad \frac{3}{8} \qquad \frac{5}{8} \qquad \frac{7}{8}$$

$$\frac{0}{1} \qquad \frac{1}{2} \qquad \frac{1}{1}$$

$$\frac{1}{3} \qquad \frac{2}{3}$$

$$\frac{1}{4} \qquad \frac{2}{5} \qquad \frac{3}{5} \qquad \frac{3}{4}$$





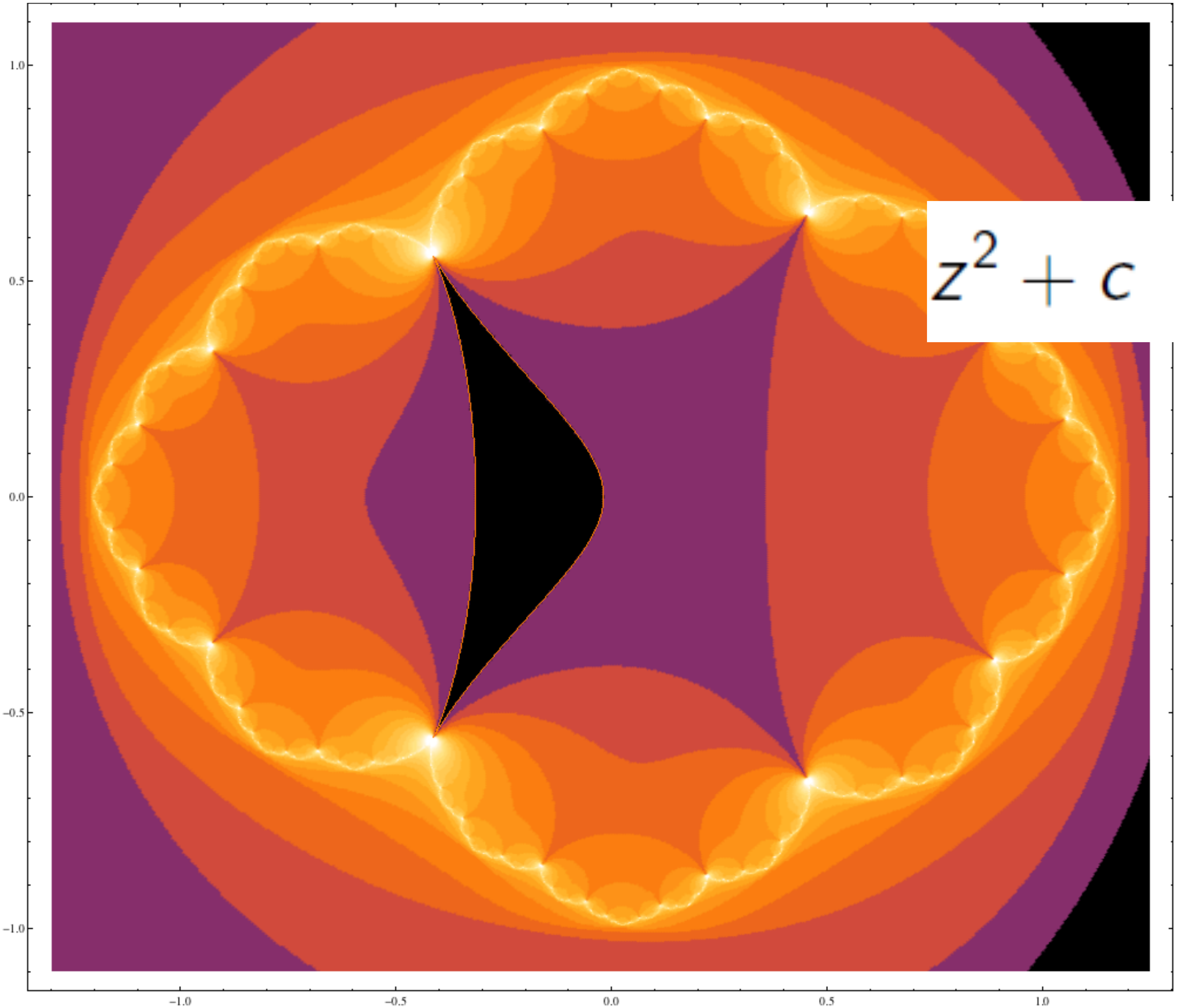
Deltoid group

Notation:

- T circular (∞, ∞, ∞)-triangle, $G \subset \mathrm{Aut}^{-}(\mathbb{D})$ is its reflection group
- K is the deltoid, $\bar{S}^{-1}\big|_K = \{R_1, R_2, R_3\}$

We have

- conformal map $f : T \to K$ extends analytically to $\mathbb{D}$
- f is univalent in $\mathbb{D}$, and $U = f(\mathbb{D})$ is John
- R_j's extend to automorphisms of U
- f conjugates G and $\langle R_1, R_2, R_2 \rangle$
- conformal map $\hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}} \setminus U$ conjugates $z \mapsto \bar{z}^2$ and $\bar{S}$
- conformal welding is given by the ?-function


