Kolmogorov equations for stochastic PDE's with multiplicative noise

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1 Introduction

We are here concerned with the following stochastic differential equation in the Hilbert space $H = L^2(0,1)$,

$$\begin{cases} dX(t,\xi) = D_\xi^2 X(t,\xi)dt + g(X(t,\xi))dW(t,\xi), \quad t \geq 0,\ \xi \in [0,1], \\ \\ X(t,0) = X(t,1) = 0, \quad t \geq 0, \\ \\ X(0,\xi) = x(\xi), \quad x \in H,\ \xi \in [0,1], \end{cases} \tag{1}$$

where g is a real function of class C^2 bounded together with its derivatives of order less or equal to 2 and W is a cylindrical Wiener process in H (see below for a precise definition). Existence and uniqueness of a solution of (1) are well known, see [16], [7]. Let us denote by R_t the corresponding transition semigroup,

$$R_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad t \geq 0,\ x \in H, \tag{2}$$

where φ is a real bounded Borel function and $X(t,x)$ is the solution of equation (1).

The Kolmogorov equation corresponding to (1) reads as follows,

$$\begin{cases} u_t(t,x) = \dfrac{1}{2}\,\mathrm{Tr}\,[\sigma^2(x)u_{xx}(t,x)] + \langle Ax, u_x(t,x)\rangle, \quad t \geq 0,\ x \in D(A), \\ \\ u(0,x) = \varphi(x), \quad x \in H, \end{cases} \tag{3}$$

where A is the linear operator,

$$Ax = D_\xi^2, \quad x \in D(A) = H^2(0,1) \cap H_0^1(0,1),$$

and for any $x \in L^2(0,1)$ the symmetric Nemitskii operator $\sigma(x) \in L(L^2(0,1))$ is defined by

$$[\sigma(x)y](\xi)=g(x(\xi))y(\xi),\quad y\in L^2(0,1),\ \xi\in[0,1]. \tag{4}$$

There is an increasing interest on infinite dimensional Kolmogorov equations, see the monographs [15], [5], [13], [8] (and references therein) and the papers [1],[17], [2], [3].

In particular, in [13] the case when the noise is additive is mainly considered with the exception of chapters 6 and 7. More precisely, Chapter 6 is devoted to Hölder continuous perturbations of the infinite dimensional Heat semigroup, see also some recent developpements in this direction in [2] and [3]. In chapter 7 of [13] the case when coefficients are of class C^3 is considered. Notice that in equation (1) the multiplicative noise can be written as $\sigma(x)dW(t)$ where for any $x\in L^2(0,1)$ the operator $\sigma(x)\in L(L^2(0,1))$ is operator defined by (4). Thus, in spite of the fact that g is C^2, the mapping $\sigma(x)$ is only once Gateaux differentiable (except when g is constant). So, the method in [13, Chapter 7] does not work and some new technique has to be used.

The first main result of the paper is that if $\varphi\in C^3_b(H)$ there is a smooth solution of (3), see Theorem 13. It is well known that a candidate for the solution of (3) is given by

$$u(t,x):=R_t\varphi(x)=\mathbb{E}[\varphi(X(t,x))], \tag{5}$$

where R_t is the transition semigroup defined by (2). So, the proof of existence of a solution of (3) will consists in showing (by justifying the chain rule) that $u(t,x)$ is twice differentiable, that the trace of $\sigma^2D^2u(t,x)$ is finite and that equation (3) is fulfilled. However, these computations are not straightforward since σ is neither regular nor of trace class. This idea was applied in the case of reaction–diffusion equations with additive noise, see [6] and in the completely different situation of Navier–Stokes equations with additive noise, see [9].

We notice that the regularity we get for the solution of (3) is not enough to apply the Itô formula and so, to prove the uniqueness of a smoth solution of (3). This would require an additional job which we plan to make in a future paper.

In the second part of the paper we assume that $1/g$ is bounded so that there is a unique invariant measure μ, see [16]. We study here the Kolmogorov equation (2) in the space $L^2(H,\mu)$. It is well known that the transition semigroup R_t can be uniquely extended to a strongly continuous semigroup of contractions in $L^2(H,\mu)$. We shall still denote by R_t this extension and by L_μ the infinitesimal generator of R_t.

As a second main result of the paper we construct a core Γ for the generator L_μ consisting of regular functions and show that on Γ the operator L_μ is in fact a differential operator given by

$$L_\mu\varphi(x)=\frac{1}{2}\,\mathrm{Tr}\,[\sigma^2(x)D^2\varphi(x)]+\langle Ax,D\varphi(x)\rangle,\quad x\in D(A),\quad \varphi\in\Gamma.$$

As it was pointed out in several previous situations, see e.g. [8] and references therein, to have an explicit expression of L_μ on the core Γ allows to prove easily the so–called "identité du carré des champs",

$$\int_H L_\mu\varphi\,\varphi\,d\mu = -\frac{1}{2}\int_H |\sigma(x)D\varphi|^2 d\mu, \quad \varphi\in D(L_\mu). \tag{6}$$

Using (6) it is possible to prove that the derivative operator is closable in $L^2(H,\mu)$. This allows to define the Sobolev space $W^{1,2}(H,\mu)$ and to show that the domain of L_μ is included in $W^{1,2}(H,\mu)$.

Finally, we prove the Poincaré inequality, by generalizing a result proved in [10] in the case of additive noise. As a standard consequence we obtain the spectral gap of L_μ and the exponential convergence to equilibrium of R_t.

It would be interesting to consider the more general problem,

$$\begin{cases} dX(t,\xi) = (D_\xi^2 X(t,\xi) + f(X(t,\xi)))dt + g(X(t,\xi))dW(t,\xi), \quad \xi\in[0,1],\\[2mm] X(t,0)=X(t,1)=0, \quad t\ge 0\\[2mm] X(0,\xi)=x(\xi), \quad x\in H,\ \xi\in[0,1], \end{cases} \tag{7}$$

where f is a suitable real function. However, problem (7) does not seem to be a straightforward generalization of (1). It will be the object of a future research.

1.1 Notations

We denote by H the Hilbert space $L^2(0,1)$ (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$). When there is the danger of confusion between the norm and the absolute value of a function x we shall write $|x|_{L^2(0,1)}$ instead of $|x|$. Moreover $L(H)$ (norm $\|\cdot\|$) will represent the Banach algebra of all linear bounded operators in H and $L_1(H)$ (norm $\|\cdot\|_{L_1(H)}$) the space of all trace class operators in H. We recall that

$$\|T\| = \sup\{|Tx|:\ x\in H,\ |x|=1\}, \quad T\in L(H).$$

For any Hilbert space K (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$), we denote by $C_b(H;K)$ the linear space of all continuous and bounded mappings $\varphi\colon H\to K$. $C_b(H;K)$ endowed with the norm

$$\|\varphi\|_0 = \sup_{x\in K}|\varphi(x)|, \quad \varphi\in C_b(H;K),$$

is a Banach space.

Moreover $C_b^1(H;K)$ will represent the subspace of $C_b(H;K)$ of all functions $\varphi\colon H\to K$ which are Fréchet differentiable on H with a continuous and bounded derivative $D\varphi$. The space $C_b^k(H;K)$ for $k\ge 2$ are defined analogously. We shall write $C_b^i(H;\mathbb{R}) = C_b^i(H)$, $i\in\mathbb{N}$.

If $\varphi \in C_b^1(H)$ and $x \in H$, we shall identify $D\varphi(x)$ with the unique element h of H such that

$$D\varphi(x)y = \langle h, y\rangle, \quad x, y \in H.$$

If $\varphi \in C_b^2(H)$ and $x \in H$, we shall identify $D^2\varphi(x)$ with the unique linear operator $T \in L(H)$ such that

$$D\varphi(x)(y, z) = \langle Ty, z\rangle, \quad x, y, z \in H.$$

1.2 An extension of Gronwall's lemma

The following result is a generalization of a well known result.

