Universality of local spectral statistics of random matrices

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"Perhaps I am now too courageous when I try to guess the distribution of the distances between successive levels (of energies of heavy nuclei). Theoretically, the situation is quite simple if one attacks the problem in a simpleminded fashion. The question is simply what are the distances of the characteristic values of a symmetric matrix with random coefficients."

Eugene Wigner, 1956

Nobel prize 1963

INTRODUCTION

Basic question [Wigner]: Consider a large matrix whose elements are random variables with a given probability law. What can be said about the statistical properties of the eigenvalues? Do some universal patterns emerge and what determines them?

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix} \Longrightarrow \quad (\lambda_1, \lambda_2, \dots, \lambda_N) \text{ Eigenvalues?}$$

$N =$ size of the matrix, will go to infinity.

Analogy: Central limit theorem: $\frac{1}{\sqrt{N}}(X_1 + X_2 + \ldots + X_N) \sim \mathcal{N}(0, \sigma^2)$

Gaussian Unitary Ensemble (GUE):

$H = (h_{jk})_{1 \le j,k \le N}$ hermitian $N \times N$ matrix with

$$h_{jk} = \bar{h}_{kj} = \frac{1}{\sqrt{N}} (x_{jk} + i y_{jk}) \quad \text{and} \quad h_{kk} = \frac{\sqrt{2}}{\sqrt{N}} x_{kk}$$

where x_{jk}, y_{jk} (for $j < k$) and x_{kk} are independent standard Gaussian

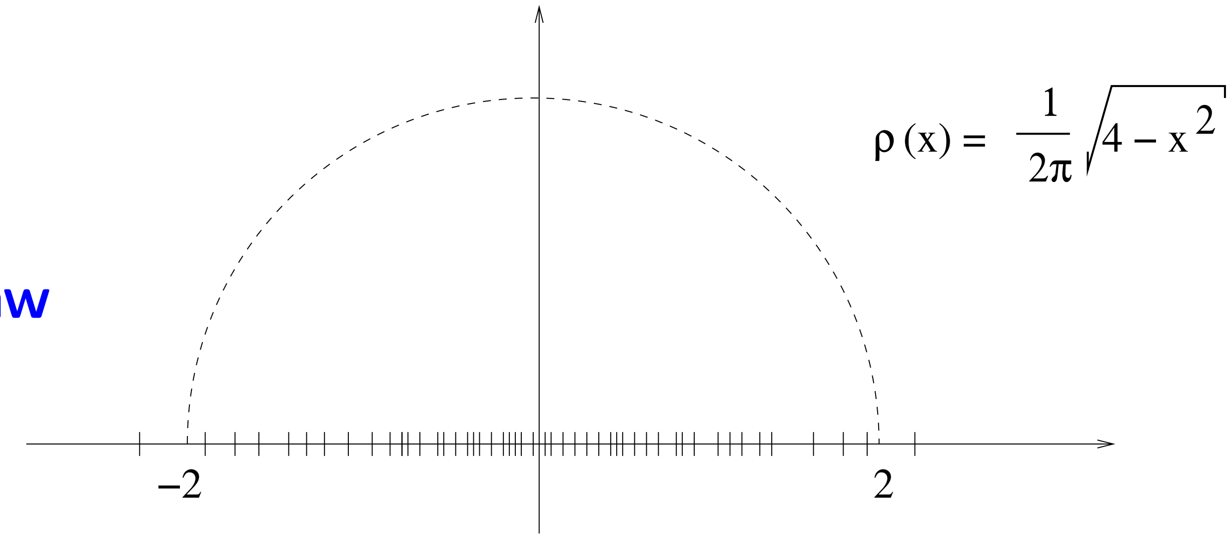
The eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$ are of order one:

$$\mathbb{E} \frac{1}{N} \sum_i \lambda_i^2 = \mathbb{E} \frac{1}{N} \mathrm{Tr} H^2 = \frac{1}{N} \sum_{ij} \mathbb{E} |h_{ij}|^2 = 2$$

at least in average sense.

Hermitian can be replaced with symmetric or quaternion self-dual (GOE, GSE)

Wigner semicircle law



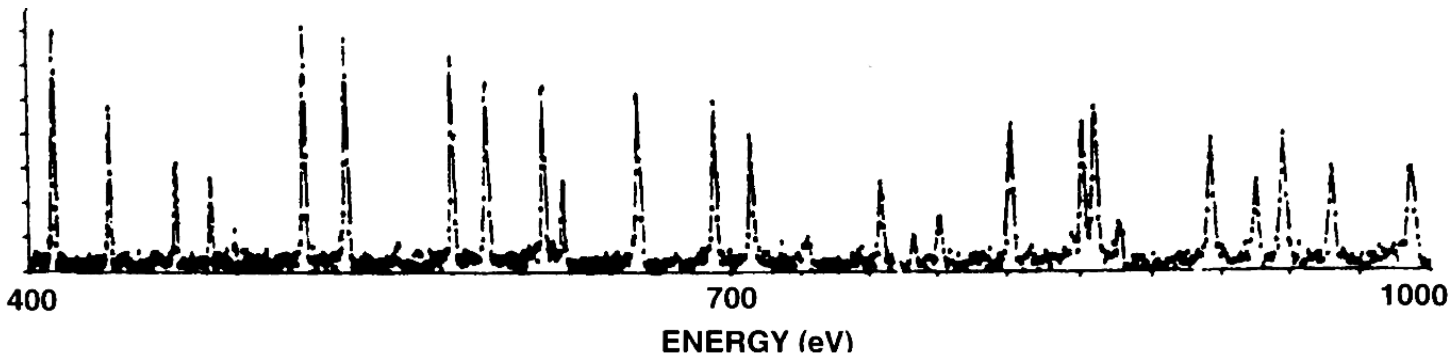
Typical evalue spacing (gap): $\delta_i = \lambda_{i+1} - \lambda_i \sim \frac{1}{N}$ (in the bulk)

Observations: i) Semicircle density. ii) Level repulsion.