1.0
0.5
0.0
−0.5
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−1.0
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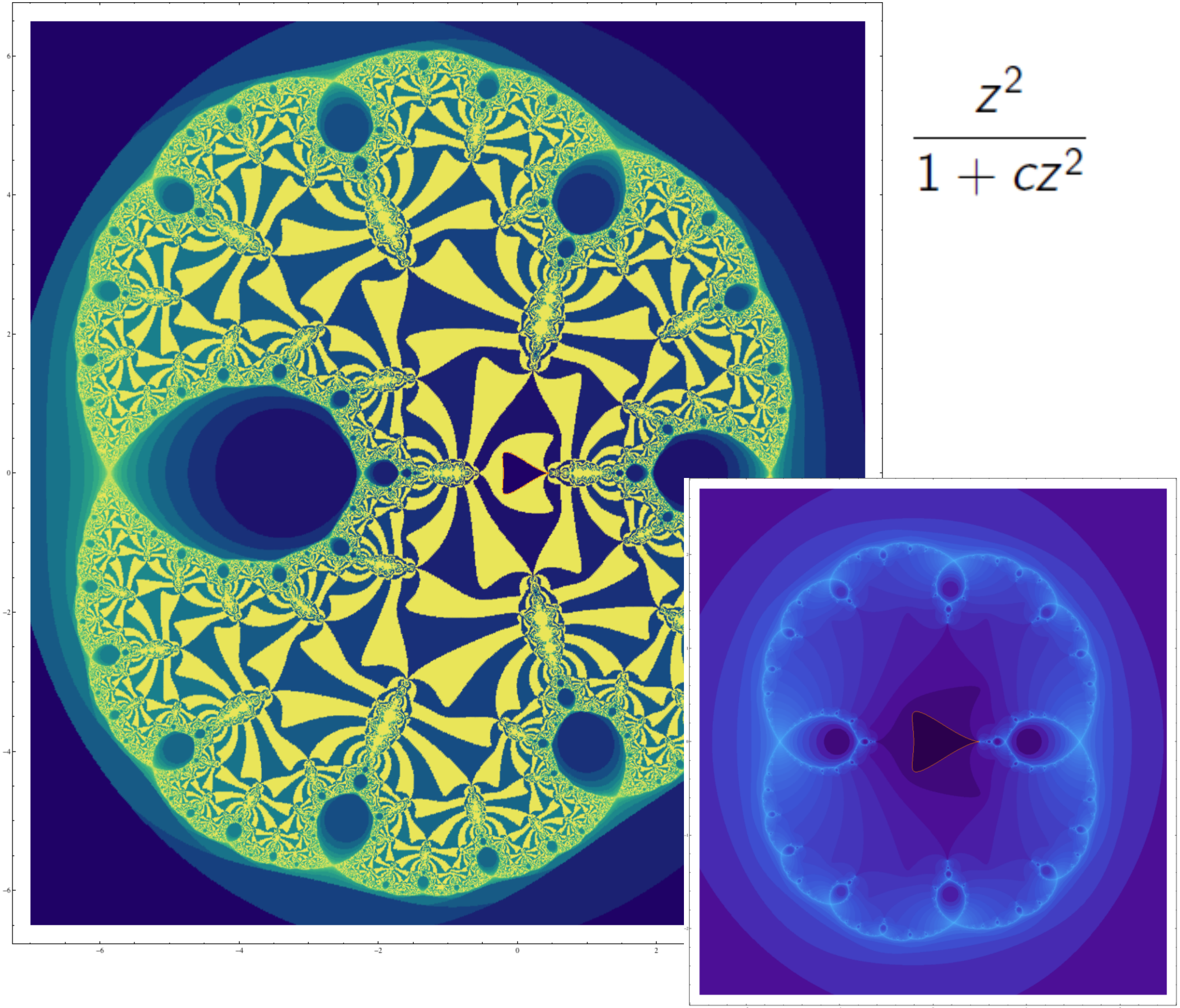
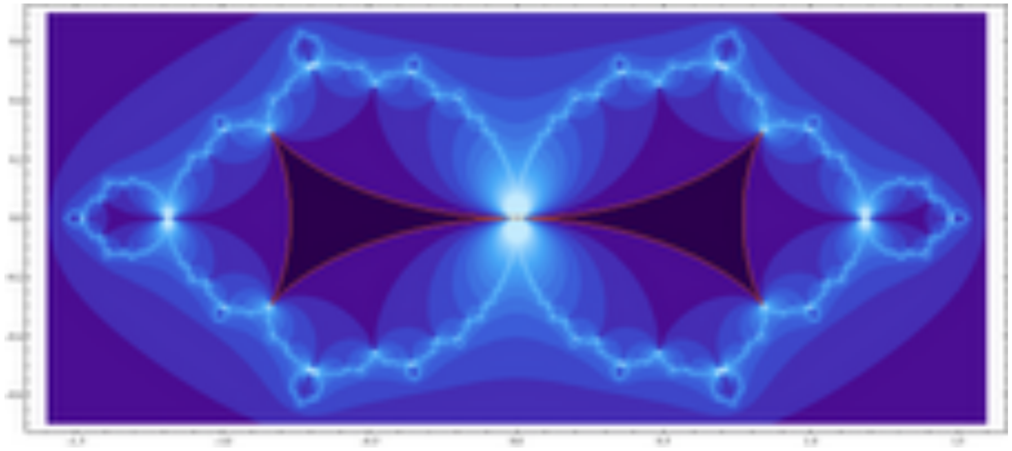
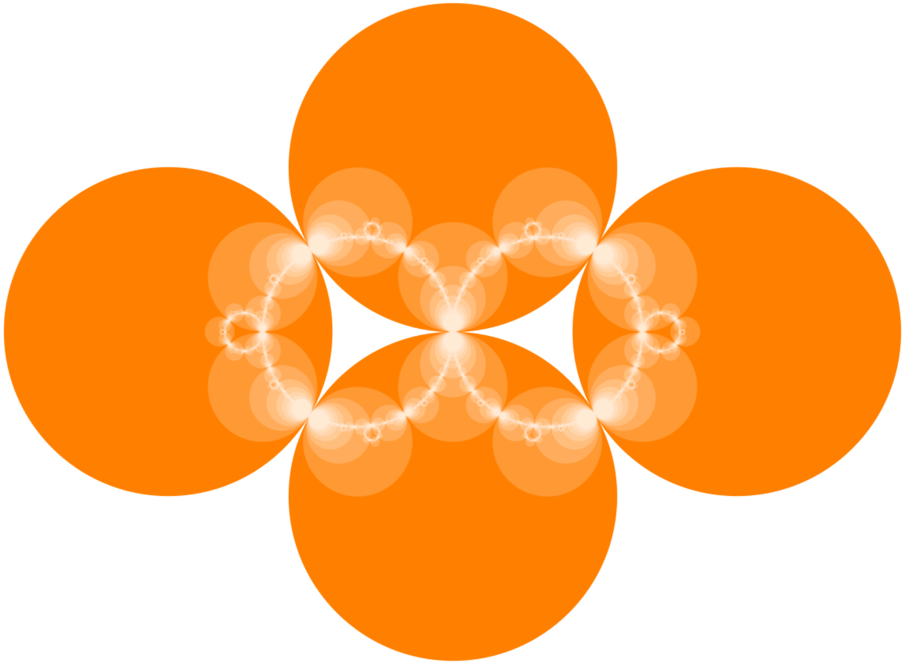
0.0
0.5
1.0
$z^2 + c$

$$\frac{z^2}{1 + cz^2}$$





centaurs



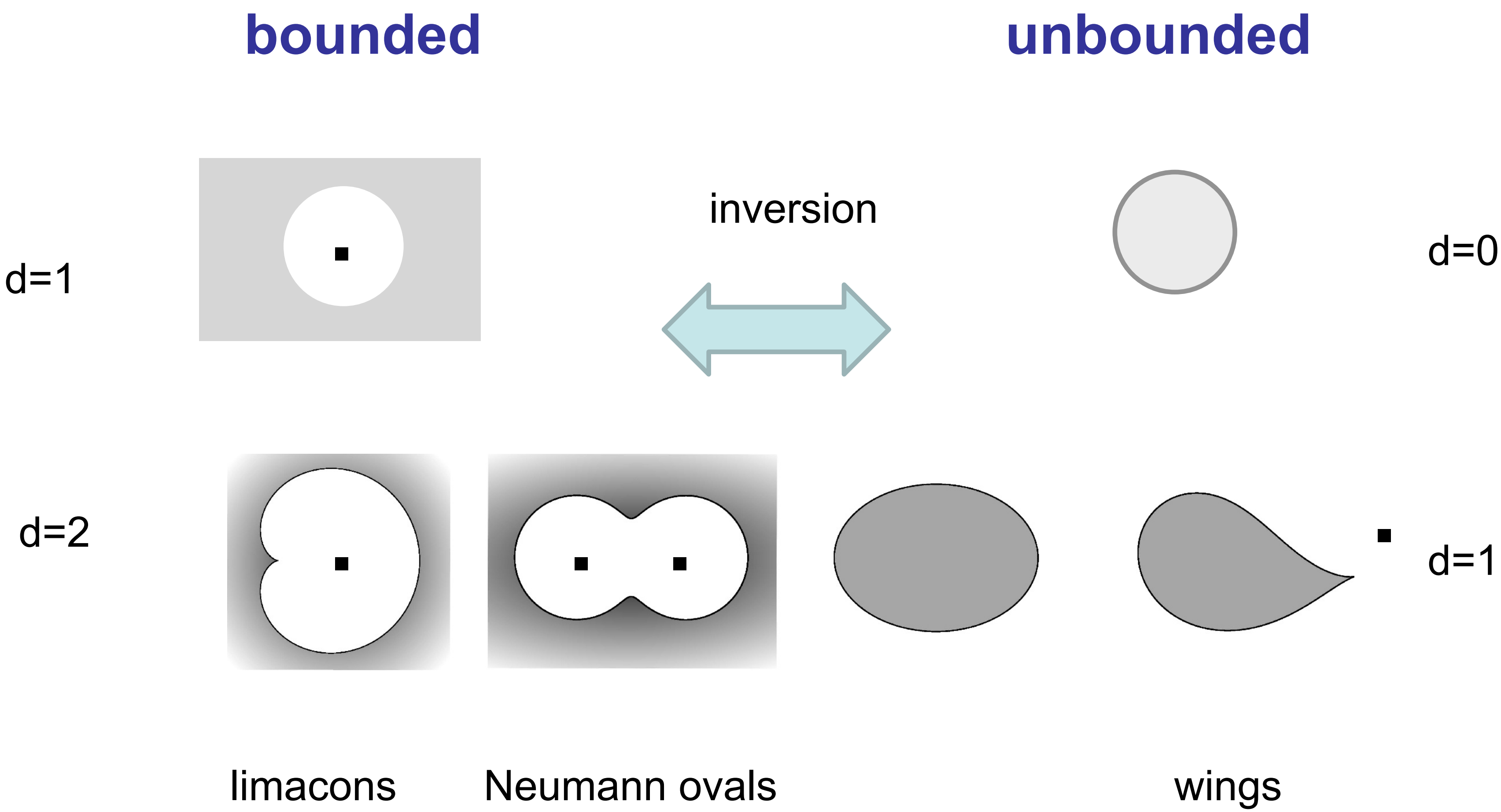
Quadrature domains

- QD is a domain which has a Schwarz function
- bounded Ω is a QD if $\exists$ rational function $r = r_\Omega$ with poles in Ω and $r(\infty) = 0$ such that for all $f \in C_A(\Omega)$,

$$\int_\Omega f dA = \frac{1}{2i}\int_{\partial\Omega} f r_\Omega = \sum c_k f^{(m_k)}(z_k)$$