Lemma 1. *Assume that* $f\colon [0, +\infty) \to [0, +\infty)$ *fulfills the inequality,*

$$f(t) \le a(t) + b \int_0^t (t-s)^{-1/2} f(s)ds, \quad t \ge 0, \tag{8}$$

where a *is continuous nonnegative and* b *is a nonnegative constant. Then we have,*

$$\begin{aligned} f(t) \le{}& a(t) + b \int_0^t (t-s)^{-1/2} a(s)ds \\ &+ \int_0^t e^{(t-s)\pi b^2} \left[a(s) + b\int_0^s (s-\sigma)^{-1/2} a(\sigma)d\sigma\right]. \end{aligned} \tag{9}$$

If, in particular, $a(t) = a$ *we have*

$$f(t) \le a e^{\pi b^2 t} + 2ab \int_0^t s^{-1/2} e^{\pi b^2 (t-s)} ds \quad t \ge 0. \tag{10}$$

and

$$f(t) \le 3a e^{\pi b^2 t} \quad t \ge 0. \tag{11}$$

Proof. We write (8) as

$$f \le a + b\psi_{-1/2} * f, \tag{12}$$

where $\psi_{-1/2}(t) = t^{-1/2}$ and $*$ denotes the convolution (^{1}). Taking the convolution of both sides of (10) with $\psi_{-1/2}$ and taking into account that

$$(\psi_{-1/2} * \psi_{-1/2})(t) = \int_0^t (t-s)^{-1/2} s^{-1/2} ds = \pi,$$

yields

$$\psi_{-1/2} * f \le a * \psi_{-1/2} + \pi b(1 * f). \tag{13}$$

Substituting this in (12) yields

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[1] $(f * g)(t) = \int_0^t f(t-s)g(s)ds.$

$$f \le a + b(a * \psi_{-1/2}) + \pi b^2 (1 * f), \tag{14}$$

which is equivalent to

$$f(t) \le a(t) + b\int_0^t (t-s)^{-1/2} a(s) ds + \pi b^2 \int_0^t f(s) ds. \tag{15}$$

Consequently

$$\begin{aligned} f(t) \le a(t) + b\int_0^t (t-s)^{-1/2} a(s) ds \\ + \int_0^t e^{(t-s)\pi b^2} \left[a(s) + b\int_0^s (s-\sigma)^{-1/2} a(\sigma) d\sigma \right] \end{aligned}$$

Now (9) follows from the classical Gronwall lemma. Finally, (10) is clear and by (10) we have

$$\begin{aligned} f(t) \le a e^{\pi b^2 t} \left(1 + 2b\int_0^t s^{-1/2} e^{-\pi b^2 s} ds \right) \\ \le a e^{\pi b^2 t} \left(1 + 2b\int_0^\infty s^{-1/2} e^{-\pi b^2 s} ds \right), \end{aligned}$$

which yields (11). □

2 Existence and uniqueness of solutions

2.1 The abstract setting

Let us write problem (1) in an abstract form introducing the linear self–adjoint operator $A\colon D(A) \subset H \to H$,

$$\begin{cases} Ax = D_\xi^2 x, \quad x \in D(A), \\ D(A) = H^2(0,1) \cap H_0^1(0,1), \end{cases}$$

where $H^i(0,1),\ i = 1,2$ denote the usual Sobolev spaces and

$$H_0^1(0,1) = \{x \in H_0^1(0,1) :\ x(0) = x(1) = 0\}.$$

We define moreover the (generally) nonlinear operator $\sigma\colon H \to L(H)$ by setting,

$$[\sigma(x)y](\xi) = g(x(\xi))y(\xi), \quad \xi \in [0,1],\ x, y \in H.$$

We denote by (e_k) the complete orthonormal sistem in H consisting of the eigenfunctions of A,

$$e_k(\xi) = \sqrt{\frac{2}{\pi}}\ \sin k\pi\xi, \quad \xi \in [0,1],\ k \in \mathbb{N},$$

so that

$$Ae_k = -k^2\pi^2 e_k, \quad k \in \mathbb{N}.$$

Notice that

$$\|e^{tA}\| \le e^{-\pi^2 t}, \quad t \ge 0.$$

Finally, we introduce the cylindrical white noise,

$$W(t) = \sum_{k=1}^{\infty} e_k \beta_k(t), \quad t \ge 0, \tag{16}$$

where (β_k) is a sequence of mutually independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$.

Now we can write problem (1) as follows,

$$\begin{cases} dX = AXdt + \sigma(X)\, dW(t) = AXdt + \displaystyle\sum_{k=1}^{\infty} [g(X)e_k]\, d\beta_k(t), \\ \\ X(0) = x. \end{cases} \tag{17}$$

We shall solve equation (17) in the space $C_W([0,T],H)$ of all mean square continuous adapted (to the filtration $(\mathcal{F})_{t\ge 0}$) stochastic process $X(\cdot)$ defined in $[0,T]$ and taking values in H. It is well known that $C_W([0,T],H)$, endowed with the norm

$$\|X\|_{C_W([0,T],H)} = \left(\sup_{t\in[0,T]} \mathbb{E}(|X(t)|^2) \right)^{1/2},$$

is a Banach space.

Definition 2. *A* mild solution *of equation* (17) *is a process* $X \in C_W([0,T],H)$ *such that*

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}\sigma(X(s))dW(s), \quad t \ge 0,\ x \in H. \tag{18}$$

In the following we shall denote by $X(\cdot, x)$ the solution of (18).

An important rôle will be played by the *stochastic convolution*,

$$W_X(t) = \int_0^t e^{(t-s)A}\sigma(X(s))dW(s) = \sum_{k=1}^{\infty} \int_0^t e^{(t-s)A}[g(X(s))e_k]d\beta_k(s),$$

where $X \in C_W([0,T],H)$.

As we shall see, though the the cylindrical white noise (16) does not leave in H (see e.g. [11, §4.3.1]), the stochastic convolution $W_X(t)$ does.

In order to study basic properties of $W_X(t)$ it is useful to introduce the function

$$F(t)=\sum_{h=1}^\infty e^{-th^2},\quad t>0. \tag{19}$$

Notice that

$$F(t)\le e^{-t}+\int_1^\infty e^{-tx^2}dx=e^{-t}\left(1+\int_1^\infty e^{-t(x^2-1)}dx\right)$$

$$=e^{-t}\left(1+\int_0^\infty e^{-ty^2}\frac{y}{\sqrt{y^2+1}}\,dy\right)\le e^{-t}\left(1+\int_0^\infty e^{-ty^2}dy\right).$$

So,

$$F(t)\le e^{-t}\left(1+2t^{-1/2}\right)\le 4t^{-1/2}e^{-t/2},\quad t>0. \tag{20}$$

Lemma 3. *Let $X\in C_W([0,T],H)$. Then we have*

$$\mathbb{E}(|W_X(t)|^2)=\int_0^t\sum_{h=1}^\infty e^{-2\pi^2h^2(t-s)}\mathbb{E}(|g(X(s))e_h|^2)ds. \tag{21}$$

Proof. Taking into account the independence of the (β_k) we have,

$$\mathbb{E}(|W_X(t)|^2)=\sum_{k=1}^\infty\int_0^t\mathbb{E}(|e^{(t-s)A}[g(X(s))e_k]|^2)ds.$$

Using the Parseval identity we find,

$$\sum_{k=1}^\infty|e^{(t-s)A}[g(X(s))e_k]|^2=\sum_{h,k=1}^\infty|\langle e^{(t-s)A}[g(X(s))e_k],e_h\rangle|^2$$

$$=\sum_{h,k=1}^\infty e^{-2(t-s)\pi^2h^2}|\langle g(X(s))e_k,e_h\rangle|^2=\sum_{h,k=1}^\infty e^{-2(t-s)\pi^2h^2}|\langle e_k,g(X(s))e_h\rangle|^2$$

$$=\sum_{h=1}^\infty e^{-2(t-s)\pi^2h^2}|\sigma(X(s))e_h|^2.$$

So, (21) follows. □

Proposition 4. *Let $X\in C_W([0,T],H)$. Then we have*

$$\mathbb{E}(|W_X(t)|^2)\le 8\sqrt{\pi}\;\|g\|_0^2,\quad t\ge 0. \tag{22}$$

Moreover, for all $X,Y\in C_W([0,T],H)$ we have,

$$\begin{aligned}&\mathbb{E}(|W_X(t)-W_Y(t)|^2)\\&=\frac{8\|g\|_1^2}{\pi\sqrt{2\pi}}\int_0^t e^{-(t-s)\pi^2}(t-s)^{-1/2}\mathbb{E}[|X(s)-Y(s)|^2)]ds.\end{aligned} \tag{23}$$

Proof. By (21) we have, recalling that $|e_k(\xi)|^2 \le \frac{2}{\pi}$ and taking into account (20),

$$\mathbb{E}|W_X(t)|^2 \le \frac{2\|g\|_0^2}{\pi} \int_0^t F(2\pi^2 s)ds \le \frac{8\|g\|_0^2}{\pi} \int_0^\infty e^{-\pi^2 s} s^{-1/2} ds$$

and (22) follows. The proof of (23) is similar. □

2.2 Existence and uniqueness

The following result is well known, see e.g. [16], we present however the short proof for the reader's convenience.

Proposition 5. *For any $x \in H$ there exists a unique solution $X(\cdot, x)$ of equation* (18)*.*

Proof. Write equation (18) in the form

$$X = e^{tA}x + \varLambda(X), \quad X \in C_W([0,T],H),$$

where

$$\varLambda(X)(t) = W_X(t), \quad t \in [0,T].$$

Then by (22) it follows that $\varLambda$ maps $C_W([0,T],H)$ in itself. Let moreover $X, Y \in C_W([0,T],H)$. Then by (23) it follows that,

$$\begin{aligned}|\varLambda(X)(t) - \varLambda(Y)(t)| &\le \frac{8\|g\|_1^2}{\pi\sqrt{\pi}} \int_0^t e^{-(t-s)\pi^2}(t-s)^{-1/2} ds \|X - Y\|_{C_W([0,T],H)} \\ &\le \frac{16\|g\|_1^2}{\pi\sqrt{2\pi}}\, t^{1/2}\, \|X - Y\|_{C_W([0,T],H)}.\end{aligned}$$

Now let $T_1 \in (0,T]$ be such that $\frac{16\|g\|_1^2}{\pi\sqrt{2\pi}}\, T_1^{1/2} < 1$. Then $\varGamma$ is a contraction on $C_W([0,T_1],H)$. Therefore, equation (18) has a unique solution on $[0,T_1]$. By a similar argument, one can show existence and uniqueness on $[T_1, 2T_1]$ and so on. □

2.3 Galerkin approximations

It is useful to consider Galerkin approximations of equation (18). For any $n \in \mathbb{N}$ we denote by P_n the projector

$$P_n x = \sum_{k=1}^n \langle x, e_k\rangle e_k, \quad x \in H$$

and set $A_n = AP_n$. Then we consider the equation

$$X^n(t,x) = e^{tA_n}x + \sum_{k=1}^{n} \int_0^t e^{(t-s)A_n}[g(X^n(s,x))e_k]d\beta_k(s). \tag{24}$$

The following result is standard.

