Buffon's needle probability of rational product Cantor sets

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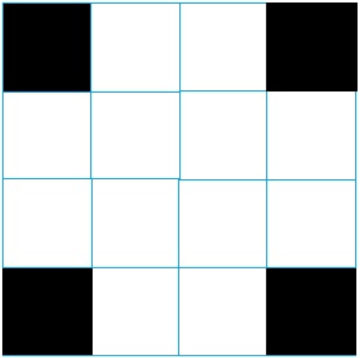
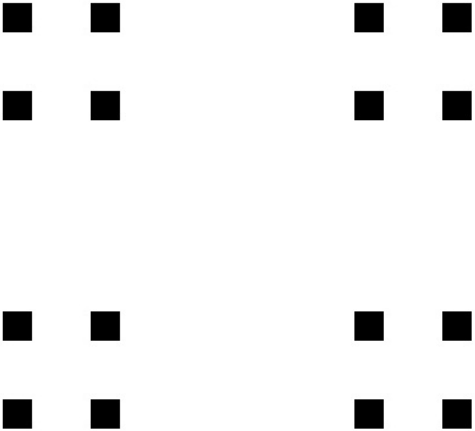
The problem is of interest in ergodic theory as well as theory of analytic functions (*analytic capacity*).

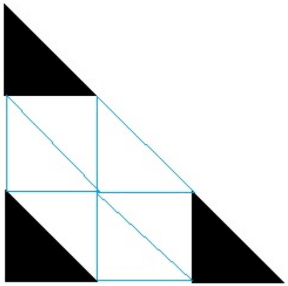
The 4-corner set, 1st iteration



The 4-corner set, 2nd iteration



The 1-dimensional Sierpiński triangle, 1st iteration



Product Cantor sets

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- Iterate the construction.

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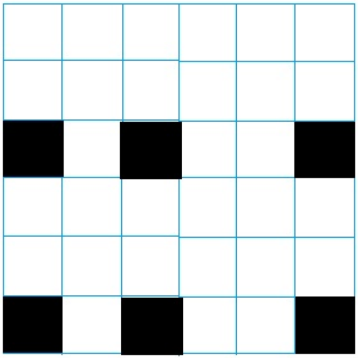

In this example, $L = 6$, $A = \{0, 2, 5\}$, $B = \{0, 3\}$

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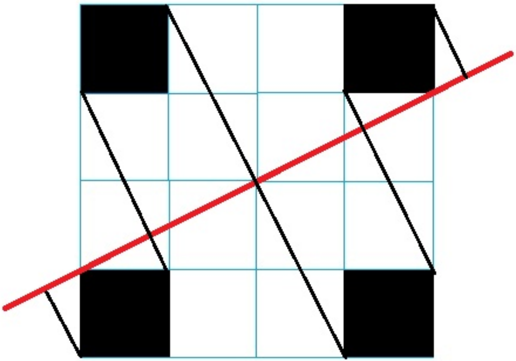
- **How fast?**

The 4-corner set, projection with $\tan \theta = 1/2$



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- **Bateman-Volberg 2008:** improvement to $F_n \geq \frac{C \log n}{n}$ for the 4-corner set.
  (The same method works for the triangle, but not for product sets.)
- **The expected asymptotics** for the above examples is $F_n \approx C/n$, possibly up to log factors (as above). But this is far from proved...

Favard length: upper bounds

- **Peres-Solomyak 2002:** $F_n \leq Ce^{-c \log^* n}$ for very general self-similar sets, including the above examples. ($\log^* n$: the number of iterations of log needed for $\log \ldots \log n \leq 10$)

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- **Bond-Volberg 2010:** $F_n \leq Ce^{-c\sqrt{\log n}}$ for general self-similar sets.

Main result

Bond-Łaba-Volberg 2011:

- $F_n \leq Cn^{-p/\log\log n}$ for product sets as above with $|A|, |B| \leq 6$. (The exponent $p > 0$ depends on the set.)

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- The same result holds under explicit number theoretic conditions on A, B without the size restriction. (These conditions always hold if $|A|, |B| \leq 6$, but they can also be checked directly if A, B are given.)
- Can improve this to $F_n \leq Cn^{-p}$ under an additional condition on A, B.

Conditions on A, B: connection to number theory

- Define trigonometric polynomials

$$\phi_A(\xi) = \frac{1}{|A|} A(e^{2\pi i \xi}),\ \ A(z) = \sum_{a \in A} z^a,$$

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- The conditions on A, B will concern the roots of $A(z), B(z)$ with $|z| = 1$.
- If there are no such roots, then $F_n \leq Cn^{-p}$ holds. Otherwise, we need to study
  - divisibility of $A(z), B(z)$ by cyclotomic polynomials
  - diophantine properties of roots $e^{2\pi i \xi_0}$, $\xi_0 \notin \mathbb{Q}$.

Fourier-analytic approach

We outline the main steps very briefly, then discuss the number theoretic part in more detail.