Holds for other symmetry classes GUE, GOE, GSE.

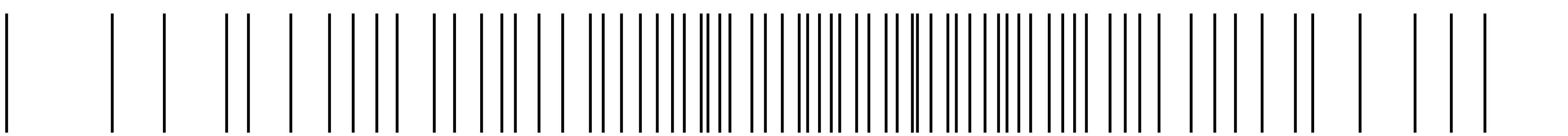
For Wishart matrices, i.e. matrices of the form $H = AA^*$, where the entries of A are i.i.d.: Marchenko-Pastur law

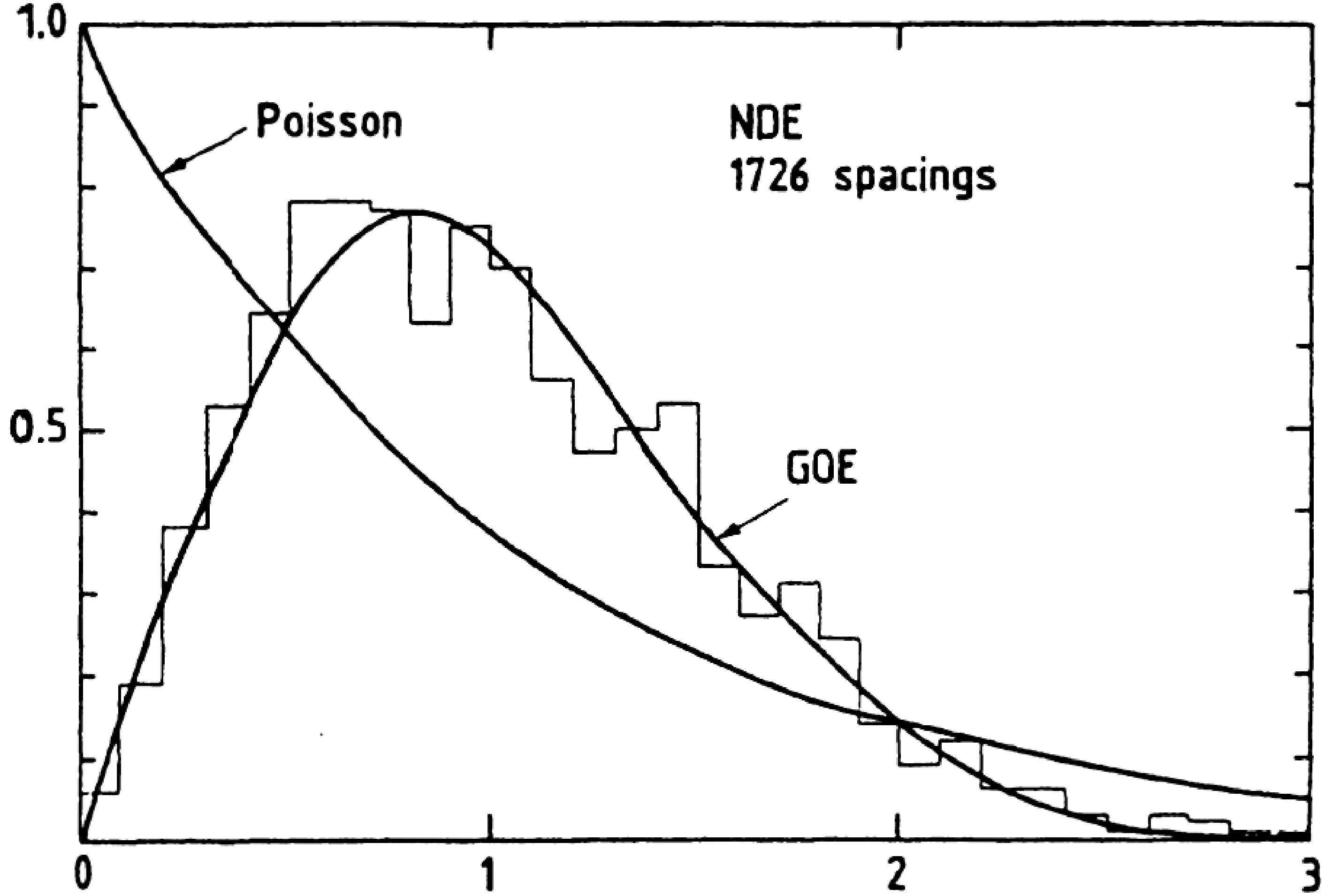
• **E. Wigner (1955):** The excitation spectra of heavy nuclei have the same spacing distribution as the eigenvalues of GOE.
Experimental data for excitation spectra of heavy nuclei: $({}^{238}U)$



typical Poisson statistics:

Typical random matrix eigenvalues





Level spacing (gap) histogram for different point processes.