Proposition 6. *For any* $T > 0$, $x \in H$ *and* $n \in \mathbb{N}$, *there exists a unique solution* $X^n(\cdot,x)$ *of equation* (24). *Moreover,*

$$\lim_{n\to\infty} X^n(\cdot,x) = X(\cdot,x), \quad \text{in } C_W([0,T],H), \tag{25}$$

where $X(\cdot,x)$ *is the solution of* (18).

3 Kolmogorov equation

3.1 Setting of the problem

We are here concerned with the following *Kolmogorov equation*,

$$\begin{cases} u_t(t,x) = \dfrac{1}{2}\,\mathrm{Tr}\,[\sigma^2(x)u_{xx}(t,x)] + \langle Ax, u_x(t,x)\rangle, \quad t \geq 0,\ x \in D(A), \\[2ex] u(0,x) = \varphi(x), \quad x \in H. \end{cases} \tag{26}$$

We are going to show that when the initial datum φ is sufficiently regular equation (26) has a solution in a classical sense. As it is well known, a candidate for the solution $u(t,x)$ of (26) is provided by the formula

$$u(t,x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in C_b(H),\ t \geq 0,\ x \in H, \tag{27}$$

where $X(t,x)$ is the solution of (18). We shall check that, under suitable assumptions on φ, formula (27) produces in fact a solution of (26).

We shall need to consider the approximating equation

$$\begin{cases} u^n_t(t,x) = \dfrac{1}{2}\,\mathrm{Tr}\,[P_n\sigma^2(x)u^n_{xx}(t,x)] + \langle A_n x, u^n_x(t,x)\rangle \\[2ex] u^n(0,x) = \varphi(P_n x), \end{cases} \tag{28}$$

which has a unique strict solution which we denote by $u^n(t,x)$.

3.2 Estimates for derivatives of $X(t,x)$

This subsection is devoted to establish some estimates concerning the derivatives X_x and X_{xx}, which will be used later. We start from the directional derivative

$$\eta^z(t,x) := X_x(t,x)z = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\,(X(t,x+\epsilon z) - X(t,x))$$

where $z \in H$. By using Galerkin approximations it is not difficult to show that the diretional derivative $\eta^z(t,x)$ does exist and it is the solution of the equation,

$$\eta^z(t,x) = e^{tA}z + \sum_{k=1}^{\infty} \int_0^t e^{(t-s)A}[g'(X(s,x))\eta^z(s,x)e_k]d\beta_k(s). \tag{29}$$

Lemma 7. *There exists two positive constants* a_1 *and* λ_1 *such that*

$$\mathbb{E}(|\eta^z(t,x)|^2) \le a_1|z|^2 e^{\lambda_1 t} \quad t \ge 0,\ x \in H. \tag{30}$$

Proof. We have

$$\mathbb{E}(|\eta^z(t,x)|^2) = |e^{tA}z|^2 + \sum_{k=1}^{\infty} \mathbb{E}\int_0^t |e^{(t-s)A}[g'(Y(s,x))\eta^z(s,x)e_k]|^2 ds.$$

Arguing as in the proof of Lemma 3 and taking into account (21), we see that,

$$\begin{aligned}\mathbb{E}(|\eta^z(t,x)|^2) &\le |e^{tA}z|^2 + \frac{2}{\pi}\,\|g'\|_0^2 \int_0^t F(2\pi^2(t-s))\mathbb{E}(|\eta^z(s,x)|^2)ds \\ &\le |e^{tA}z|^2 + \frac{8\|g'\|_0^2}{\sqrt{2}\pi^2}\, \int_0^t (t-s)^{-1/2}e^{(t-s)/2}\mathbb{E}(|\eta^z(s,x)|^2)ds.\end{aligned}$$

By Lemma 1 it follows that

$$\mathbb{E}(|\eta^z(t,x)|^2) \le |e^{tA}z|^2 e^{\frac{16\|g'\|_0^2}{\sqrt{2}\pi^2}\, t^{1/2}}, \quad t \ge 0$$

and the conclusion follows. □

We want now to estimate $\zeta^z(t,x) := X_{xx}(t,x)(z,z)$ where

$$X_{xx}(t,x)(z,z) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\,(\eta^z(t,x+\epsilon z) - \eta^z(t,x))$$

and $z \in H$. Formally $\zeta^z(t,x)$ is the solution of the equation,

$$\begin{aligned}\zeta^z(t,x) &= \sum_{k=1}^{\infty}\int_0^t e^{(t-s)A}[g'(Y(s,x))\zeta^z(s,x)e_k]d\beta_k(s) \\ &\quad + \sum_{k=1}^{\infty}\int_0^t e^{(t-s)A}[g''(Y(s,x))(\eta^z(s,x))^2 e_k]d\beta_k(s).\end{aligned} \tag{31}$$

Here a problem arises since the term

$$g''(Y(s,x))(\eta^z(s,x))^2 e_k, \tag{32}$$

which appears in the second integral, belongs to $L^4(0,1)$ and not to $L^2(0,1)$ in general. For this reason we first need an estimate for

$$\mathbb{E}(|\eta^z(t,x)|^4_{L^4(0,1)}) = \mathbb{E}(|[\eta^z(t,x)]^2|^2_{L^2(0,1)}).$$

To get this estimate we shall proceed in two steps.

(i) We shall estimate $\mathbb{E}(|\eta^z(t,x)|^4_{L^2(0,1)})$.
(ii) We shall estimate $\mathbb{E}(|(-A)^{1/8}\eta^z(t,x)|^4_{L^2(0,1)})$.

Then we notice that, by the Sobolev embedding theorem, we have

$$D((-A)^{1/8}) \subset L^4(0,1)$$

and so we end up with the required estimate for $\mathbb{E}(|\eta^z(t,x)|^4_{L^4(0,1)})$.

Let us write (29) as

$$\eta^z(t,x) = e^{tA}z + \int_0^t \Phi(t-s)dW(s), \tag{33}$$

where

$$\Phi(t-s) = e^{(t-s)A}\sigma(s) = \sum_{k=1}^{\infty} e^{(t-s)A}[g'(X(s,x))\eta^z(s,x)e_k]. \tag{34}$$

We shall use the following Burkholder estimate, see [11].

$$\mathbb{E}\left[\left|\int_0^t \Phi(t-s)dW(s)\right|^4\right] \le c\mathbb{E}\left[\left(\int_0^t \|\Phi(t-s)\|^2_{HS}ds\right)^2\right], \tag{35}$$

where c is a given positive constant and

$$\begin{aligned}\|\Phi(t-s)\|^2_{HS} &= \sum_{h,k=1}^{\infty} \langle e^{(t-s)A}[g'(X(s,x))\eta^z(s,x)e_k], e_h\rangle|^2\\ &= \sum_{h,k=1}^{\infty} e^{-2\pi^2h^2(t-s)}\langle g'(X(s,x))\eta^z(s,x)e_k, e_h\rangle|^2\\ &= \sum_{h=1}^{\infty} e^{-2\pi^2h^2(t-s)}|g'(X(s,x))\eta^z(s,x)e_h\rangle|^2 \qquad (36)\\ &\le \frac{2}{\pi}\ \|g\|_1^2F(2\pi^2(t-s))|\eta^z(s,x)|^2\\ &\le \frac{32}{\pi^2}\ \|g\|_1^2(t-s)^{-1/2}|\eta^z(s,x)|^2.\end{aligned}$$

where we have used (21). Now we are ready to prove

Lemma 8. *Let $z \in L^2(0,1)$. Then there exists constants $a_2 > 0$ and $\lambda_2 > 0$ such that*

$$\mathbb{E}(|\eta^z(t,x)|^4_{L^2(0,1)}) \le a_2e^{\lambda_2t}|z|^4_{L^2(0,1)}, \quad t \ge 0,\ x \in H. \tag{37}$$

Proof. Let $z \in L^2(0,1)$. By (33) we have

$$\mathbb{E}(|\eta^z(t,x)|^4_{L^2(0,1)}) \le 8|e^{tA}z|^4_{L^2(0,1)} + 8\mathbb{E}\left[\left|\int_0^t \varPhi(t-s)dW(s)\right|^4_{L^2(0,1)}\right].$$

which, taking into account (35), yields

$$\mathbb{E}(|\eta^z(t,x)|^4_{L^2(0,1)}) \le 8|e^{tA}z|^4_{L^2(0,1)} + 8c\mathbb{E}\left[\left(\int_0^t \|\varPhi(t-s)\|^2_{HS}ds\right)^2\right]. \quad (38)$$