- similar for unbounded QDs
- Notation: $d = \deg r_\Omega =$ order of QD, $n =$ number of nodes (=distinct poles of r_Ω)

Examples of QDs



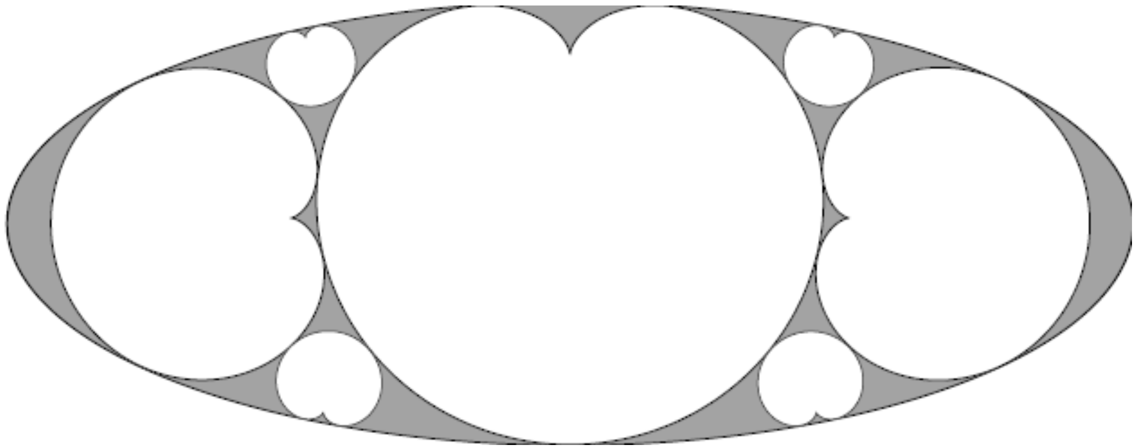
Combination theorem

- If K is a local droplet of algebraic potential $Q(z) = |z|^2 - H(z)$ and $h_Q = \partial H$, then

$$h_Q = \sum r_{\Omega_j}, \qquad K^c = \bigsqcup \Omega_j.$$

- If K^c is disjoint union of QDs, then K is a local droplet of some algebraic potential

Example: 7 cardioids in ellipse



- one UQD of order 1 plus 7 BQDs of order 2, so

$$\deg h = 15$$

- Hele-Shaw flow preserves h. At time $t = T - \epsilon$ we get a local droplet with 21 components whose complement is UQD with $d = 15$, $n = 8$. Note:

$$21 = d + n - 2$$

Connectivity bounds

Theorem. If Ω is UQD of order $d \geq 2$, then

$$\mathrm{conn}\Omega \leq \min\{d+n-1, 2d-2\}$$

If in addition ∞ is a node, then

$$\mathrm{conn}\Omega \leq d+n-2$$

Theorem. If Ω is BQD of order $d \geq 3$, then

$$\mathrm{conn}\Omega \leq \min\{d+n-2, 2d-4\}$$

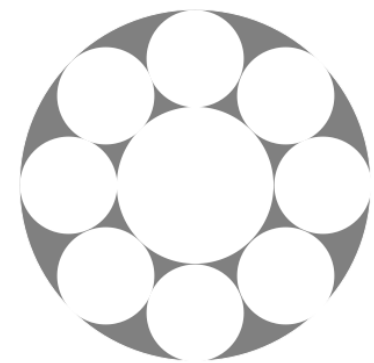
If no nodes of multiplicity ≥ 3, then

$$\mathrm{conn}\Omega \leq \min\{d+n-3, 2d-4\}$$

Examples

- k discs inside a disc: $d = n = k$, so

$$\#\text{components}(\text{int } K) \leq \min\{d + n - 1, 2d - 2\} = 2k - 2$$

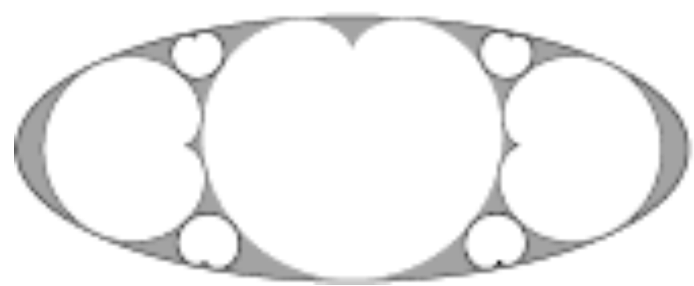


- k discs inside an ellipse: $d = n = k + 1$, so

$$\# \leq \text{components}(\text{int } K) \min\{d + n - 2, 2d - 2\} = 2k$$

- k limacons inside an ellipse: $d = 2k + 1$, $n = k + 1$,

$$\#\text{components}(\text{int } K) \min\{d + n - 2, 2d - 2\} = 3k$$



Gravitational lensing

- Cor: if $\deg r = d$, then $\#\mathrm{Fix}(\bar{r}) \leq 5d - 5$ (Khavinson et al)

  $\#\mathrm{Fix}_{\mathrm{attr}} \leq \#\mathrm{comp}K_{t=\epsilon} \leq 2d-2, \quad \#\mathrm{Fix}_{\mathrm{attr}} - \#\mathrm{Fix}_{\mathrm{rep}} = 1-d$ (Lefschetz)