NDE – Nuclear Data Ensemble, resonance levels of 30 sequences of 27 different nuclei.

SINE KERNEL FOR CORRELATION FUNCTIONS

Probability density of the eigenvalues: $p(x_1, x_2, \ldots, x_N)$

The k-point correlation function is given by

$$p_N^{(k)}(x_1, x_2, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} p(x_1, \ldots x_k, x_{k+1}, \ldots, x_N) \mathrm{d}x_{k+1} \ldots \mathrm{d}x_N$$

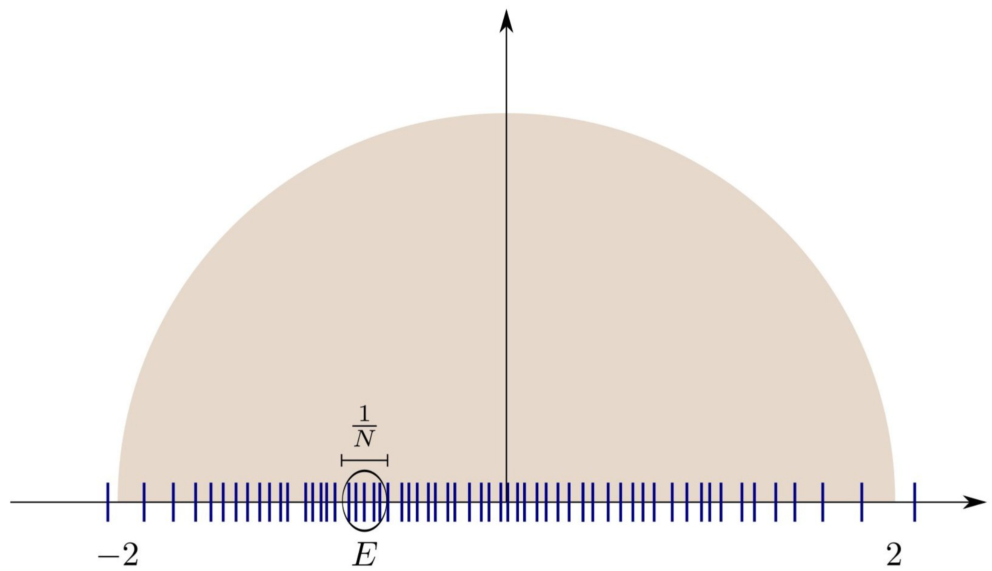
Special case: $k = 1$ (density)

$$\varrho_N(x) := p_N^{(1)}(x) = \int_{\mathbb{R}^{N-1}} p(x, x_2, \ldots, x_N) \mathrm{d}x_2 \ldots \mathrm{d}x_N$$

It allows to compute expectation of observables with one eigenvalue:

$$\mathbb{E}\frac{1}{N}\sum_{i=1}^{N} O(\lambda_i) = \int O(x)\varrho_N(x)\mathrm{d}x \to \frac{1}{2\pi}\int O(x)\sqrt{4-x^2}\mathrm{d}x$$

Higher k computes observables with k evalues.



Local level correlation statistics for GUE [Gaudin, Dyson, Mehta]

$$\lim_{N\to\infty} \frac{1}{[\rho(E)]^2}\, p_N^{(2)}\Big(E + \frac{x_1}{N\rho(E)}, E + \frac{x_2}{N\rho(E)}\Big) = \det\Big\{S(x_i - x_j)\Big\}_{i,j=1}^{2}$$

for any $|E| < 2$ (bulk spectrum), where $S(x) := \frac{\sin \pi x}{\pi x}$

$$= 1 - \Big(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)}\Big)^2 \qquad (\Longrightarrow \quad \text{Level repulsion})$$

k-point correlation functions are given by $k \times k$ determinants:

$$\lim_{N\to\infty} \frac{1}{[\rho(E)]^k} \, p_N^{(k)}\Big(E + \frac{x_1}{N\rho(E)}, E + \frac{x_2}{N\rho(E)}, \ldots, E + \frac{x_k}{N\rho(E)}\Big)$$

$$= \det\Big\{S(x_i - x_j)\Big\}_{i,j=1}^k$$

The limit is independent of E as long as E is in the bulk spectrum, i.e. $|E| < 2$.

Gap distribution (original question of Wigner) is obtained from correlation functions by the exclusion-inclusion formula.

Main question: going beyond Gaussian towards universality!

There are two almost disjoint directions of generalization: Gaussian is the common intersection.

GENERALIZATION NO.1: INVARIANT ENSEMBLES

Unitary ensemble: Hermitian matrices with density

$$\mathcal{P}(H)\mathrm{d}H \sim e^{-\mathrm{Tr}\,V(H)}\mathrm{d}H$$

Invariant under $H \to UHU^{-1}$ for any unitary U

Joint density function of the eigenvalues is explicitly known

$$p(\lambda_1, \ldots, \lambda_N) = \text{const.} \prod_{i<j} (\lambda_i - \lambda_j)^{\beta} e^{-\sum_j V(\lambda_j)}$$

classical ensembles $\beta = 1, 2, 4$ (orthogonal, unitary, symplectic symmetry classes; GOU, GUE, GSE for Gaussian case, $V(x) = x^2/2$)

Correlation functions can be explicit computed via orthogonal polynomials due to the Vandermonde determinant structure.

large N asymptotic of orthogonal polynomials $\Longrightarrow$ local statistics is indep of V. But density depends on V.