Let us estimate the term $J := \mathbb{E}\left[\left(\int_0^t \|\varPhi(t-s)\|^2_{HS}ds\right)^2\right]$. We have by (36)

$$J \le \frac{2^{10}}{\pi^4}\ \|g\|_1^4\mathbb{E}\left[\left(\int_0^t (t-s)^{-1/2}|\eta^z(s,x)|^2_{L^2(0,1)}ds\right)^2\right]$$

$$= \frac{2^{10}}{\pi^4}\ \|g\|_1^4 \int_0^t\int_0^t (t-s)^{-1/2}(t-s_1)^{-1/2}$$

$$\mathbb{E}\left[|\eta^z(s,x)|^2_{L^2(0,1)}|\eta^z(s_1,x)|^2_{L^2(0,1)}\right]dsds_1$$

$$\le \frac{2^9}{\pi^4}\ \|g\|_1^4 \int_0^t\int_0^t (t-s)^{-1/2}(t-s_1)^{-1/2}\mathbb{E}\left[|\eta^z(s,x)|^4_{L^2(0,1)}\right]dsds_1$$

$$+\frac{2^9}{\pi^4}\ \|g\|_1^4 \int_0^t\int_0^t (t-s)^{-1/2}(t-s_1)^{-1/2}\mathbb{E}\left[|\eta^z(s_1,x)|^4_{L^2(0,1)}\right]dsds_1$$

$$\le \frac{2^{11}}{\pi^4}\ \|g\|_1^4 t^{1/2}\int_0^t (t-s)^{-1/2}\mathbb{E}\left[|\eta^z(s,x)|^4_{L^2(0,1)}\right]ds.$$

Substituting in (38) yields

$$\mathbb{E}(|\eta^z(t,x)|^4_{L^2(0,1)}) \le 8|e^{tA}z|^4_{L^2(0,1)}$$

$$+c\ \frac{2^{14}}{\pi^4}\ \|g\|_1^4 t^{1/2}\int_0^t (t-s)^{-1/2}\mathbb{E}\left[|\eta^z(s,x)|^4_{L^2(0,1)}\right]ds.$$

Now by the Gronwall Lemma it follows that there exist positive constants ρ, l such that

$$\mathbb{E}(|\eta^z(t,x)|^4_{L^2(0,1)}) \le 8|e^{tA}z|^4_{L^2(0,1)} + \rho\int_0^t (t-s)^{-1/2}e^{l(t-s)}|e^{sA}z|^4_{L^2(0,1)}ds, \quad (39)$$

which implies the conclusion. □

Lemma 9. *Let $z \in L^2(0,1)$. Then there exists constants $a_3 > 0$ and $\lambda_3 > 0$ such that*

$$\mathbb{E}(|(-A)^{1/8}\eta^z(t,x)|^4_{L^2(0,1)}) \le a_3 e^{\lambda_3 t}|(-A)^{1/8}e^{tA}z|^4_{L^2(0,1)}, \quad t \ge 0,\ x \in H. \tag{40}$$

Proof. Let $z \in L^2(0,1)$. Then we have

$$\begin{aligned}\mathbb{E}(|(-A)^{1/8}\eta^z(t,x)|^4_{L^2(0,1)}) &\le 8|(-A)^{1/8}e^{tA}z|^4_{L^2(0,1)}\\ &+8\mathbb{E}\left[\left|\int_0^t \Phi_1(t-s)dW(s)\right|^4_{L^2(0,1)}\right].\end{aligned}$$

where

$$\Phi_1(t-s) = (-A)^{1/8}e^{(t-s)A}\sigma(s) = \sum_{k=1}^\infty (-A)^{1/8}e^{(t-s)A}[g'(X(s,x))\eta^z(s,x)e_k].$$

We have

$$\begin{aligned}\|\Phi_1(t-s)\|^2_{HS} &= \sum_{h,k=1}^\infty \langle(-A)^{1/8}e^{(t-s)A}[g'(X(s,x))\eta^z(s,x)e_k], e_h\rangle|^2\\ &= \sum_{h,k=1}^\infty (\pi h)^{1/2}e^{-2\pi^2h^2(t-s)}\langle g'(X(s,x))\eta^z(s,x)e_k, e_h\rangle|^2\\ &= \sum_{h=1}^\infty (\pi h)^{1/2}e^{-2\pi^2h^2(t-s)}|g'(X(s,x))\eta^z(s,x)e_h\rangle|^2.\end{aligned}$$

It is not difficult to show that there is a positive constant d_1 such that

$$\|\Phi_1(t-s)\|^2_{HS} \le d_1(t-s)^{-3/4}|\eta^z(s,x)|^2. \tag{41}$$

Now we have

$$\begin{aligned}\mathbb{E}(|(-A)^{1/8}\eta^z(t,x)|^4_{L^2(0,1)}) &\le 8|(-A)^{1/8}e^{tA}z|^4_{L^2(0,1)}\\ &+8c\mathbb{E}\left[\left(\int_0^t \|\Phi_1(t-s)\|^2_{HS}ds\right)^2\right].\end{aligned} \tag{42}$$

By proceeding as in the proof of the previous lemma we find that there is $d_2 > 0$ such that

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^t \|\Phi_1(t-s)\|^2_{HS}ds\right)^2\right]\\ &\qquad\le d_2\ \|g\|_1^4 t^{1/4}\int_0^t (t-s)^{-3/4}\mathbb{E}\left[|\eta^z(s,x)|^4_{L^2(0,1)}\right]ds\end{aligned}$$

which yields

$$\mathbb{E}(|(-A)^{1/8}\eta^z(t,x)|^4_{L^2(0,1)}) \le 8|(-A)^{1/8}e^{tA}z|^4_{L^2(0,1)}$$

$$+d_2t^{1/2}\int_0^t (t-s)^{-3/4}\mathbb{E}\left[|\eta^z(s,x)|^4_{L^2(0,1)}\right]ds.$$

So, the conclusion follows from Lemma 8. □

Now, using the Sobolev embedding $D((-A)^{1/8}) \subset L^4(0,1)$ we can conclude that

Lemma 10. *Let* $z \in L^2(0,1)$. *Then there exists constants* $a_4 > 0$ *and* $\lambda_4 > 0$ *such that*

$$\mathbb{E}(|\eta^z(t,x)|^4_{L^4(0,1)}) \le a_4e^{\lambda_4 t}|z|^4_{L^2(0,1)}, \quad t \ge 0,\ x \in H. \tag{43}$$

Now we are in position to estimate ζ^z for $z \in L^2(0,1)$.

Lemma 11. *There exists constants* $a_5 > 0$ *and* $\lambda_5 > 0$ *such that for all* $z \in L^2(0,1)$, *we have*

$$\mathbb{E}(|\zeta^z(t,x)|^2) \le a_5|z|^4_{L^2(0,1))}e^{\lambda_5 t}, \quad t \ge 0,\ x \in H. \tag{44}$$

Proof. Let $z \in L^2(0,1)$. Using Galerkin approximations we can show that (31) holds. ¿From (31) we deduce that,

$$\mathbb{E}(|\zeta^z(t,x)|^2) \le 2\mathbb{E}\sum_{k=1}^\infty \int_0^t |e^{(t-s)A}[g'(X(s,x))\zeta^z(s,x)e_k]|^2ds$$

$$+2\mathbb{E}\sum_{k=1}^\infty \int_0^t |e^{(t-s)A}[g''(X(s,x))(\eta^z(s,x))^2e_k]|^2ds:$$

$$= 2J_1 + 2J_2.$$

Arguing as in the proof of Lemma 3 we find

$$\begin{aligned}
J_1 &= \mathbb{E}\sum_{h,k=1}^\infty |\langle e^{(t-s)A}[g'(X(s,x))\zeta^z(s,x)e_k], e_h\rangle|^2 \\
&= \mathbb{E}\sum_{h,k=1}^\infty e^{-2(t-s)\pi^2h^2}|\langle g'(X(s,x))\zeta^z(s,x)e_k], e_h\rangle|^2 \\
&= \mathbb{E}\sum_{h=1}^\infty e^{-2(t-s)\pi^2h^2}|g'(X(s,x))\zeta^z(s,x)e_h|^2 \\
&\le \frac{2}{\pi}\ \|g\|_1^2F(2(t-s)\pi^2)\mathbb{E}|\zeta^z(s,x)|^2 \le \frac{4}{\pi^2}(t-s)^{-1/2}\ \mathbb{E}|\zeta^z(s,x)|^2.
\end{aligned} \tag{45}$$

Similarly for J_2 we find, taking into account Lemma 10,

$$
\begin{aligned}
J_2 &= \mathbb{E} \sum_{h,k=1}^{\infty} |\langle e^{(t-s)A}[g''(X(s,x))(\eta^z(s,x))^2 e_k]|, e_h\rangle|^2 \\
&= \mathbb{E} \sum_{h,k=1}^{\infty} e^{-2(t-s)\pi^2 h^2} |\langle g'(X(s,x))(\eta^z(s,x))^2 e_k], e_h\rangle|^2 \\
&= \mathbb{E} \sum_{h=1}^{\infty} e^{-2(t-s)\pi^2 h^2} |g'(X(s,x))(\eta^z(s,x))^2 e_h|^2 \\
&\le \frac{2}{\pi} \, \|g\|_1^2 F(2(t-s)\pi^2) \mathbb{E}|(\eta^z(s,x))^2|^2 \\
&\le \frac{4}{\pi^2}(t-s)^{-1/2} \, \mathbb{E}|(\eta^z(s,x))^2|^2 \le a_4 \frac{4}{\pi^2}(t-s)^{-1/2} \, e^{\lambda_4 s} |z|^4_{L^2(0,1)}.
\end{aligned}
\tag{46}
$$

Now the conclusion follows from (45), (46) and the Gronwall lemma. □

Remark 12. By Lemma 11 it follows that if $\varphi \in C_b^2(H)$ the function $u(t,\cdot)$ possesses bounded second order derivatives in all directions of H for any $t \ge 0$. So, it is Frécher differentiable and belongs to $C_b^1(H)$ (more precisely to $C_b^{1+\varepsilon}(H)$ for all $\varepsilon \in (0,1)$).

