Classical "$ax+b$" - group

$$G = \left\{ \begin{pmatrix} a, & b \\ 0, & 1 \end{pmatrix} : \begin{matrix} a, b \in \mathbb{R} \\ a > 0 \end{matrix} \right\}$$

$$\begin{pmatrix} a, & b \\ 0, & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ 1 \end{pmatrix}$$

closed linear envelope

$$A = C_0(G) = \left\{ \varphi(b)\psi(a) : \begin{matrix} \varphi \in C_0(\mathbb{R}) \\ \psi \in C_\infty(]0,\infty[) \end{matrix} \right\}$$

$$\Delta \in \mathrm{Mor}(A, \underbrace{A \otimes A}_{C_\infty(G \times G)})$$

$$(\Delta c)(g_1, g_2) = c(g_1 g_2)$$

$$\Delta a = a \otimes a$$

$$\Delta b = a \otimes b + b \otimes I$$

Remark: a, b do not belong to A. They are affiliated with A:

$$a, b \notin A$$

$$a, b \;\eta\; A.$$

Quantum "ax + b" group.

(A, Δ) - locally compact quantum group

$\underbrace{a, b}_{\text{selfadjoint elements}} \;\eta\; A$; $\quad a > 0$

$$\Delta a = a \otimes a$$

$$\Delta b = a \otimes b + b \otimes I$$

Elements a and b satisfy the specific commutation relation:

$$ab = q^2 ba$$

$|q^2| = 1$

$q^2 = e^{-i\hbar}$

$\hbar \in \mathbb{R}$.

More precisely

$$\boxed{\forall_{t \in \mathbb{R}} \quad a^{it}\, b\, a^{-it} = e^{\hbar t}\, b}$$

Then we write

$$a \overset{\hbar}{\longrightarrow\!\!\circ}\, b$$

and say that a, b satisfy Zakrzewski relation

$\hbar \in \mathbb{R}$

Zakrzewski relation

R, S - selfadjoint operators, $R > 0$.

We say that $R \xrightarrow{\hbar} \circ S$ if

$$R^{it} S R^{-it} = e^{\hbar t} S$$

for all $t \in \mathbb{R}$

Interesting facts:

Let $R \xrightarrow{\hbar} \circ S$ with R, S strictly positive.

Then:

1. $R + S$ is selfadjoint if $|\hbar| < 2\pi$

2. $R - S$ is never selfadjoint. It has deficiency indeces $(n, 0)$ if $|\hbar| < 4\pi$

THEOREM.

Let $0 < \hbar < \frac{\pi}{2}$

(A, Δ) be a locally compact quantum group

$a, b \;\eta\; A$ $\quad a, b$ - selfadjoint

$a > 0$

$a \xrightarrow{\hbar} b$

Assume that

$\Delta(b) \supset a \otimes b + b \otimes I$

$\Delta(b)$ strongly commute with $I \otimes |b|$

A is of minimal size i.e there exists a faithful representation of A with multiplicity 2 with respect to $(a, |b|)$

Then $\hbar = \dfrac{\pi}{2k+3}$ where k is a nonnegative integer.

Remark: If A is not of minimal size then all values of $\hbar$ are allowed.

Definition:

Let $\hbar$ be a real number. We say that $\hbar$ is admissible if there exists a locally compact quantum group (A, Δ) and selfadjoint elements $a, b \;\eta\; A$ such that $a > 0$, $a \overset{\hbar}{\multimap} b$ and

$$\Delta(b) \supset a \otimes b + b \otimes I$$

$\Delta(b)$ strongly commutes with $I \otimes |b|$

Theorem: The following numbers are admissible:

$$\frac{\pi}{3},\ \frac{\pi}{5},\ \frac{\pi}{7},\ \frac{\pi}{9},\ \ldots\ldots$$

$$3\pi,\ 5\pi,\ 7\pi,\ 9\pi,\ \ldots\ldots$$

$$\frac{\pi}{4},\ \frac{\pi}{6},\ \frac{\pi}{8},\ \frac{\pi}{10},\ \ldots\ldots$$

$$4\pi,\ 6\pi,\ 8\pi,\ 10\pi,\ \ldots\ldots$$

THEOREM:

Let Q - symmetric operator

τ - unitary selfadjoint ($\tau^* = \tau$, $\tau^2 = I$)

Assume that

$$\tau Q = - Q \tau$$

Then

$$[Q]_\tau = Q^* \Big|_{\{x \in D(Q^*) : (\tau - I) x \in D(Q)\}}$$

is a selfadjoint extension of Q.

Proof:

$$\tau = \begin{pmatrix} -I, & 0 \\ 0, & I \end{pmatrix} \qquad Q = \begin{pmatrix} 0, & Q_0^+ \\ Q_0, & 0 \end{pmatrix}$$

where $Q_0^+ \subset Q_0^*$. Then

$$[Q]_\tau = \begin{pmatrix} 0, & Q_0^* \\ Q_0, & 0 \end{pmatrix}$$

Q.E.D.