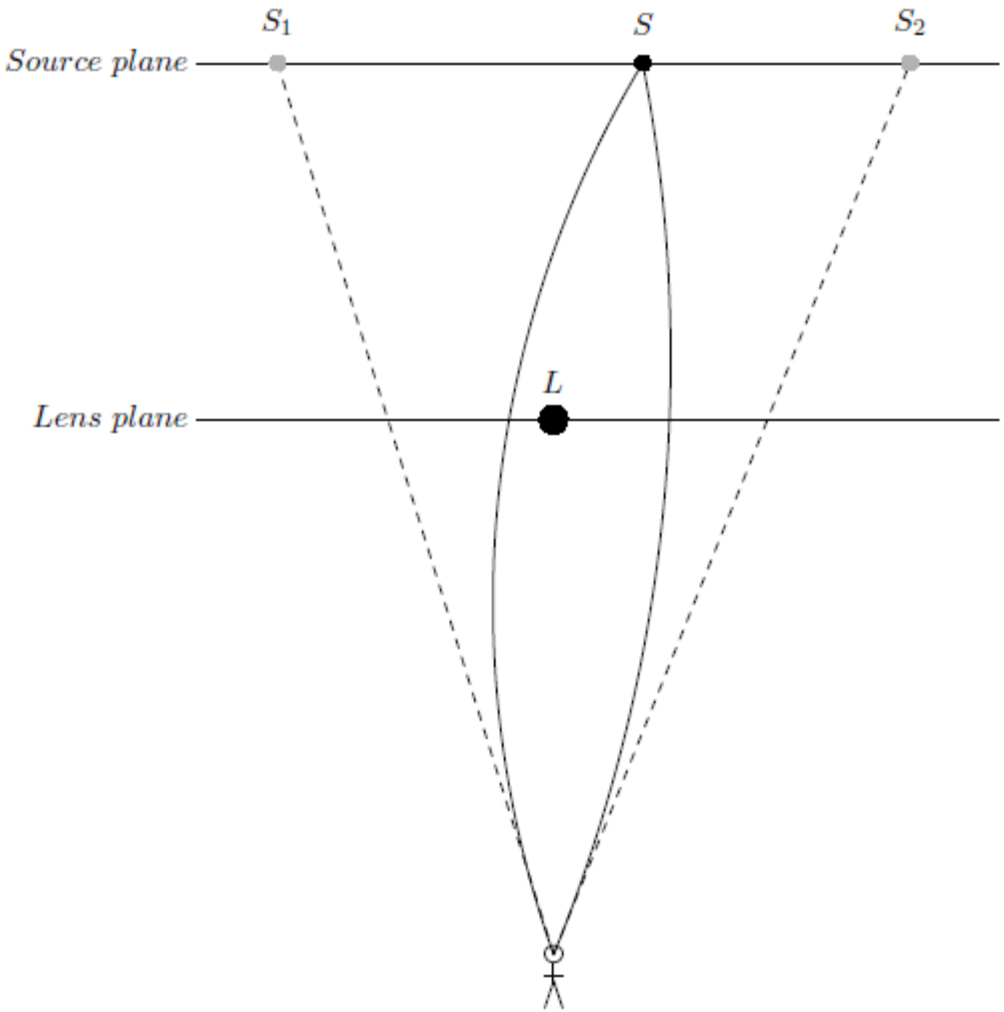
- Lens equation: $z = \overline{r(z)}$,

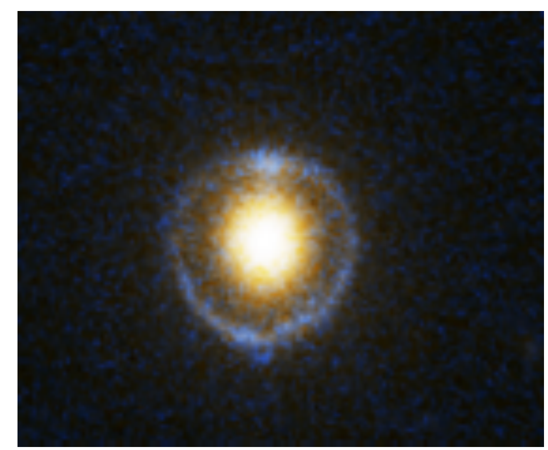
$$r(z) = \sum_{1}^{N} \frac{\sigma_j}{z - z_j} + w - \gamma z$$

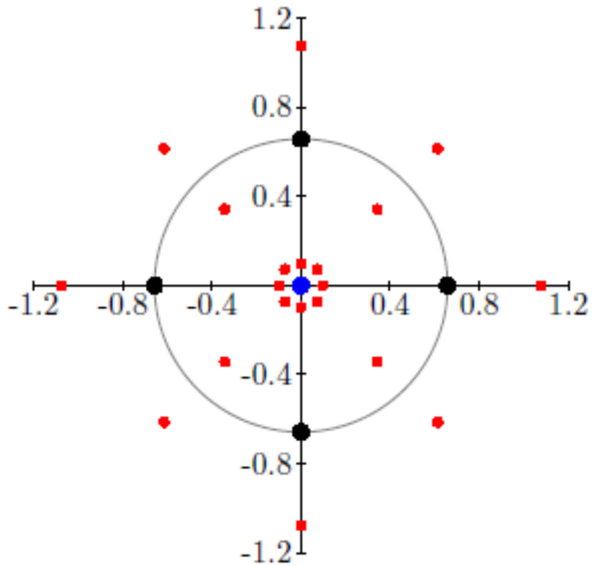
  ($\sigma_j, w \in \mathbb{C}, \gamma \in \mathbb{R}$ are parameters; z_j positions of masses, $\bar{w}$ of source,

  fixedpoints of $\bar{r}$ are images)

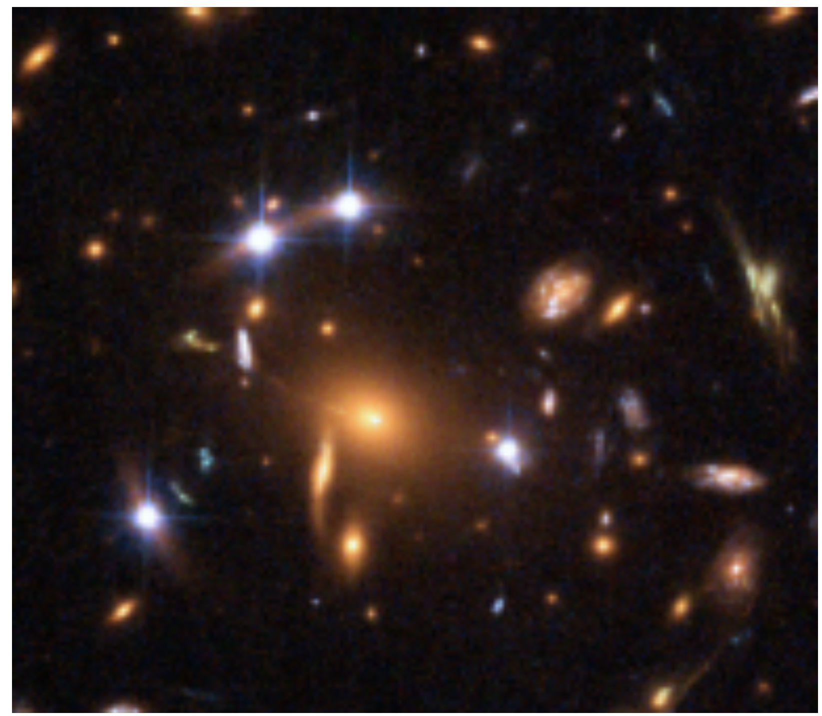
- Conclusion: $\#\mathrm{Fix}_{\mathbb{C}}(\bar{r}) \leq 5N - 5$ if $\gamma = 0$, $\leq 5N - 1$ if $\gamma \neq 0$. Sharp!


S_1
S
S_2
Source plane
L
Lens plane





[Rhie: $\gamma = 0$, $N = 5$, 20 images]



Ovals of real algebraic curves

Corollary. Given d and given some configuration of ovals O in the plane, there exists a local droplet K of an algebraic potential of degree d such that ∂K is isotopic to O iff

$$q_1 + 3q_2 + 4q_3 + 4q_4 + 5q_5 + 5q_6 + \cdots \leq d + 1$$

where q_j is the number of QDs of connectivity j

Example: $d = 4$, $q_1 = 2$, $q_2 = 1$

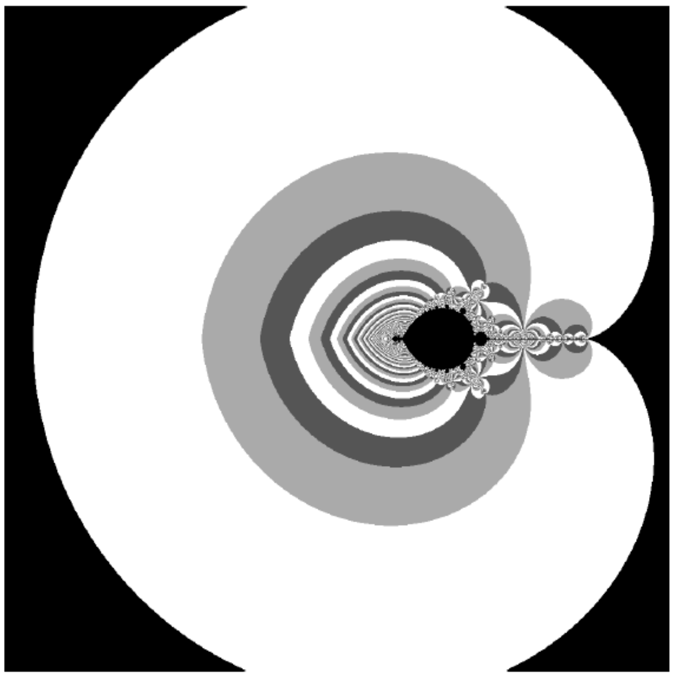
Comments:

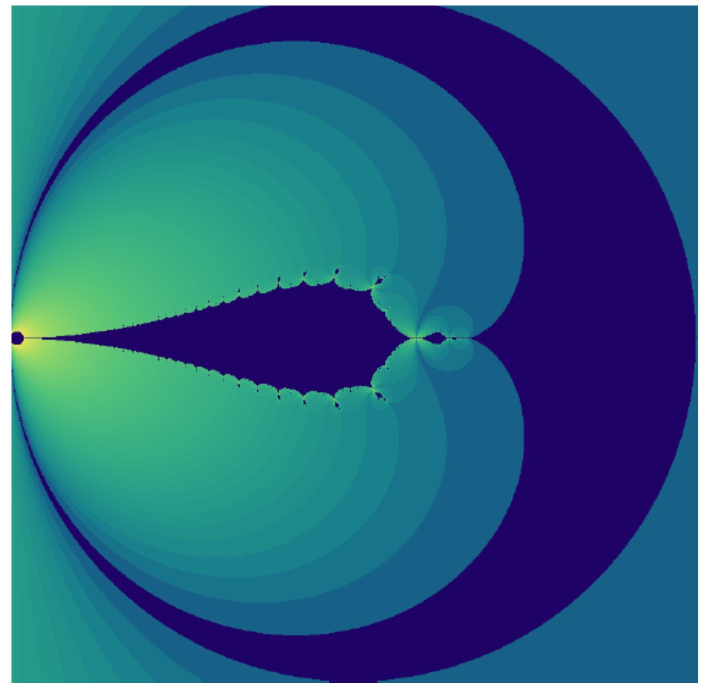
- $\#(\text{ovals}) \leq 2d-2$
- If h is a polynomial, then the degree of the algebraic curve is $2d$, so Harnack's bound gives $2d^2-3d+2$, e.g. $\#(2) \leq 4$
- More general class of algebraic curves comes from $Q = |R|^2 - H$ where R is rational, H harmonic

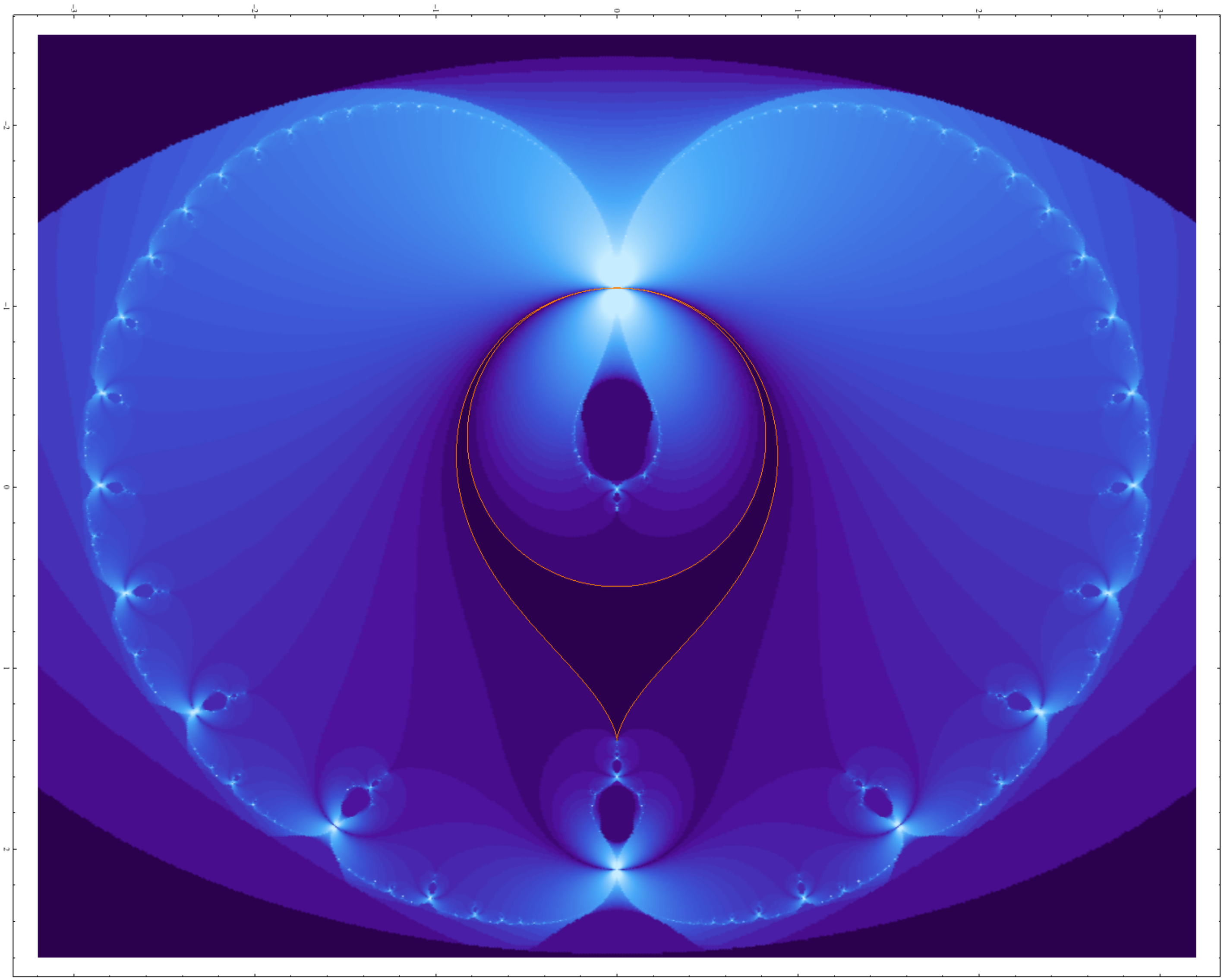
Teichmuller spaces of local droplets, an example

$$\heartsuit := \{z/2 - z^2/4 : \ |z| \leq 1\}$$

- Parameter space: points $c \in \heartsuit$
- Dynamical plane: droplet $K_c = B(c, \rho) \setminus \heartsuit$ where ρ is such that the circle is tangent to the cardioid
- J_c is "Julia set" of S-function of K_c
- Connectivity locus in parameter space: $\mathcal{M} = \{c : \ J_c \,\mathrm{connected}\}$
- "Reflection groups" remain the same: $\Gamma_c \equiv \Gamma(2)$