3.3 Strict solutions of the Kolmogorov equation

We are now in position to show existence of a strict solution $u(t,x)$ (in the sense that $u(t,x)$ fulfills conditions (i)-(iv) of Proposition 13 below) of equation (26) for all $\varphi \in C_b^2(H)$. Let us define,

$$\overline{\lambda} = \frac{1}{2} \, \max\{\lambda_i : \ i = 1,..,5\}$$

and

$$\kappa = \frac{1}{2} \, \max\{a_i^{1/2} : \ i = 1,..,5\}$$

Theorem 13. *Assume that $\varphi \in C_b^2(H)$, $D^2\varphi(x)$ is of trace class for any $x \in H$ and Tr $[D^2\varphi] \in C_b(H)$. Let*

$$u(t,x) = \mathbb{E}[\varphi(X(t,x))], \quad t \ge 0, \ x \in H,$$

where $X(t,x)$ is the mild solution of (17). Then the following statements hold.

(i) For all $t \ge 0$, $u(t,\cdot) \in C_b^1(H)$ and possesses second order derivatives in all directions of H.

(ii) For all $t > 0$ *and any* $x \in H$ *we have*

$$|u_x(t,x)| \le \kappa\, e^{\overline{\lambda}t}\|\varphi\|_1 \tag{47}$$

and

$$|u_{xx}(t,x)| \le \kappa\, e^{\overline{\lambda}t}\|\varphi\|_2. \tag{48}$$

(iii) There exists $\kappa_1 > 0$ *such that for all* $t \ge 0$ *and any* $x \in H$ *we have*

$$\begin{aligned} \left|\mathrm{Tr}\,[\sigma^2(x)u_{xx}(t,x)]\right| &= \left|\sum_{k=1}^{\infty}\langle u_{xx}(t,x)(\sigma(x)e_k,\sigma(x)e_k)\rangle\right| \\ &\le \kappa_1 e^{\overline{\lambda}t}\|\varphi\|_2(1+\sup_{x\in H}\|D^2\varphi(x)\|_{L_1(H)}). \end{aligned} \tag{49}$$

(iv) For all $x \in D(A)$*,* $u(\cdot,x)$ *is differentiable in* $(0,+\infty)$ *and fulfills* (26)*.*

Proof. Let us prove (i). For any $x, z \in H$ and $t \ge 0$ we have

$$\langle u_x(t,x),z\rangle = \mathbb{E}[\langle D\varphi(X(t,x),\eta^z(t,x)\rangle]$$

Therefore

$$|\langle u_x(t,x),z\rangle| \le \|\varphi\|_1|\eta^z(t,x)|.$$

By Lemma 7 it follows that

$$|\langle u_x(t,x),z\rangle| \le \|\varphi\|_1 a_1^{1/2}e^{\lambda_1 t/2}.$$

Thus (47) follows from the arbitrariness of z. Moreover $u(t,\cdot) \in C_b^1(H)$ in view of Remark 12.

Let us prove (ii). For any $x, z \in H$ and $t \ge 0$ we have

$$\langle u_{xx}(t,x)z,z\rangle = \mathbb{E}[\langle D\varphi(X(t,x),\zeta^z(t,x)\rangle] + \mathbb{E}[\langle D^2\varphi(X(t,x)\eta^z(t,x),\eta^z(t,x)\rangle].$$

Therefore

$$\begin{aligned} |\langle u_{xx}(t,x)z,z\rangle| &\le \|\varphi\|_1\mathbb{E}[|\zeta^z(t,x)|] + \|\varphi\|_2\mathbb{E}[|\eta^z(t,x)|^2] \\ &\le \|\varphi\|_1\sqrt{a_5}\; e^{\lambda_5 t/2}|z|^2 + \|\varphi\|_2 a_1\; e^{\lambda_1 t/2}|z|^2 \end{aligned}$$

and (ii) follows.

Let us prove (iii). For any $x, z \in H$ and $t \ge 0$ we have

$$\begin{aligned} \mathrm{Tr}\,[\sigma^2(x)u_{xx}(t,x)] &= \sum_{k=1}^{\infty}\langle u_{xx}(t,x)(\sigma(x)e_k,\sigma(x)e_k)\rangle \\ &= \sum_{k=1}^{\infty}\mathbb{E}[\langle D\varphi(X(t,x),\zeta^{\sigma(x)e_k}(t,x)\rangle] \\ &\quad + \mathrm{Tr}\;\mathbb{E}[\sigma^2(x)X_x(t,x)D^2\varphi(X(t,x)X_x^*(t,x)] \\ &\qquad\qquad := J_1(t,x) + J_2(t,x). \end{aligned}$$

Therefore

$$|J_2(t,x)| \le a_1 e^{\lambda_1 t} \|D^2\varphi\|_{HS}.$$

Concerning J_1 we have

$$|J_1(t,x)| \le \|\varphi\|_1 |T(t,x)|$$

where

$$T(t,x) = \sum_{k=1}^{\infty} \zeta^{\sigma(x)e_k}(t,x).$$

Then one can checks that $T(t,x)$ is the solution of the equation

$$\begin{aligned} T(t,x) &= \int_0^t e^{(t-s)A}[g'(X(s,x))T(s,x)]dW(s) \\ &\quad + \int_0^t e^{(t-s)A}[g''(X(s,x))K(s,x)]dW(s), \end{aligned} \tag{50}$$

where

$$K(t,x) = \sum_{k=1}^{\infty} (\eta^{\sigma(x)e_k}(t,x))^2.$$

Using estimate (39), it is not difficult to show that equation (50) has a solution and estimate (49) holds.

Let us prove finally (iv). Assume that $x \in D(A)$ and let $u^n(t,x)$ be the solution of (28). Then, taking into account that estimates from Lemmas 7 and 11 can also be proved for the function u^n with constants independent of n, it is not difficult to check that,

$$\lim_{n\to\infty} u^n(t,x) = u(t,x), \quad t > 0,\ x \in H,$$

$$\lim_{n\to\infty} u^n_x(t,x) = u_x(t,x), \quad t > 0,\ x \in H,$$

and

$$\lim_{n\to\infty} \text{Tr}\,[\sigma(x)C\sigma(x)u^n_{xx}(t,x)] = \text{Tr}\,[\sigma(x)C\sigma(x)u_{xx}(t,x)], \quad t > 0,\ x \in H.$$

Consequently,

$$\lim_{n\to\infty} u^n_t(t,x) = u_t(t,x), \quad t > 0,\ x \in D(A),$$

and the conclusion follows. □

We consider finally the elliptic Kolmogorov equation

$$\lambda\varphi(x) - \frac{1}{2}\,\text{Tr}\,[\sigma^2(x)\varphi_{xx}(x)] - \langle Ax, \varphi_x(x)\rangle = f(x), \quad x \in D(A), \tag{51}$$

where $\lambda > 0$ and $f \in C_b(H)$ are given.

Theorem 14. *Assume that* $\lambda > \overline{\lambda}$, $f \in C_b^2(H)$, $D^2f(x)$ *is of trace class for any* $x \in H$ *and* $Tr\,[D^2f] \in C_b(H)$. *Define*

$$\varphi(x) = \int_0^\infty e^{-\lambda t}\mathbb{E}[f(X(t,x))]dt, \quad t \geq 0,\ x \in H,$$

Then the following statements hold.

(i) $\varphi \in C_b^1(H)$ *and possesses second order derivatives in all directions of* H.