GENERALIZATION NO.2: (GENERALIZED) WIGNER ENSEMBLES

$$H = (h_{ij})_{1 \le i,j \le N}, \quad \bar{h}_{ji} = h_{ij} \quad \text{independent}$$

$$\mathbb{E} h_{ij} = 0, \quad \mathbb{E}|h_{ij}|^2 = \sigma_{ij}^2, \qquad \sum_i \sigma_{ij}^2 = 1,$$

$$\frac{c}{N} \le \sigma_{ij}^2 \le \frac{C}{N}$$

$$\text{Moment condition:} \quad \mathbb{E}|\sqrt{N} h_{ij}|^{4+\varepsilon} < C$$

If h_{ij} are i.i.d. then it is called Wigner ensemble.

Universality conjecture (Dyson, Wigner, Mehta etc) : If h_{ij} are independent, then the local eigenvalues statistics are the same as for the Gaussian ensembles.

Several previous results for invariant ensembles

Dyson (1962-76), Gaudin-Mehta (1960-) classical Gaussian ensembles via Hermite polynomials

General case by Deift etc. (1999), Pastur-Schcherbina (2008), Bleher-Its (1999), Deift etc (2000-, GOE and GSE), Lubinsky (2008)

All these results are limited to invariant ensembles and to the classical values of $\beta = 1, 2, 4$ (OP Method). For non-classical values, there is no underlying matrix ensemble, but the Gibbs measure

$$p(\lambda_1, \ldots, \lambda_N) = \text{const.} \prod_{i<j} (\lambda_i - \lambda_j)^{\beta} e^{-\beta N \sum_j V(\lambda_j)}$$

can still be studied ("log-gas"). $\Longrightarrow$ PROBLEM 1.

No previous results for Wigner (apart from Johansson's for hermitian matrices with Gaussian convolution)

Universality of Wigner matrices? $\Longrightarrow$ PROBLEM 2.

PROBLEM 1: NON-CLASSICAL β-ENSEMBLES

$$p(\lambda_1, \ldots, \lambda_N) = \text{const.} \prod_{i<j} (\lambda_i - \lambda_j)^{\beta} e^{-\beta N \sum_j V(\lambda_j)}$$

Limit density ϱ is the unique minimizer of

$$I(\nu) = \int_{\mathbb{R}} V(t)\nu(t)\mathrm{d}t - \int_{\mathbb{R}}\int_{\mathbb{R}} \log|t-s|\nu(s)\nu(t)\mathrm{d}t\mathrm{d}s.$$

Theorem [Bourgade-E-Yau, 2011] Let $\beta > 0$ and V be real analytic. Let $p^{(k)}_{V,N}$ and $p^{(k)}_{G,N}$ be the k-point correlation functions for V and for the Gaussian case, $V(x) = x^2/2$.
Fix $E \in \text{int}(\text{supp}\varrho)$, $E' \in \text{int}(\text{supp}\varrho_{sc})$ and $\varepsilon := N^{-1/2}$, then

$$\int_{E-\varepsilon}^{E+\varepsilon} \frac{\mathrm{d}x}{2\varepsilon} \frac{1}{\varrho(E)^k} p^{(k)}_{V,N}\Big(x + \frac{\alpha_1}{N\varrho(E)}, \ldots, x + \frac{\alpha_k}{N\varrho(E)}\Big)$$

$$- \int_{E'-\varepsilon}^{E'+\varepsilon} \frac{\mathrm{d}x}{2\varepsilon} \frac{1}{\varrho_{sc}(E')^k} p^{(k)}_{\mathrm{G},N}\Big(x + \frac{\alpha_1}{N\varrho_{sc}(E')}, \ldots, x + \frac{\alpha_k}{N\varrho_{sc}(E')}\Big) \to 0.$$

weakly in $\alpha_1, \ldots, \alpha_k$ as $N \to \infty$.

PROBLEM 2: NON-INVARIANT WIGNER ENSEMBLES

Theorem [E-Schlein-Yau-Yin, 2009-2010] The bulk universality holds for generalized Wigner ensembles i.e., for $|E| < 2$, $\varepsilon = N^{-1+\delta}, \delta > 0$

$$\lim_{N\to\infty} \int_{E-\varepsilon}^{E+\varepsilon} \frac{\mathrm{d}x}{2\varepsilon} \Big(p_{F,N}^{(k)} - p_{\mu,N}^{(k)} \Big) \Big(x + \frac{b_1}{N}, \dots, x + \frac{b_k}{N} \Big) = 0 \quad \text{weakly}$$

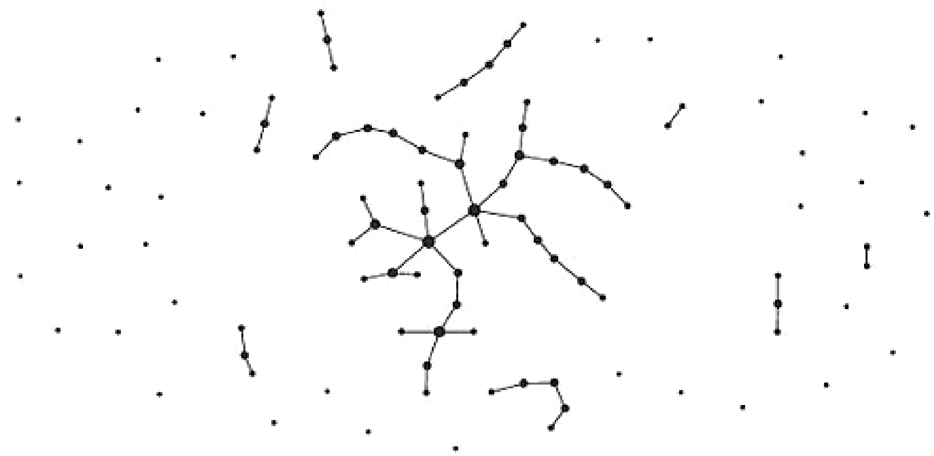
F	μ
generalized symmetric matrices	GOE
generalized hermitian	GUE
generalized self-dual quaternion	GSE
real covariance	real Gaussian Wishart
complex covariance	complex Gaussian Wishart

Variances can vary in this theorem.