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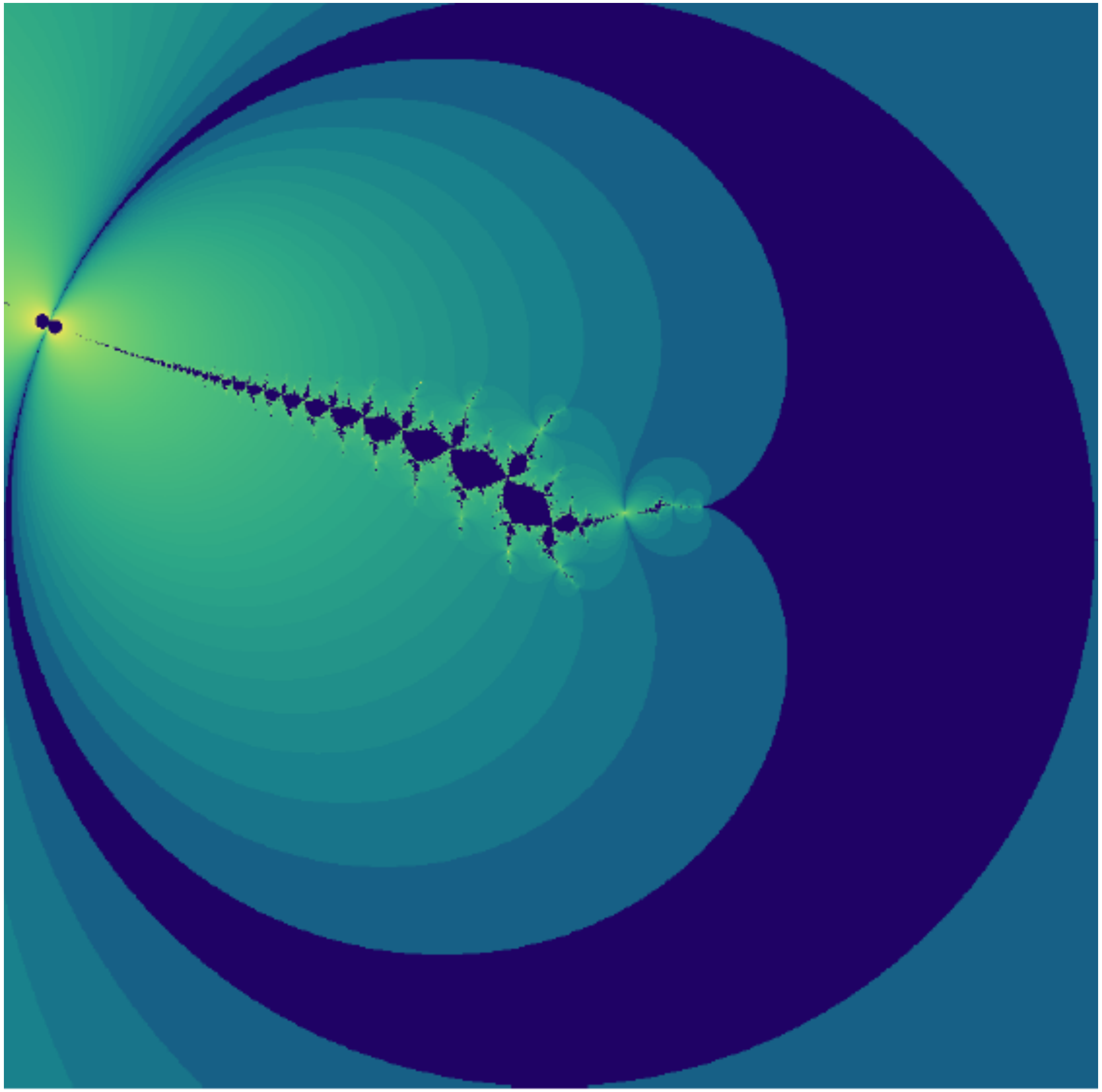
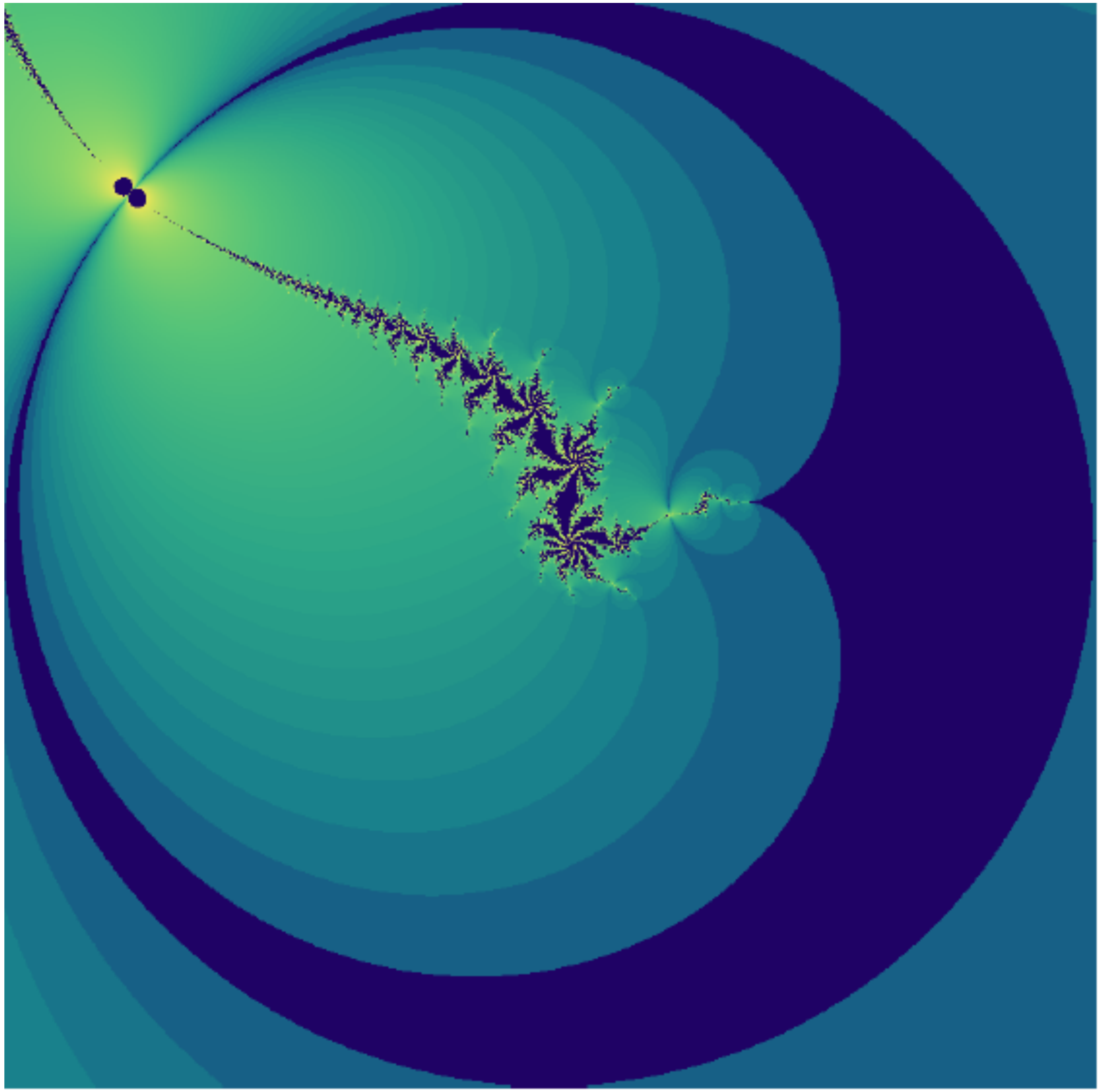