- Change convention: consider only $\theta \in [0, \pi/4]$, let

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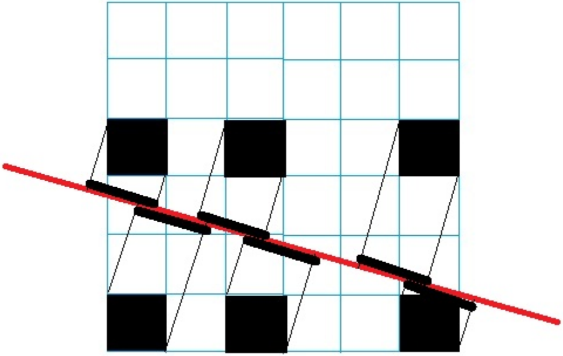
- Let μ_∞, μ_n be the natural probability measures on E_∞, E_n (so that $\mu_n \to \mu_\infty$ weakly)
- Let $\pi_t \mu_\infty$, $\pi_t \mu_n$ be their projections:

$$\pi_t \mu_n(X) = \mu_n(\pi_t^{-1}(X))$$

The projected measure $\pi_t\mu_1$ for a product Cantor set



$\pi_t\mu_1$ is a measure on the real line, its density $\approx$ the sum of characteristic functions of intervals.

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- Fix t, let f_n = density of $\pi_t(\mu_n)$. Heuristically, $\|f_n\|_2$ large corresponds to a lot of overlap between the projected squares.

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- For self-similar sets, the implication can be reversed (NPV argument)
- Therefore, we want $\|f_n\|_2$ large for most directions θ. But $\|f_n\|_2 = \|\widehat{f_n}\|_2$, so take Fourier transforms...

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- The last term acts as a cut-off function on $[-L^n, L^n]$.

Estimating L^2 norms from below

- After rescaling, pigeonholing, calculations... we need a bound from below on

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- Main difficulty: controlling the damage from the low frequency terms $\prod_{j=1}^{m}$.

Sets of small values

- Need to study the set where

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- If there are no such roots, the estimate is complete at this point. Otherwise, continue...

Roots of $A(x^{L^j})$ on the unit circle

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- But this still leaves a large set where the low frequency part is not small. Can we integrate on that set?
- **BŁV:** Salem's argument does not work on arbitrary sets. But it does work on difference sets $\Gamma - \Gamma$. We need to find a large enough difference set free of high multiplicity roots.

Background: cyclotomic polynomials

The argument depends on the divisibility of $A(x)$ by cyclotomic polynomials.

- For $s = 1, 2, \ldots,$ the s-**th cyclotomic polynomial** Φ_s is

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- The roots of Φ_s are exactly the s-th primitive roots of unity.
- We have $x^n - 1 = \prod_{s|n} \Phi_s(x)$.

Factorization of $A(x)$

Write $A(x) = A_1(x)A_2(x)A_3(x)A_4(x)$, where

- $A_1(x) = \prod \Phi_s(x)$: $\Phi_s | A$, $(s, L) \neq 1$
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- $A_3(x) = \prod (x - e^{2\pi i \xi_j})$: $A(e^{2\pi i \xi_j}) = 0$, $\xi_j \in \mathbb{R} \setminus \mathbb{Q}$
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Each of the factors A_1, A_2, A_3 requires a different method. (A_4 is harmless.)

To simplify the exposition, we will discuss 3 examples where only one type of roots is present. For more general sets, the arguments need to be combined.

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- Roots of low frequency part are (a subset of) roots of $x^{12m} - 1$. This controls their multiplicity.
- Argument first used in NPV for 4-corner set, extended in ŁZ, BŁV to cyclotomic divisors Φ_s with $(s, L) \neq 1$.

Example 2: Recurrent roots

Let $L = 25$, $A = B = \{0, 3, 4, 8, 9\}$.

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- $A(x)$ is divisible by $\Phi_{12}(x) = 1 - x^2 + x^4$.
- Since $(12, 25) = 1$, $A(x^{25^j})$ is also divisible by Φ_{12} for $j = 1, 2, \dots$ (This is because if ω is a primitive 12-th root of 1, then so is ω^{25^j}.)

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Let $L = 25$, $A = B = \{0, 3, 4, 8, 9\}$.

- $A(x)$ is divisible by $\Phi_{12}(x) = 1 - x^2 + x^4$.
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- The low frequency part has roots of very high multiplicity. The first method fails.
- We need to use the second method: construct a large set $\Gamma \subset [0, 1]$ such that $\Gamma - \Gamma$ is away from "bad" roots.

Vanishing sums of roots of unity

The construction relies on classical results on vanishing sums of roots of unity (Rédei, de Bruijn, Schoenberg, Mann, Lam-Leung, Poonen-Rubinstein, ...)

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- Can rotate and add regular polygons for more examples.
- All vanishing sums of roots of unity are linear combinations of regular polygons with integer ($+$ or $-$) coefficients.

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- But we need a quantitative result.
- We can prove the result we need for all sets with $|A|, |B| \leq 6$, and for larger sets with additional conditions.

Example 3: Non-cyclotomic roots

Let $L = 25$, $A = B = \{0, 3, 4, 5, 8\}$.

- $A(x)$ has 4 roots $e^{2\pi i \lambda_j}$ with $\lambda_j \in \mathbb{R} \setminus \mathbb{Q}$.

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- Use this to prove that the roots of the low frequency part do not accumulate too closely.
- This part causes the loss of $\log \log n$ in the final estimate on F_n. If there are no such roots, the stronger power estimate holds.

Thank you!