(ii) For all $x \in H$ *we have*

$$|u_x(t,x)| \leq \frac{\kappa}{\lambda - \overline{\lambda}}\ \|\varphi\|_1 \tag{52}$$

and

$$|u_{xx}(t,x)| \leq \frac{\kappa}{\lambda - \overline{\lambda}}\ \|\varphi\|_2. \tag{53}$$

(iii) There exists $\kappa_1 > 0$ *such that for all* $x \in H$ *we have*

$$\begin{aligned}\left|\mathrm{Tr}\,[\sigma^2(x)u_{xx}(t,x)]\right| &= \left|\sum_{k=1}^\infty \langle u_{xx}(t,x)(\sigma(x)e_k,\sigma(x)e_k)\rangle\right| \\ &\leq \frac{\kappa_1}{\lambda - \overline{\lambda}}\ \|\varphi\|_2(1 + \|D^2\varphi\|_{HS}).\end{aligned} \tag{54}$$

(ii) We have

$$|\varphi_x(x)| \leq \frac{\sqrt{a_1}}{\lambda - \overline{\lambda}}\ \|f\|_1, \quad x \in H, \tag{55}$$

and

$$\left|\mathrm{Tr}\,[\sigma^2(x)\varphi_{xx}(t,x)]\right| \leq \frac{\kappa}{(\lambda - \overline{l})}\ \|f\|_2, \quad x \in H. \tag{56}$$

(iv) For all $x \in D(A)$ *the equation* (51) *is fulfilled.*

Proof. The conclusion follows from Proposition 13 and estimates (47), (48) and (49). □

3.4 The Kolmogorov operator

It is well known that the semigroup R_t is not in general strongly continuous in $C_b(H)$. However, we can define its infinitesimal generator by proceeding as in [4]. Namely, for any $\lambda > 0$ and any $f \in C_b(H)$ we define

$$F_\lambda(f)(x) = \int_0^\infty e^{-\lambda t}R_tf(x)dt, \quad x \in H.$$

Proposition 15. *For any* $f \in C_b(H)$ *and any* $\lambda > 0$ *we have* $F_\lambda(f) \in C_b(H)$ *and the following estimate holds*

$$\|F_\lambda(f)\|_0 \leq \frac{1}{\lambda}\, \|f\|_0. \tag{57}$$

Moreover there exists a unique closed operator $L\colon D(L) \subset C_b(H) \to C_b(H)$ *such that for any* $\lambda > 0$ *and any* $f \in C_b(H)$ *we have* $F_\lambda(f) = R(\lambda, L)f$.

Proof. Let first $f \in C_b^1(H)$; then it is obvious that if $F_\lambda(f) \in C_b(H)$ the inequality (57) holds. Moreover for all $x, y \in H$ we have

$$\begin{aligned} |F_\lambda(f)(x) - F_\lambda(f)(y)| &\leq \int_0^\infty e^{-\lambda t}\mathbb{E}(|f(X(t,x)) - f(X(t,y))|)dt \\ &\leq \|f\|_1 \int_0^\infty e^{-\lambda t}\mathbb{E}|X(t,x) - X(t,y)|dt. \end{aligned} \tag{58}$$

On the other hand we have

$$X(t,x) - X(t,y) = \int_0^1 X_x(t,(1-r)x + ry)(x-y)dr,$$

so that, recalling Lemma 7, we find

$$|X(t,x) - X(t,y)| \leq \sqrt{a_1}\, e^{\frac{1}{2}\,\lambda_1 t}|x-y|. \tag{59}$$

Now, substituting this inequality in (58) yields

$$|F_\lambda(f)(x) - F_\lambda(f)(y)| \leq \sqrt{a_1}\, \int_0^\infty e^{(\frac{1}{2}\,\lambda_1 - \lambda)t}dt|x-y|.$$

Thus, if $\lambda > \frac{1}{2}\,\lambda_1$ we have proved that $F_\lambda(f) \in C_b(H)$ (it is even Lipschitz). Since $C_b^1(H)$ is dense in $C_b(H)$ we can conclude that $F_\lambda(f) \in C_b(H)$ for all $f \in C_b(H)$ (and $\lambda > \frac{1}{2}\,\lambda_1$).

Now it is easy to see that F_λ fulfills the resolvent identity

$$F_\lambda - F_\mu = (\mu - \lambda)F_\lambda F_\mu, \quad \lambda, \mu > 0.$$

So, by a classical result, see e. g. [18], there exists a unique closed operator $L\colon D(L) \subset C_b(H) \to C_b(H)$ such that for any $\lambda > \frac{1}{2}\,\lambda_1$ and any $f \in C_b(H)$ we have $F_\lambda(f) = R(\lambda, L)f$.

Finally, by (57) we see that L is m-dissipative so that condition $\lambda > \frac{1}{2}\,\lambda_1$ can be replaced by $\lambda > 0$. $\Box$

Remark 16. Assume that $f \in C_b^2(H)$, $D^2f(x)$ *is of trace class for any* $x \in H$ *and* Tr $[D^2f] \in C_b(H)$. *Let moreover* $\lambda > 0$ *and* $\varphi = R(\lambda, L)$. *Then by Proposition 14 it follows that*

$$L\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[\sigma^2(x)\varphi_{xx}(x)] + \langle Ax, \varphi_x(x)\rangle, \quad x \in D(A). \tag{60}$$

Now we are going to prove, following an argument in [10], that if $\varphi \in R(\lambda, L)(C_b^2(H))$ we have $\varphi^2 \in D(L)$.

Proposition 17. *Assume that* $f \in C_b^2(H)$, $D^2 f(x)$ *is of trace class for any* $x \in H$ *and Tr* $[D^2 f] \in C_b(H)$. *Let moreover* $\lambda > 0$ *and* $\varphi = R(\lambda, L)$. *Then* $\varphi^2 \in D(L)$ *and*

$$L(\varphi^2) = 2\varphi\, L\varphi + |\sigma D\varphi|^2. \tag{61}$$

Proof. Let L_n be the approximating Kolmogorov operator

$$L_n\varphi(x) = \frac{1}{2}\, \mathrm{Tr}\, [\sigma^2(x) P_n \varphi_{xx}(x)] + \langle A_n x, \varphi_x(x)\rangle, \quad \varphi \in C_b(H),\ x \in H \tag{62}$$

and let $\varphi^n = R(\lambda, L_n)f$. Then, by a straightforward computation, it follows that

$$L_n((\varphi^n)^2) = 2\varphi^n\, L_n\varphi^n + |\sigma P_n D\varphi^n|^2.$$

Now, multiplying both sides of the equation $\lambda\varphi^n - L_n\varphi^n = f$ by φ^n, yields

$$\lambda(\varphi^n)^2 - L_n\varphi^n\, \varphi^n = f\varphi^n,$$

which is equivalent to

$$2\lambda(\varphi^n)^2 - L_n((\varphi^n)^2) = 2f\varphi^n - |\sigma P_n D\varphi^n|^2.$$

Therefore,

$$(\varphi^n)^2 = R(2\lambda, L_n)(2f\varphi^n - |\sigma P_n D\varphi^n|^2).$$

Letting $n \to \infty$ yields

$$\varphi^2 = R(2\lambda, L)(2f\varphi - |\sigma D\varphi|^2).$$

Consequently

$$2\lambda\varphi^2 - L\varphi^2 = 2f\varphi - |\sigma(x) D\varphi|^2,$$

which yields (61). □

4 Invariant measures

4.1 Existence and uniqueness

We denote by $\mathcal{P}(H)$ the set of all Borel probability measures on H. We recall that a probability measure $\mu \in \mathcal{P}(H)$ is said to be *invariant* for the transition semigroup R_t defined by (7) if

$$\int_H R_t\varphi d\mu = \int_H \varphi d\mu \quad \text{for all } \varphi \in C_b(H). \tag{63}$$

Theorem 18. *There is an invariant measure* μ *for* R_t*. Moreover, for any* $\beta \in [0, 1/4)$ *we have*

$$\int_H |(-A)^\beta x|^2 \mu(dx) < +\infty. \tag{64}$$

Finally, if $1/g$ *is bounded the invariant measure* μ *is unique.*

Proof. Let $X(t,x)$ be the solution of (18). Using Lemma 3 and inequality (21), we find that

$$\begin{aligned}\mathbb{E}(|X(t,x)|^2) &\le 2e^{-2\pi^2 t}|x|^2 + 2\sum_{h=1}^{\infty}\int_0^t e^{-2\pi^2 h^2 (t-s)}\mathbb{E}(|\sigma(X(s,x))e_h|^2)ds \\ &\le 2e^{-2\pi^2 t}|x|^2 + \|g\|_0^2 \int_0^t F(2\pi^2(t-s))ds \\ &\le 2e^{-2\pi^2 t}|x|^2 + 4\|g\|_0^2 \int_0^t e^{-\pi^2(t-s)}(2\pi^2(t-s))^{-1/2}ds.\end{aligned}$$

So,

$$\mathbb{E}(|X(t,x)|^2) \le 2e^{-2\pi^2 t}|x|^2 + \frac{4}{\sqrt{\pi}}\,\|g\|_0^2. \tag{65}$$

Now let $\beta \in (0, 1/4)$. Using the well known estimate

$$\|(-A)^\beta e^{tA}\| \le c_\beta t^{-\beta} e^{-\pi^2 t}, \quad t \ge 0, \tag{66}$$

where c_β is a suitable constants, we find,

$$\begin{aligned}&\mathbb{E}(|(-A)^\beta X(t,x)|^2) \\ &\le 2c_\beta t^{-2\beta} e^{-2\pi^2 t}|x|^2 + 2\int_0^t \sum_{h=1}^{\infty}(\pi h)^{4\beta} e^{-2\pi^2 h^2 (t-s)}\mathbb{E}(|\sigma(X(s,x))e_h|^2)ds \\ &\le 2c_\beta t^{-2\beta} e^{-2\pi^2 t}|x|^2 + 2\|g\|_0^2 \int_0^t F_\beta(2\pi^2(t-s))ds\end{aligned}$$

where F_β is defined by

$$F_\beta(t) = \sum_{h=1}^{\infty} h^{4\beta} e^{-h^2 t}, \quad t > 0.$$

It is not difficult to show that there is $k_\beta > 0$ such that

$$F_\beta(t) \le k_\alpha t^{-1/2-2\beta} e^{-t}, \quad t \ge 0.$$

Now we have

$$\begin{aligned}&\mathbb{E}(|(-A)^{\beta}X(t,x)|^2)\le 2c_\beta t^{-2\beta}e^{-2\pi^2 t}|x|^2\\ &+2\|g\|_0^2k_\beta\int_0^t(2\pi^2 s)^{-1/2-2\beta}e^{-2\pi^2 s}ds \qquad (67)\\ &\le 2c_\beta t^{-2\beta}e^{-2\pi^2 t}|x|^2+c_\beta k_\beta\|g\|_0^2.\end{aligned}$$

Since the embedding $D(A)\subset H$ is compact, the existence of an invariant measure follows from the Krylov–Bogoliubov theorem.