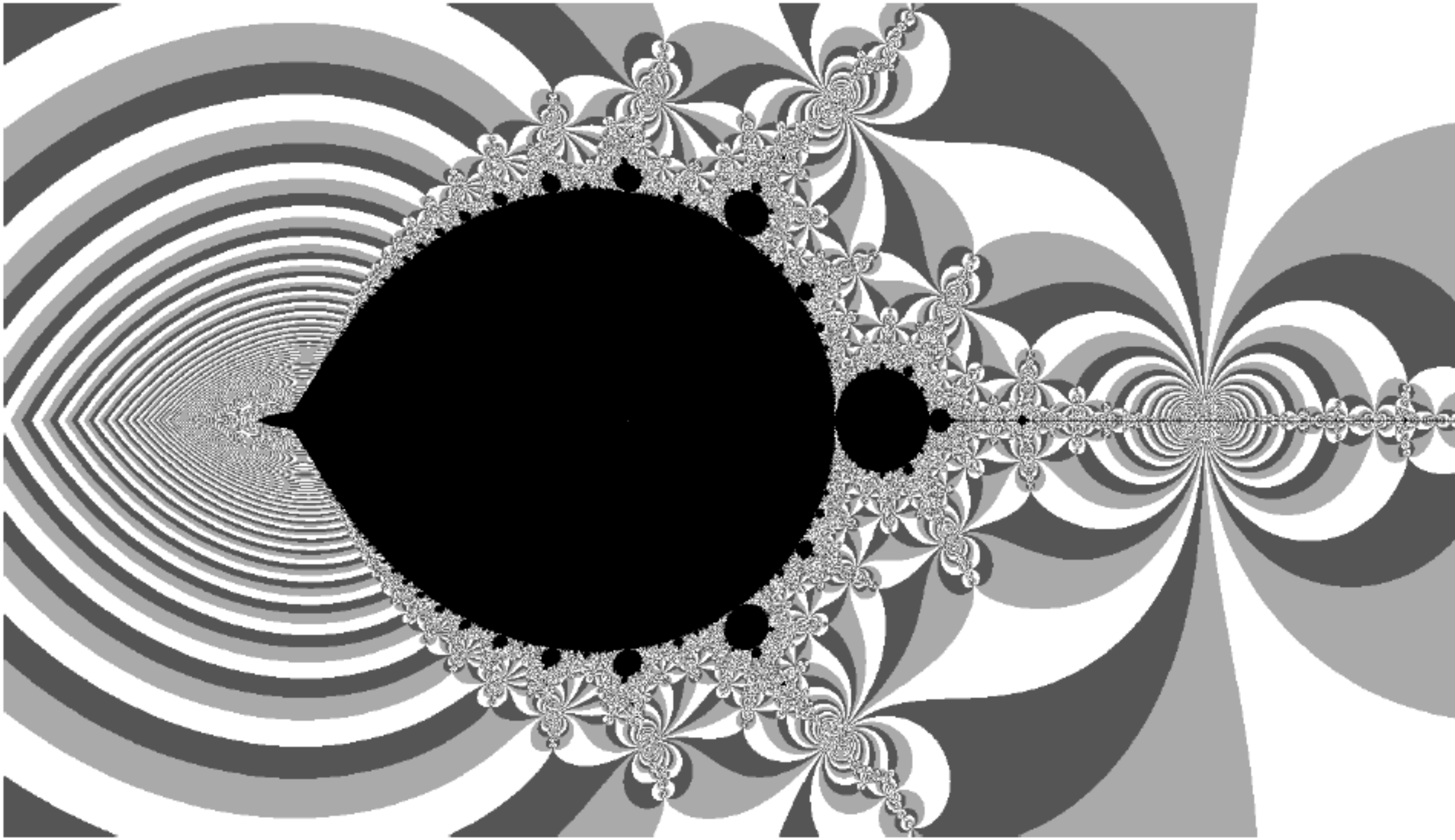
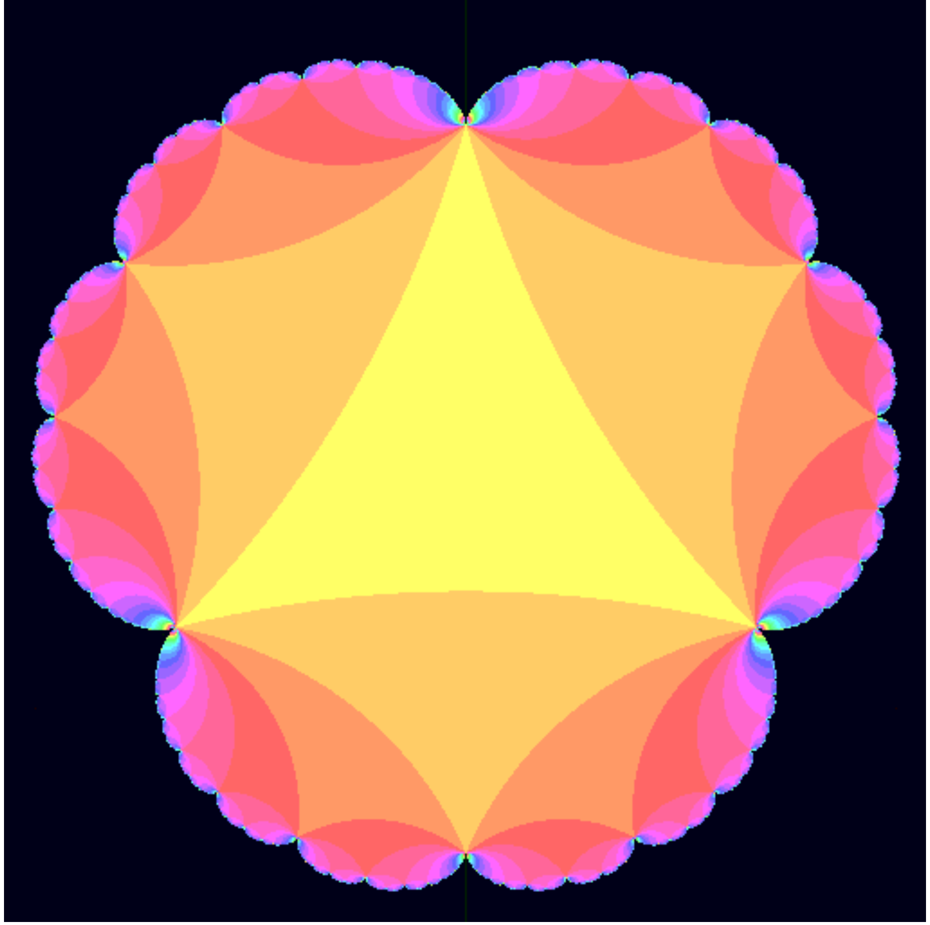
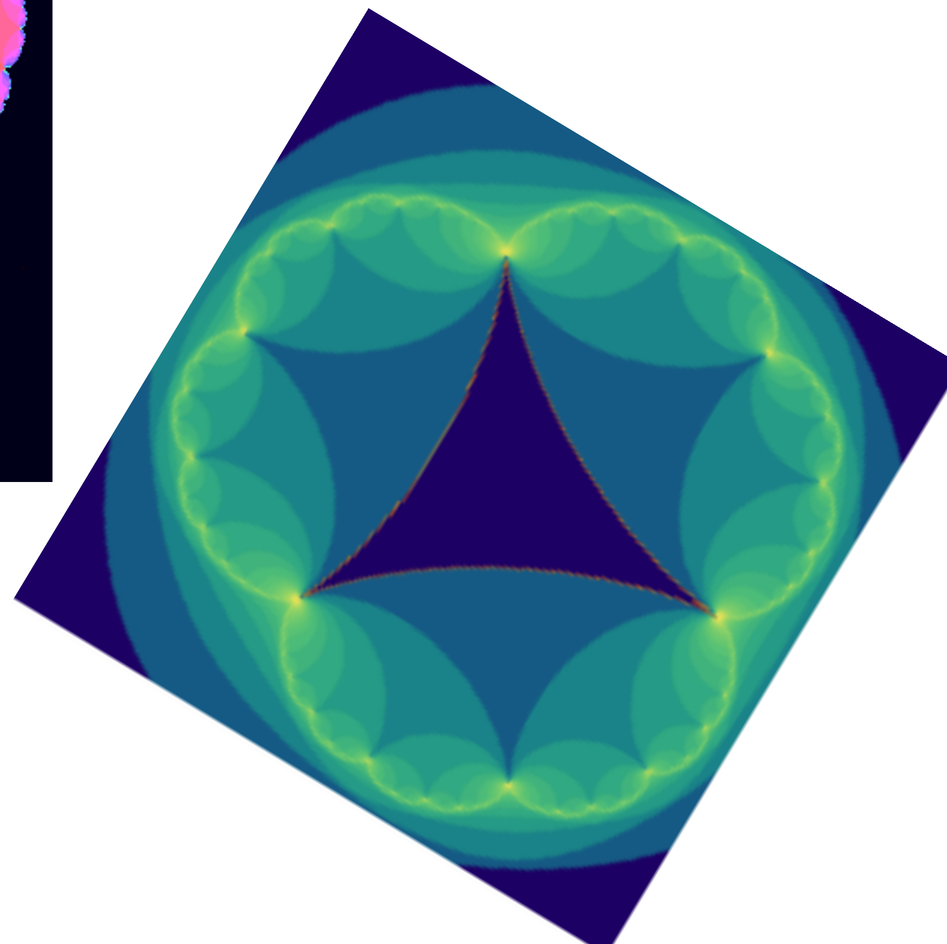