We also have a similar result at the spectral edge (universality of Tracy-Widom distribution)

ERDŐS-RÉNYI RANDOM GRAPHS

$N = 100$, $p = 0.01$



Adjacency matrix $A = (a_{ij})$, real symmetric with

$$a_{ij} = \frac{\gamma}{q} \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p\,, \end{cases}$$

where $q := \sqrt{pN}$ and $\gamma = (1-p)^{-1/2}$ so that Var $a_{ij} = N^{-1}$.

Note that $\mathbb{E}\, a_{ij} \neq 0 \Longrightarrow$ there is a large eigenvalue.

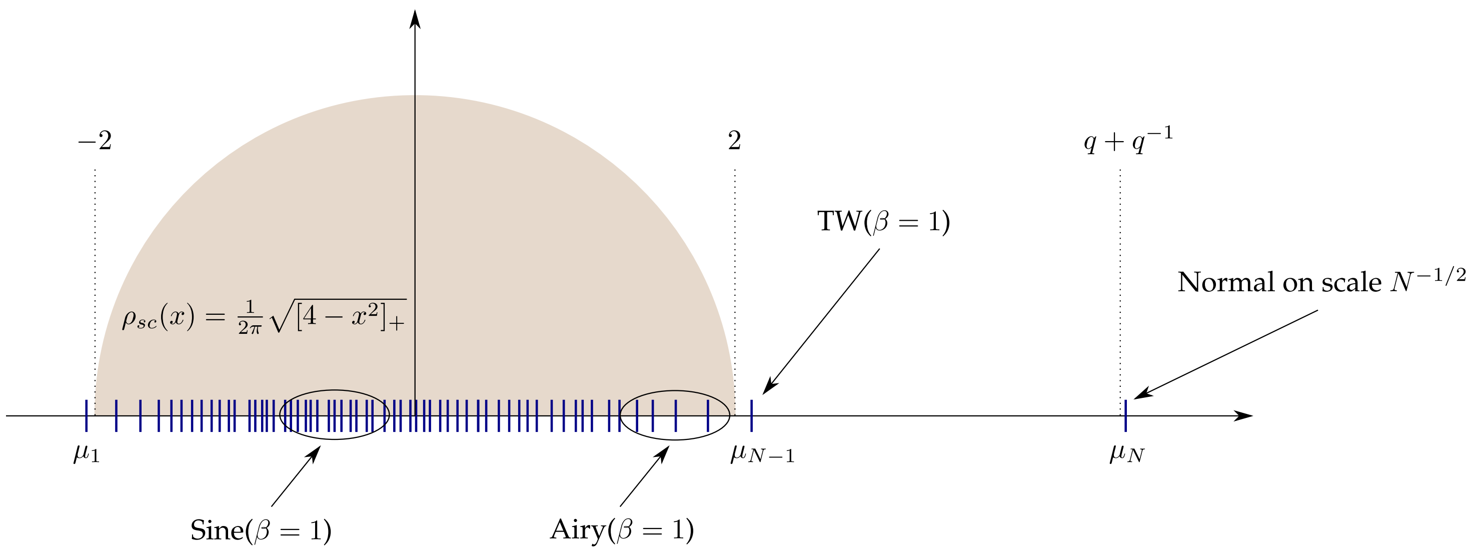
Each column typically has $pN = q^2$ nonvanishing entries.

Theorem [E-Knowles-Yau-Yin, 2011] Bulk and edge universality for Erdős-Rényi sparse matrices with $pN \gg N^{2/3}$.

Bulk is given by the (analogue of) the sine-kernel.

Edge is given by the Airy kernel and Tracy-Widom.

Single outlier is Gaussian.



RECENT RESULTS ON BULK UNIVERSALITY

1. Hermitian ensemble with C^6 distribution. [EPRSY 2009]. (Brezin-Hikami, contour integral and reverse heat flow approach)

2. Hermitian Wigner ensemble with probability law supported on at least three points [Tao-Vu] (Extension to Bernoulli in [ERSTVY]). Symmetric ensemble with the first 4 moments of matrix elements matching the GOE [Tao-Vu] (4-moment approach)

3. Symmetric ensemble with three point condition [E-Schlein-Yau]. (Dyson Brownian Motion (DBM) flow approach)

4. Generalized symmetric or hermitian Wigner ensembles (the variances were allowed to vary) [E-Yau-Yin].

5. Erdős-Rényi sparse matrices with $pN \gg N^{2/3}$ [E-Knowles-Yau-Yin]

Similar development for real and complex sample covariance ensembles [E-Schlein-Yau-Yin], [Tao-Vu], [Peche], and also for edge univ.

KEY STEPS IN OUR PROOF FOR THE WIGNER CASE

Step 1. Good local semicircle law including a control near the edge.

Method: System of self-consistent equations for the Green function, control the error by large deviation methods.