Let us show (64). Let $\gamma>0$ and set

$$\varphi_\gamma(x)=\frac{|x|^2}{1+\gamma|x|^2},\quad x\in H.$$

Then $\varphi_\gamma\in C_b(H)$ and, proceeding as in the proof of (65) we see that there exists a constant $\kappa>0$ (independent on λ) such that,

$$R_t(\varphi_\gamma)(x)=\mathbb{E}[\varphi_\gamma(X(t,x))]\le e^{-2\pi^2 t}\varphi_\gamma(x)+\kappa. \qquad (68)$$

Integrating both sides of (68) with respect to x over H and taking into account the invariance of μ yields,

$$\int_H\varphi_\gamma(x)\mu(dx)\le e^{-2\pi^2 t}\int_H\varphi_\gamma(x)\mu(dx)+\kappa.$$

Therefore, there exists $\kappa_1>0$ (independent on λ) such that

$$\int_H\varphi_\gamma(x)\mu(dx)\le\kappa_1.$$

Letting γ tend to 0 yields,

$$\int_H|x|^2\mu(dx)\le\kappa_1. \qquad (69)$$

Now, integrating both sides of (67) with respect to x over H and taking into account again the invariance of μ yields,

$$\int_H|(-A)^{-\beta}x|^2\mu(dx)\le 2c_\beta t^{-2\beta}e^{-2\pi^2 t}\kappa_1+c_\beta\|g\|_0^2,$$

and so, (64) follows. Finally, if $1/g$ is bounded the uniqueness of μ follows from from the Doob Theorem since R_t is irreducible and strong Feller by [16]. □

Remark 19. More general results of existence of invariant measures can be found in the paper [7].

4.2 Existence of a core of smooth functions for L_μ

Let us fix an invariant measure μ for R_t. It is well known that R_t can be uniquely extended to a strongly continuous semigroup of contractions on $L^2(H,\mu)$ which we shall still denote by R_t. The infinitesimal generator of R_t in $L^2(H,\mu)$ will be denoted by L_μ. Since R_t is a contraction semigroup, L_μ is m-dissipative in $L^2(H,\mu)$.

In this subsection we want to define a core of L_μ consisting of regular functions.

Proposition 20. *Set*

$$\Lambda = \{\varphi \in C_b^2(H): \; D^2\varphi(x) \in L_1(H) \text{ for all } x \in H \text{ and } \mathrm{Tr}\,[D^2\varphi] \in C_b(H)\}$$

and

$$\Gamma := \bigcup_{\lambda>0} R(\lambda, L)(\Lambda).$$

Then Γ is a core for L_μ.

Moreover if $\varphi \in \Gamma$ we have $\varphi^2 \in D(L_\mu)$ and the following identity holds

$$L_\mu(\varphi^2) = 2\varphi\, L_\mu\varphi + |\sigma D\varphi|^2. \tag{70}$$

Proof. Let $\lambda > 0$. It is clear that any $\varphi \in D(L)$ belongs to $D(L_\mu)$ as well, so that we have

$$(\lambda - L_\mu)(\Gamma) = (\lambda - L)(\Gamma) \supset \Lambda.$$

Since Λ is dense in $L^2(H,\mu)$ (by a standard argument of monotone classes), we can conclude that $(\lambda - L_\mu)(\Gamma)$ is dense in $L^2(H,\mu)$. Now the Lumer-Phillips theorem implies that Γ is a core for L_μ.

Finally, the last statement follows from (61). □

5 The basic integration by parts formula

In this section we assume that g^{-1} is bounded. We recall that in this case μ is the unique invariant measure of the transition semigroup R_t.

Proposition 21. *The operator*

$$D : \Gamma \subset L^2(H,\mu) \to L^2(H,\mu;H),\; \varphi \mapsto D\varphi, \tag{71}$$

is uniquely extendible to a linear bounded operator $D\colon D(L_\mu) \to L^2(H,\mu;H)$, where $D(L_\mu)$ is endowed with the graph norm of L_μ. Moreover, the following identity holds

$$\int_H L_\mu\varphi\,\varphi\, d\mu = -\frac{1}{2}\int_H |\sigma D\varphi|^2 d\mu, \qquad \varphi \in D(L_\mu). \tag{72}$$

Identity (72) is called in French “identité du carré du champs”. It will play an important rôle in what follows.
Proof. Let $\varphi \in \Gamma$, then $\varphi^2 \in D(L_\mu)$ in view of Proposition 20 we have

$$L_\mu \varphi^2 = 2\varphi L_\mu \varphi + |gD\varphi|^2.$$

Integrating this identity with respect to μ over H and taking into account that

$$\int_H L_\mu \varphi^2 d\mu = 0$$

by the invariance of μ, implies(72) when $\varphi \in \Gamma$. Let now $\varphi \in D(L_\mu)$. Since Γ is a core for L_μ, there exists a sequence $\{\varphi_n\} \subset \Gamma$ such that

$$\varphi_n \to \varphi, \quad L_0\varphi_n \to L_\mu \varphi \quad \text{in} \ \ L^2(H,\mu).$$

By (72) it follows that

$$\int_H |\sigma D(\varphi_n - \varphi_m)|^2 d\mu = -2\int_H L_\mu(\varphi_n - \varphi_m)\ (\varphi_n - \varphi_m)\ d\mu.$$

So, the sequence $\{\sigma D\varphi_n\}$ is Cauchy in $L^2(H,\mu;H)$ and the conclusion follows. □

Proposition 22. *Let $\varphi \in L^2(H,\mu)$ and $t \geq 0$. Then, for any $T > 0$, the linear operator*

$$\sigma DR_t : D(L_\mu) \subset L^2(H,\mu) \to L^2(0,T;L^2(H,\mu;H)),\ \varphi \to \sigma DR_t\varphi,$$

is uniquely extendible to a linear bounded operator, still denoted by σDR_t, from $L^2(H,\mu)$ into $L^2(0,T;L^2(H,\mu;H))$. Moreover the following identity holds

$$\int_H (R_t\varphi)^2\ d\mu + \int_0^t ds \int_H |\sigma DR_s\varphi|^2 d\mu = \int_H \varphi^2\ d\mu. \tag{73}$$

Proof. We first establish (73) for $\varphi \in D(L_\mu)$. In this case we have

$$\frac{d}{dt}\ R_t\varphi = L_\mu\varphi.$$

Multiplying scalarly this identity by $R_t\varphi$, integrating with respect to μ over H and using (73), yields,

$$\frac{d}{dt}\ \int_H (R_t\varphi)^2\ d\mu + \int_H |\sigma DR_s\varphi|^2 d\mu = 0. \tag{74}$$

Now (73) follows integrating (74) with respect to t. The case when $\varphi \in L^2(H,\mu)$ can be handled by approximating φ by elements of $D(L_\mu)$. □

5.1 The Sobolev space $W^{1,2}(H,\mu)$

To define the Sobolev space we first show that the mapping

$$D_\mu : \varGamma \subset L^2(H,\mu) \to L^2(H,\mu;H),\ \varphi \to D_\mu\varphi \tag{75}$$

is closable. Notice the difference between the map D defined by (71) and the map D_μ. The first one is a bounded operator in $D(L_\mu)$ (endowed with the graph norm of L_μ) whereas the second will be a closable operator in $L^2(H,\mu)$.

To prove closability of D_μ we recall the following estimate from Lemma 7.

$$\mathbb{E}(|X_x(t,x)h|^2) \le a_1 e^{-2\omega t}|h|^2 \quad t\ge 0,\ h,x\in H, \tag{76}$$

where $\omega = \pi^2 - 8\pi^{-2}\|g'\|_0^2$.

Lemma 23. *Let $\{\varphi_n\} \subset \varGamma$ and let $G \in L^2(H,\mu;H)$ be such that*

$$\lim_{n\to\infty} D\varphi_n = G \quad \text{in } L^2(H,\mu;H).$$

Then, for any $t \ge 0$ we have

$$\lim_{n\to\infty} DR_t\varphi_n = \mathbb{E}[X_x^*(t,x)G(X(t,x))] \quad \textit{in } L^2(H,\mu;H).$$

In particular, if $D\varphi_n \to 0$ in $L^2(H,\mu;H)$ we have $DR_t\varphi_n \to 0$ in $L^2(H,\mu;H)$ for all $t > 0$.