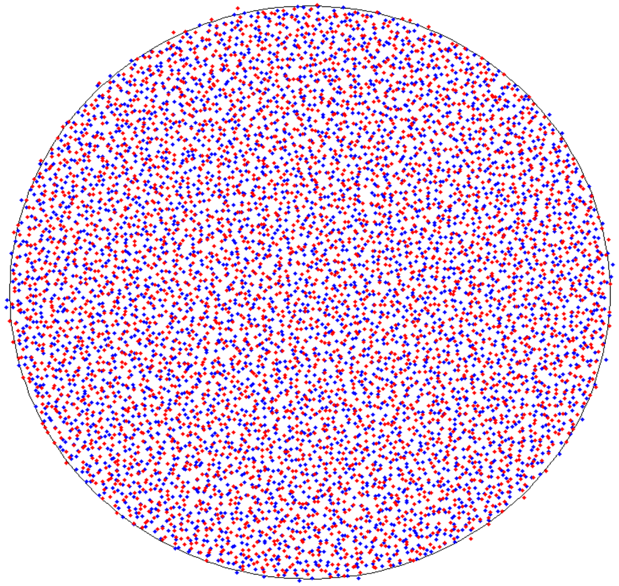


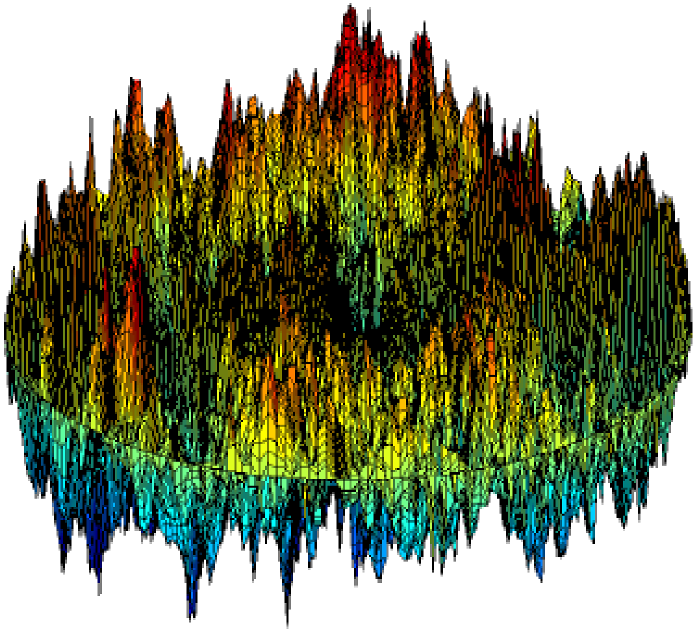


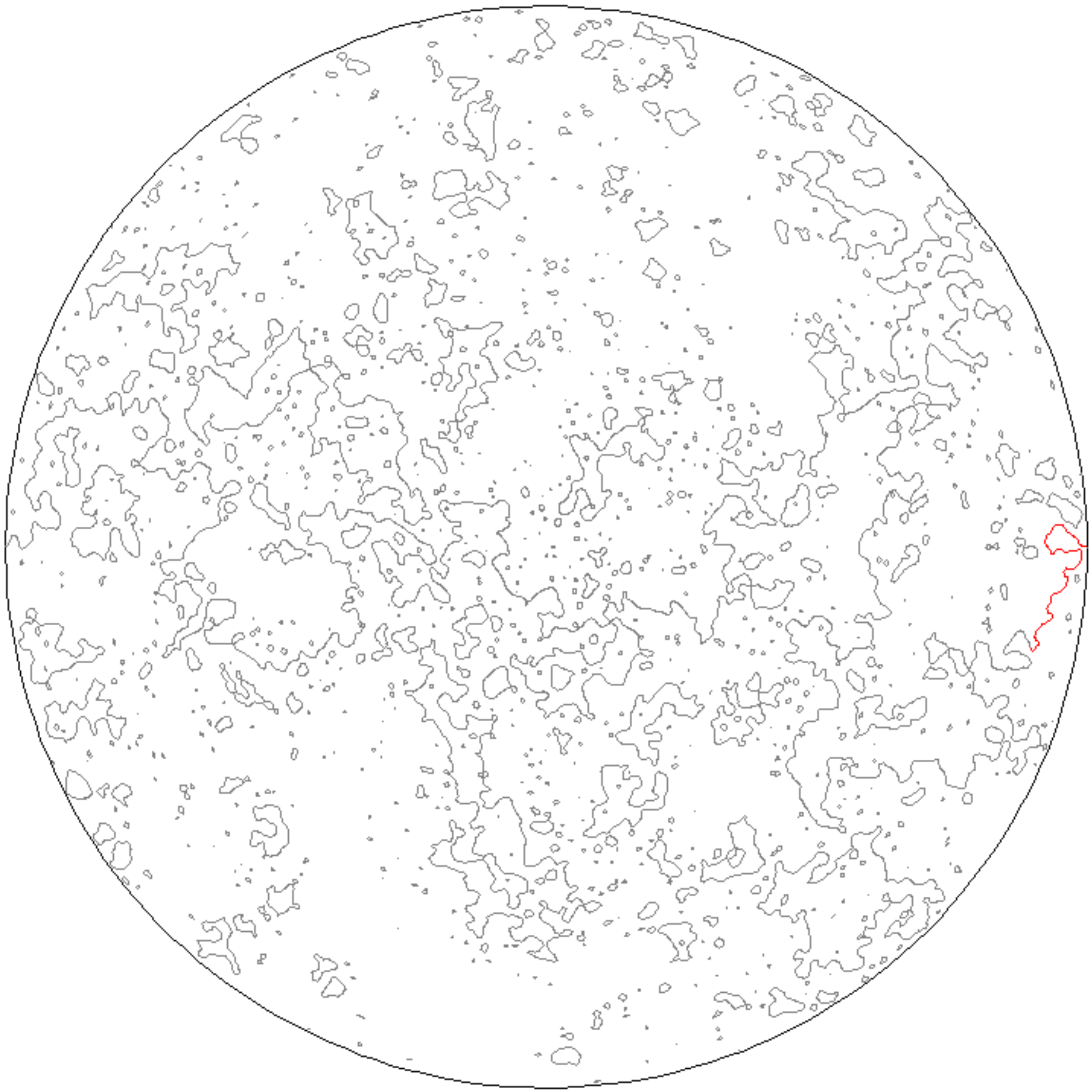
Bers slice for hexagonal torus

Inside the droplet

- Convergence of fluctuations to a Gaussian field. Use of RNM model to approximate various objects related to GFF
- Universality laws at regular points in the bulk and boundary, some types of singular points in the bulk







References

- Based on joint work with Seung Yeop Lee. I thank Seung Yeop for pictures and crucial contributions
- Use of conformal dynamics was inspired by a paper by Khavinson and Swiatek
- Interesting analogy with some work in dimer model (Kenyon, Okounkov, Sheffield)