Step 2. Universality for Wigner matrices with a small ($\sim N^{-\varepsilon}$) Gaussian component.

Method: Modify DBM to speed up its local relaxation, then show that the modification is irrelevant for statistics involving differences of eigenvalues.

Step 3. Universality for arbitrary Wigner matrices.

Method: Remove the small Gaussian component in Step 2 by resolvent perturbation theory and moment matching.

Step 1: LOCAL SEMICIRCLE LAW

Green function : $G_{ij} = \frac{1}{H-z}(i,j), \quad m(z) = \frac{1}{N}\mathrm{Tr}G = \frac{1}{N}\sum_i G_{ii}$

Let m_{sc} be the Stieltjes transform of the semicircle measure, i.e.,

$$m_{sc}(z) = \int \frac{\varrho_{sc}(x)\mathrm{d}x}{x-z}, \qquad \varrho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}$$

Theorem [Erdős-Y-Yin, 2010] For any $z = E + i\eta$ with $\eta \gtrsim N^{-1}$ the following holds with exponentially high probability:

$$|m(z) - m_{sc}(z)| \lesssim \frac{1}{N\eta}$$

where $\lesssim$ means up to $(\log N)^{\#}$ factors. Estimates are optimal.

Step 2: DYSON BROWNIAN MOTION

Gaussian convolution matrix interpolates between Wigner and GUE.

Evolve the matrix elements with an OU process:

$$\mathrm{d}H_t = \frac{1}{\sqrt{N}}\mathrm{d}B_t - \frac{1}{2}H_t\mathrm{d}t \qquad H_t \sim e^{-t/2}H_0 + (1-e^{-t})^{1/2}V.$$

$$\mathrm{d}\lambda_i = \frac{1}{\sqrt{N}}\mathrm{d}B_i + \Big(-\frac{1}{2}\lambda_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\Big)\mathrm{d}t$$

Idea: Equilibrium is the invariant ensemble (GUE, etc.) with known local statistics.

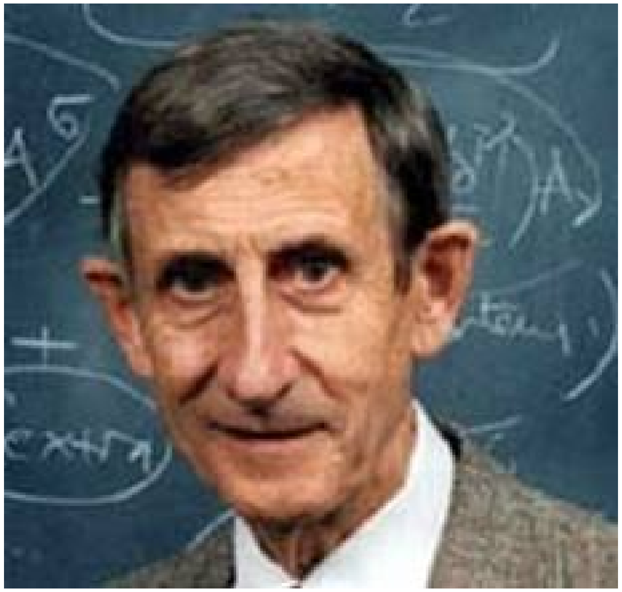
Global equilibrium is reached in time $O(1)$ (convexity, Bakry-Emery).

For local statistics, only local equilibrium needs to be achieved which is much faster. Our main result proves Dyson's conjecture on Dyson's Brownian motion:

"The picture of the gas coming into equilibrium in two well-separated stages, with microscopic and macroscopic time scales, is suggested with the help of physical intuition. A rigorous proof that this picture is accurate would require a much deeper mathematical analysis."

Freeman Dyson, 1962

on the approach to equilibrium of Dyson Brownian Motion



Global equilibrium is reached in time scale of $O(1)$.
Local equilibrium was believed to be reached in $O(N^{-1})$.

OUTLOOK: UNIVERSALITY CONJECTURES

- **Quantum Chaos Conjecture** (vague)

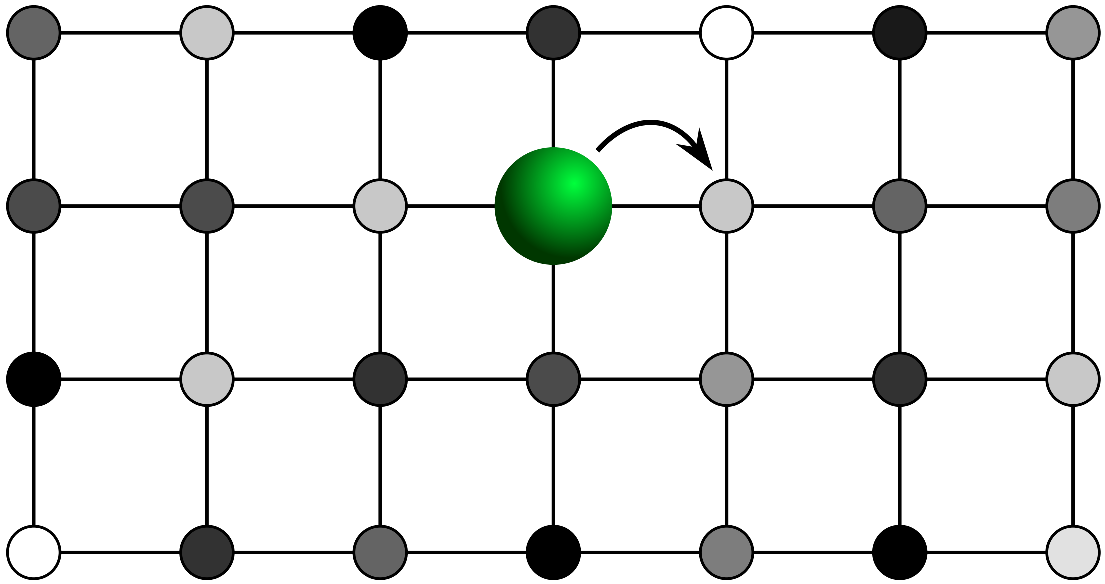
classical dynamics with potential V	e.v. gap of $-\Delta + V$
chaotic	GOE statistics
integrable	Poisson statistics