$a^* = a > 0$

$b^* = b \neq 0$

$a \overset{*}{-\!\circ}\, b$

$\beta^2 = I \quad \beta^* = \beta$

$\beta a = a\beta$

$\beta b = -b\beta$

norm closed linear envelope

$$A = \left\{ \left(f_1(b) + \beta f_2(b)\right) g(a) \; : \; \begin{array}{c} g \in C_\infty(]0,\infty[) \\ f_1, f_2 \in C_\infty(\mathbb{R}) \\ f_2(0) = 0 \end{array} \right\}$$

Then A is a C^*-algebra

$a, b, \beta b \;\eta\; A$

If $\hbar = \frac{\pi}{2k+3}$ $(k = 0, 1, 2, \ldots)$ then there exists $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that (A, Δ) is a **locally compact quantum group** and

$$\Delta(a) = a \otimes a$$

$$\Delta(b) = \left[a \otimes b + b \otimes I\right]_{(-1)^k \beta \otimes \beta}$$

$$\Delta(i\beta b^{2k+3}) = \left[a^{2k+3} \otimes i\beta b^{2k+3} + i\beta b^{2k+3} \otimes I\right]_{-\mathrm{sign}(b \otimes b)}$$

Special functions:

$$F_\hbar(r,\rho) = \begin{cases} V_\theta(\log r) & \text{for } r>0 \\ \left(1+i\rho|r|^{\frac{\pi}{\hbar}}\right) V_\theta(\log|r| - \pi i) & \text{for } r<0 \end{cases}$$

where $\theta = \frac{2\pi}{\hbar}$, V_θ is a meromorphic function on $\mathbb{C}$ such that

$$V_\theta(x) = \exp\left\{\frac{1}{2\pi i}\int_0^\infty \log(1+a^{-\theta})\frac{da}{a+e^{-x}}\right\}$$

for x in the strip $|\Im x| < \pi$

BASIC FORMULA

$$[R+S]_\tau = F_\hbar(e^{\frac{i\hbar}{2}}S^{-1}R,\tau)^* \, S \, F_\hbar(e^{\frac{i\hbar}{2}}S^{-1}R,\tau)$$

THEOREM: $\qquad \alpha = i \exp\left(\frac{i\pi^2}{2\hbar}\right)$

$R \multimap S \qquad \ker S = \{0\} \qquad \rho, \sigma$ - selfadjoint

$$\left.\begin{array}{l} \rho^2 = \chi(R<0) \\ \rho \ \text{commutes with } R \\ \quad \text{anticommutes with } S \end{array}\right\}$$

$$\left.\begin{array}{l} \sigma^2 = \chi(S<0) \\ \sigma \ \text{commutes with } S \\ \quad \text{anticommutes with } R \end{array}\right\}$$

We set: $\qquad T = e^{i\hbar/2} S^{-1} R$

$$\tau = \alpha\rho\sigma + \bar{\alpha}\sigma\rho$$

Then

1. T - selfadjoint $\qquad T \multimap R, \quad T \multimap S$

$$\operatorname{sign} T = (\operatorname{sign} R)(\operatorname{sign} S)$$

2. τ - selfadjoint and

$$\left.\begin{array}{l} \tau^2 = \chi(T<0) \\ \tau \text{ - commutes with } T \\ \tau \text{ - anticommutes with } S \text{ and } R \end{array}\right\}$$

3.

$$F_h(R,\rho)\, F_h(S,\sigma) = F_h\left([R+S]_\tau, \tilde{\sigma}\right)$$

where

$$\tilde{\sigma} = F_\hbar(T,\tau)^* \sigma\, F_h(T,\tau).$$

General case: $0 < \hbar < \frac{\pi}{2}$

$$\Delta b = [a \otimes b + b \otimes I]_\sigma$$

$\sigma^* = \sigma$ $\qquad \sigma^2 = \chi(b \otimes b < 0)$

σ commutes with $a \otimes I$, $I \otimes a$,
anticommutes with $b \otimes I$, $I \otimes b$

$$(\Delta \otimes id)\Delta b = (id \otimes \Delta)\Delta b$$

$$\Downarrow$$

$(id \otimes \Delta)\sigma = \dots\dots\dots$
$(\Delta \otimes id)\sigma = \dots\dots\dots$

$$\varphi(c) = F_\hbar(e^{i\hbar/2} b^{-1} a \otimes b, \sigma)\,\Delta(c)\,F_\hbar(e^{i\hbar/2} b^{-1} a \otimes b, \sigma)^*$$

$\varphi(\text{sign } b) = \text{sign } b \otimes I$
$(id \otimes \varphi)\sigma = \sigma \otimes I$
$(\varphi \otimes id)\sigma = \bar{\alpha}\, \sigma_{23}\sigma_{12} + \alpha\, \sigma_{12}\sigma_{23}$

$$\alpha = i e^{\frac{i\pi^2}{2\hbar}}$$

A_1 - *-algebra generated by

$$\{(\mu \otimes id)\sigma \; : \; \mu \in B(K)_*\}$$

A_2 - *-algebra generated by

$$\{(id \otimes \nu)\sigma \; : \; \nu \in B(K)_*\}$$

$$\varphi(c) = c \otimes I \qquad \text{for} \qquad c \in A_1$$

$$\varphi(c) = U^*(I \otimes c)U \qquad \text{for} \qquad c \in A_2$$

$$U = \chi(b \otimes b > 0) + \bar{\alpha}\, \chi(b \otimes b < 0)$$

Anzatz:

$$\sigma = (u \otimes v)\,\chi(b \otimes b < 0)$$

$u^2 = I \qquad u^* = u$

$v^2 = I \qquad v^* = v$

$ub = -bu$

$vb = -bv$

$uv = vu \qquad \Rightarrow \qquad \alpha = \pm 1 \qquad \hbar = \dfrac{\pi}{2k+3}$

$uv = -vu \qquad \Rightarrow \qquad \alpha = \pm i \qquad \hbar = \dfrac{\pi}{2k+4}$

No relation $\qquad \alpha$ - any $\qquad \hbar$ any.

$$\varphi(\operatorname{sign} b) = \operatorname{sign} b \otimes I$$

$$\varphi(v) = v \otimes I$$

$$\varphi(u) = \alpha\,(u \otimes vu)\,\chi(b \otimes b > 0) + \bar{\alpha}\,(u \otimes uv)\,\chi(b \otimes b < 0).$$