Proof. Write

$$DR_t\varphi_n(x) = E[X_x^*(t,x)D\varphi_n(X(t,x)],\quad t\ge 0,\ x\in H.$$

Taking into account estimate (76) and the invariance of μ, yields

$$\begin{aligned}
&\int_H |DR_t\varphi_n(x) - \mathbb{E}[X_x^*(t,x)G(X(t,x))]|^2\,\mu(dx)\\
&= \int_H |\mathbb{E}[X_x^*(t,x)(D\varphi_n(X(t,x)) - G(X(t,x))]|^2\,\mu(dx)\\
&\le a_1 e^{-2\omega t}\int_H \mathbb{E}[|D\varphi_n(X(t,x)) - G(X(t,x))|^2]\mu(dx)\\
&= a_1 e^{-2\omega t}\int_H R_t(|D\varphi_n - G|^2)(x)\mu(dx)\\
&= a_1 e^{-2\omega t}\int_H |D\varphi_n(x) - G(x)|^2\mu(dx).
\end{aligned}$$

The conclusion of the lemma follows. □

Proposition 24. *D_μ is closable. Moreover, if φ belongs to the domain of the closure $\overline{D_\mu}$ of D_μ and $\overline{D_\mu}\varphi = 0$ we have that $\overline{D_\mu} R_t \varphi = 0$ for any $t > 0$.*

Proof. Let $\{\varphi_n\} \subset \varGamma$ and $G \in L^2(H,\mu;H)$ be such that

$$\varphi_n \to 0 \quad \text{in } L^2(H,\mu), \quad D\varphi_n \to G \quad \text{in } L^2(H,\mu;H).$$

By (73) we have that

$$\int_H (R_t\varphi_n)^2 \, d\mu + \int_0^t ds \int_H |\sigma D R_s \varphi_n|^2 d\mu = \int_H \varphi_n^2 \, d\mu.$$

Letting $n \to \infty$ and taking into account that g is bounded below, yields

$$\lim_{n\to\infty} \int_0^t ds \int_H |DR_s\varphi_n|^2 d\mu = 0.$$

Consequently, by Lemma 23, it follows that

$$\int_0^t ds \int_H \left(\mathbb{E}[X_x^*(s,x) G(X(s,x))]\right)^2 \mu(dx) = 0, \quad h \in H.$$

Then for almost all $t \geq 0$ we have that

$$\mathbb{E}[X_x^*(t,x) G(X(t,x))] = 0. \tag{77}$$

Now fix $h \in H$. Then we have,

$$\begin{aligned}
&|\mathbb{E}[\langle G(X(t,x)), h\rangle]| \\
&\leq |\mathbb{E}[\langle G(X(t,x)), X_x(t,x)\cdot h\rangle]| + |\mathbb{E}[\langle G(X(t,x)), h - X_x(t,x)\cdot h\rangle]| \\
&= |\mathbb{E}[\langle G(X(t,x)), h - X_x(t,x)\cdot h\rangle]|.
\end{aligned}$$

Taking into account the invariance of μ and (77), we find that

$$\begin{aligned}
&\int_H |R_t(\langle G(x), h\rangle)|\mu(dx) = \int_H |\mathbb{E}[\langle G(X(t,x)), h\rangle]|\mu(dx) \\
&= \int_H |\mathbb{E}[\langle G(X(t,x)), h - X_x(t,x)\cdot h\rangle]|\mu(dx) \\
&\leq \left[\int_H \mathbb{E}[|G(X(t,x))|^2]\mu(dx)\right]^{1/2} \left[\int_H \mathbb{E}[|h - X_x(t,x)\cdot h|^2]\mu(dx)\right]^{1/2}.
\end{aligned}$$

Therefore, as $t \to 0$ we find by the strong continuity of R_t in $L^1(H,\mu)$

$$\int_H |\langle G(x), h\rangle|\mu(dx) = 0$$

and by the arbitrariness of h it follows that $G=0$ as required. Finally, the last statement follows from by Lemma 23. □

By Proposition 24 it follows that the mapping

$$D_\mu : \mathit{\Gamma} \subset L^2(H,\mu) \to L^2(H,\mu;H),\ \varphi \mapsto D\varphi,$$

is closable, let $\overline{D_\mu}$ its closure. We shall denote by $W^{1,2}(H,\mu)$ the domain of $\overline{D_\mu}$ and, if there is not possibility of confusion, we shall set $\overline{D_\mu}=D$.

Proposition 25. *We have $D(L_\mu)\subset W^{1,2}(H,\mu)$ with continuous embedding. Moreover, the following identity holds*

$$\int_H L_\mu\varphi\ \varphi\ d\mu = -\frac{1}{2}\int_H |\sigma D\varphi|^2 d\mu, \qquad \varphi\in D(L_\mu). \tag{78}$$

Proof. Let $\varphi\in D(L_\mu)$. Since $\mathit{\Gamma}$ is a core for L_μ, there exists a sequence $\{\varphi_n\}\subset\mathit{\Gamma}$ such that

$$\varphi_n\to\varphi, \quad L_0\varphi_n\to L_\mu\varphi \quad \text{in } L^2(H,\mu).$$

By (72) it follows that

$$\int_H |\sigma D(\varphi_n-\varphi_m)|^2 d\mu \le 2\int_H |L_0(\varphi_n-\varphi_m)|\ |\varphi_n-\varphi_m|\ d\mu.$$

Therefore the sequence $(D\varphi_n)$ is Cauchy in $L^2(H,\mu;H)$. Since D is closed it follows that $\varphi\in W^{1,2}(H,\mu)$ as required. □

5.2 The Poincaré inequality

Since $1/g$ is bounded, by Theorem 18 there is a unique invariant measure μ for R_t and by the Doob theorem, see e.g. [12], we have that,

$$\lim_{n\to\infty} R_t\varphi(x) = \int_H \varphi(y)\mu(dy), \quad x\in H, \tag{79}$$

for all $\varphi\in C_b(H)$.

Let us prove now the Poincaré inequality.

Proposition 26. *Assume that $\|g'\|_0^2\le\frac{1}{8}\,\pi^4$. Then, for any $\varphi\in W^{1,2}(H,\nu)$ we have*

$$\int_H |\varphi-\overline{\varphi}|^2 d\mu \le \frac{a_1}{\omega}\ \|g\|_0^2\ \|1/g\|_0^2 \int_H |g(x)D\varphi|^2 d\mu, \tag{80}$$

where $\overline{\varphi}=\int_H\varphi d\mu$ and $\omega=\pi^2-8\pi^{-2}\|g'\|_0^2$.

Proof. Let first $\varphi \in \varGamma$. Then by (73) we have

$$\int_H |R_t\varphi(x) - \overline{\varphi}|^2 \mu(dx) = \int_0^t ds \int_H |\sigma D R_s \varphi|^2 d\mu. \tag{81}$$

Moreover by (76) it follows that

$$\begin{aligned}&\mathbb{E}[|DR_s\varphi(x)|^2] \le \mathbb{E}\left[|D\varphi(X(s,x))|^2 \ |X_x(s,x)|^2\right]\\ &\le a_1 e^{-2\omega s}\mathbb{E}\left[|D\varphi(X(s,x))|^2\right] = a_1 e^{-2\omega s} R_s(|D\varphi|^2)(x).\end{aligned}$$

Taking into account (79) and the invariance of μ we obtain,

$$\begin{aligned}&\int_H |R_t\varphi(x) - \overline{\varphi}|^2 \mu(dx)\\ &\le a_1 \|g\|_0^2 \int_0^{+\infty} e^{-2\omega s} ds \int_H R_s(|D\varphi|^2)(x)\mu(dx)\\ &\le a_1 \|g\|_0^2 \ \|1/g\|_0^2 \int_0^{+\infty} e^{-2\omega s} ds \int_H |g(x) D\varphi(x)|^2 \mu(dx),\end{aligned}$$

and the conclusion follows. If $\varphi \in W^{1,2}(H,\mu)$, we proceed by density. □

Remark 27. If g *is constant the condition* $\|g'\|_0^2 \le \frac{1}{8}\,\pi^4$ *is trivially fulfilled and we recover a result in [10].*

Remark 28. It is well known, see e.g. [8], that the Poincaré inequality implies that the spectrum $\sigma(L_\mu)$ *of* L_μ *consists of* 0 *and a set included in the half-space*

$$\{\lambda \in \mathbb{C} : \ \Re e\ \lambda \le -\omega_1\},$$

*(*spectral gap*) where* ω_1 *is a positive constant.*

The spectral gap in turn implies an exponential convergence of $R_t\varphi$ *to the equilibrium*

$$\int_{\mathbb{R}} |R_t\varphi - \overline{\varphi}|^2 d\nu \le c e^{-2\omega_1 t} \int_{\mathbb{R}} |\varphi|^2 d\nu, \quad \varphi \in L^2(\mathbb{R},\nu),. \tag{82}$$

where c *is a suitable constant.*

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