Geodesic flow: Bohigas-Giannoni-Schmit (1984), Berry-Tabor (1977)

- **Anderson Model (1958):** V_ω random potential on $\mathbb{R}^d$ or $\mathbb{Z}^d$

random Schrödinger operator: $H = -\Delta + \lambda V_\omega$

Depending on λ and d, there are two distinct regimes.



I: Strong disorder regime: Localization, Poisson local statistics, insulator

II: Weak disorder regime: Delocalization, random matrix (GUE, GOE) local statistics, conductor.

I. is relatively well understood, II. is not.

Conjectured Dichotomy: There are essentially two different behaviors for local eigenvalue statistics of disordered quantum systems:

A: Poisson statistics, for systems with little or no correlations.
B: Random matrix statistics: for systems with high correlations.

Fundamental belief of universality: The macroscopic statistics (like density of states) depend on the models, but the microscopic statistics are independent of the details of the systems except the symmetries.

Our results on Wigner matrices verify this conjecture for random matrices, but it is still mean field.

Major goal: move towards random Schrödinger.

INTERMEDIATE MODEL: BAND MATRICES

Matrix rows/columns indexed by a d-dimensional box $\Lambda \subset \mathbb{Z}^d$.
$H_{xy} = H^*_{yx}$, $x, y \in \Lambda$, are independent, centered,

$$s_{xy} = \mathbb{E}|H_{xy}|^2, \quad \sum_y s_{xy} = 1 \qquad \forall y$$

Bandwidth W: $s_{xy} = 0$ if $|x-y| \geq W$. E.g. $(d = 1, W = 3, N = 7)$:

$$H = \begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ 0 & * & * & * & * & * & 0 \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \end{pmatrix},$$

$(W = O(1) \sim$ Random Schrödinger; $W = \Lambda$, $d = 1$ is Wigner)

Physics prediction [with SUSY, Fyodorov-Mirlin (91)]

In d=1 dimensions, the localization length $\ell \sim W^2$, i.e.

Narrow band, $W \ll N^{1/2} \Longrightarrow$ localization, Poisson statistics

Broad band, $W \gg N^{1/2} \Longrightarrow$ delocalization, GOE statistics

Proven:

- Localization for $W \ll N^{1/8}$. [Schenker]
- Delocalization: $W \gg N^{4/5}$ [E-Knowles-Yau-Yin, '12]

We also show that the resolvent kernel $G_{xy} = (H - z)^{-1}_{xy}$ exhibits diffusion in a certain regime.

SUMMARY

1. We proved bulk universality for general $\beta > 0$ ensemble with real analytic potential.

2. We proved bulk and edge universality for generalized Wigner matrices (varying variance and even singular distribution – Erdős-Rényi sparse matrices)

3. We showed (non-optimal) delocalization for band matrices.

OPEN PROBLEMS

1. Universality for sparser matrices (eventually $pN \sim O(1)$).

2. **Random band matrices:** H is symmetric with independent but not identically distributed entries with mean zero and variance

$$\mathbb{E}\,|h_{k\ell}|^2 = W^{-1} e^{-|k-\ell|/W}$$

Conjecture (even Gaussian case is open)

Narrow band, $W \ll \sqrt{N} \Longrightarrow$ localization, Poisson statistics
Broad band, $W \gg \sqrt{N} \Longrightarrow$ delocalization, GOE statistics

3. **Spectral statistics for Anderson model.**

Zeros of the Riemann-zeta function (Detour)

$$\zeta(s) = \sum_n \frac{1}{n^s}, \qquad \zeta(\frac{1}{2} + i\gamma_n) = 0, \qquad \widehat{\gamma}_n = \frac{1}{2\pi}\gamma_n \log \gamma_n, \qquad \delta_n = \widehat{\gamma}_{n+1} - \widehat{\gamma}_n$$

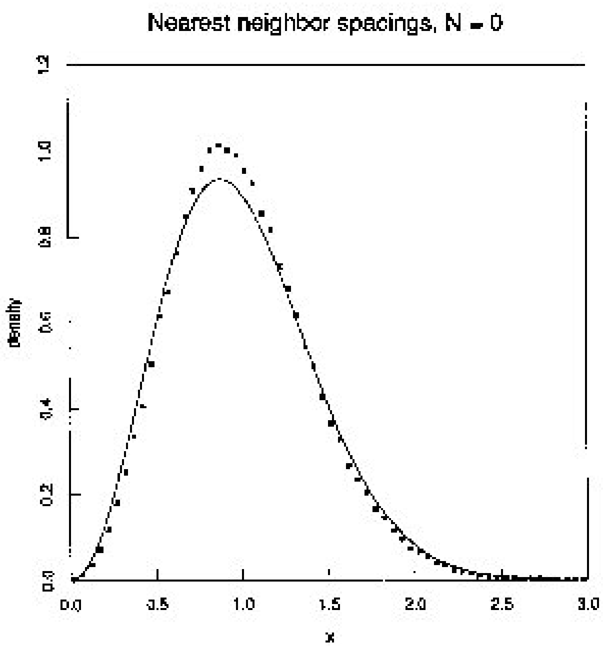


FIGURE 3

Probability density of the normalized spacings δ_n. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $1 \leqslant n \leqslant 10^5$.

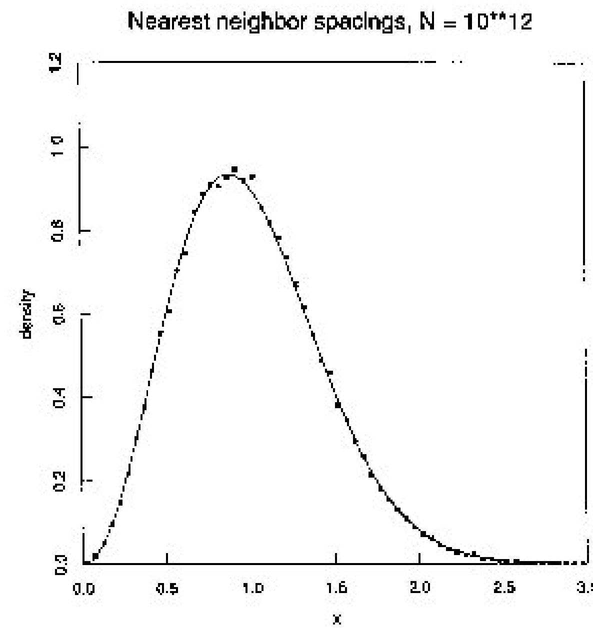


FIGURE 4

Probability density of the normalized spacings δ_n. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $10^{12} + 1 \leqslant n \leqslant 10^{12} - 10^5$.

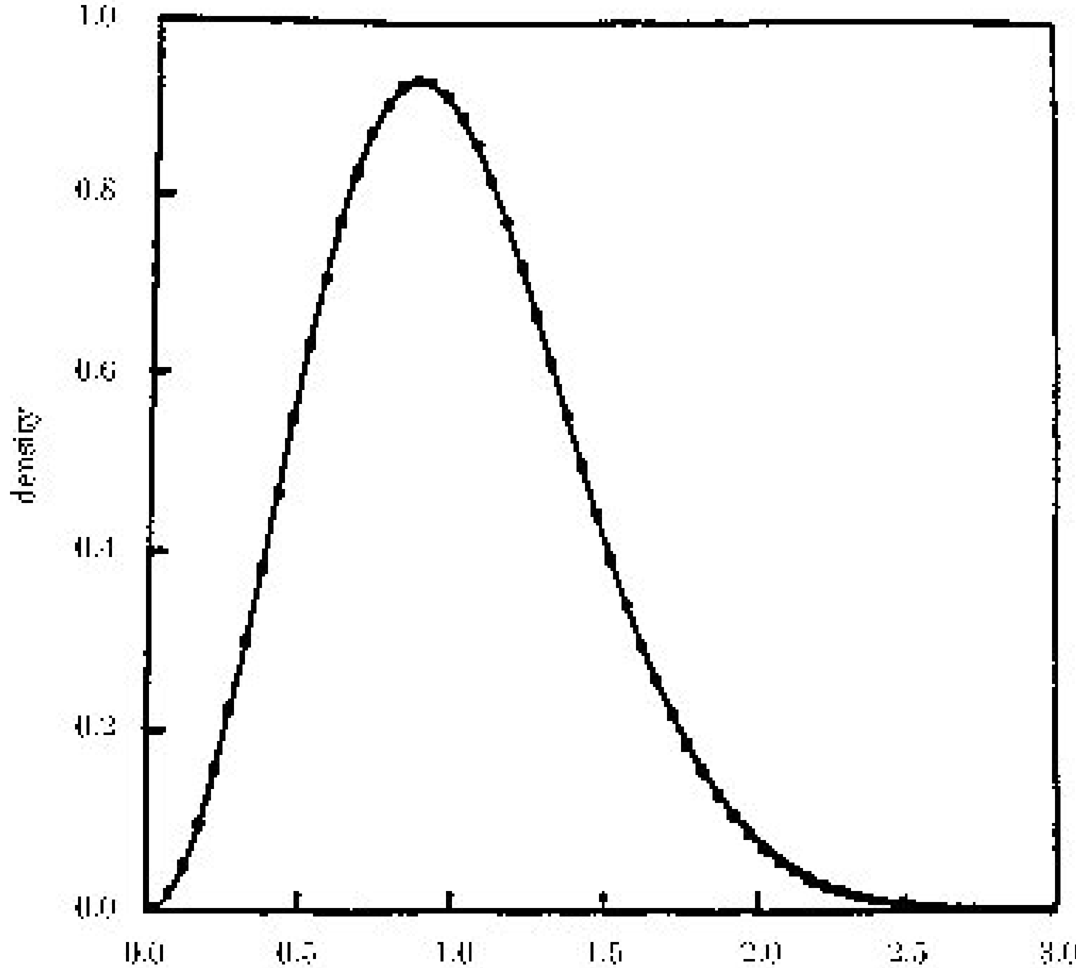


FIGURE 1. Nearest neighbor spacings among 70 million zeroes beyond the 10^{20}-th zero of zeta, verses μ_1(GUE).

[Odlyzko, ATT, 1989]