Reduced HNN Extensions of von Neumann Algebras

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1. Objective

Given Data:

- v.N.algebra N;
- v. N. subalgebra D;
- injective normal $*$-homo. $\theta : D \to N$;
- $\exists$ f.n.conditional expectations

$$E_D^N : N \to D, \quad E_{\theta(D)}^N : N \to \theta(D).$$

Main objective is to introduce and investigate **the Reduced HNN Extension:**

$$\left(M, E_D^M\right) := \left(N, E_D^N\right) *_D \left(\theta, E_{\theta(D)}^N\right),$$

natural von Neumann algebra analog of the group theoretic one.

2. HNN Extensions in Group Theory

HNN extensions were first appeared in:

<u>H-N-N's Thm.</u> (1949) Each countable discrete group G has a two generator group $\widetilde{G}$, into which G can be embedded.

H-N-N = G.Higman, H.Neumann & B.Neumann

The proof of H-N-N:

1st Step:

Enlarge G to a three generator group $\widehat{G}$ by the **amalgamated free product** construction.

2nd Step:

Enlarge $\widehat{G}$ to a two generator group $\widetilde{G}$ as an **HNN extension**.

Q.E.D.

<u>Def.</u> (HNN Extensions)

Given data:

- a countable group G;
- a subgroup $H (\subseteq G)$;
- an injective $*$-homo. $\theta : H \to G$.

The universal group G^* generated by G and a new element t with the relation

$$t\theta(h)t^{-1} = h, \quad h \in H,$$

i.e.,

$$G^* = \Big\langle G, t : t\theta(h)t^{-1} = h \ (h \in H) \Big\rangle,$$

is called the **HNN extension** with **base group** G and **stable letter** t. The group G^* is usually denoted by $G \bigstar_H \theta$.

Two Examples.

1. Baumslag-Solitar Group BS(n, m):

$$\mathsf{BS}(n,m) := \left\langle a, t : ta^n t^{-1} = a^m \right\rangle,$$

i.e., the HNN extension of

- the base group $G = \mathbb{Z} = \langle a \rangle$;
- the subgroup $H = \langle a^m \rangle \cong \mathbb{Z}$;
- the injective $*$-homo. $\theta^n_m (a^m) = a^n$.

Namely,

$$\mathsf{BS}(n,m) = \mathbb{Z} \bigstar_{\langle a^m \rangle} \theta^n_m.$$

2. Richard Thompson's Group F:

$$F := \left\langle x_1, x_2, \ldots \ : \ x_i x_j x_i^{-1} = x_{j+1} \ (i < j) \right\rangle .$$

Define

- the "shift" $\varphi : F \to F$ by

$$\varphi\left(x_i\right) := x_{i+1}, \quad i = 1, 2, \ldots ;$$

- $F^2 =$ the subgroup generated by $x_2, x_3, \ldots$

So-called *Britton's lemma* (Reduced Word Thm for HNN extensions) enables us to see

$$F = F^2 \bigstar_{F^2} \left(\varphi\big|_{F^2} \right)$$

with the stable letter x_1, and the right-hand is apparently isomorphic to $F \bigstar_F \varphi$.

Such a kind of HNN extensions, i.e., HNN extensions by proper endmorphisms are called

Ascending HNN Extensions.

A Group Theoretic Realization:

(cf. Serre's book [Tree, Springer])

Let

$$\widehat{G} := \cdots {}_{\mathrm{id}}\mathop{\bigstar}_{H}{}_{\theta} \overset{-1\mathrm{th}}{G} {}_{\mathrm{id}}\mathop{\bigstar}_{H}{}_{\theta} \overset{0\mathrm{th}}{G} {}_{\mathrm{id}}\mathop{\bigstar}_{H}{}_{\theta} \overset{1\mathrm{th}}{G} {}_{\mathrm{id}}\mathop{\bigstar}_{H}{}_{\theta} \cdots$$

$$= \left\langle \mathop{\bigstar}_{\mathbb{Z}} G \ : \ \iota_n(h) = \iota_{n+1}\left(\theta(h)\right), \ (h \in H) \right\rangle .$$

Here, ι_n = the embedding map of G onto nth component of $\mathop{\bigstar}_{\mathbb{Z}} G$; and

$$ {}_{\mathrm{id}}\mathop{\bigstar}_{H}{}_{\theta} \quad = \quad \begin{bmatrix} G & & \bigstar & & G \\ & \mathrm{id} \nwarrow & & \nearrow \theta & \\ & & H & & \end{bmatrix} .$$

Let

$\widehat{\theta} :=$ Right Shift on $\widehat{G}$ i.e., $\iota_n(g) \mapsto \iota_{n+1}(g)$,

well-defined because

$$\widehat{\theta}\left(\left\{\iota_n(h)\iota_{n+1}\left(\theta(h)\right)^{-1}\right\}\right) = \left\{\iota_n(h)\iota_{n+1}\left(\theta(h)\right)^{-1}\right\} .$$

The HNN extension $G \bigstar_H \theta$ is realized as the semi-direct product

$$G \bigstar_H \theta = \langle \iota_0(G), t \rangle = \widehat{G} \rtimes_{\widehat{\theta}} \mathbb{Z},$$

where

- ι_n = the embedding map of G onto the nth free component of $\widehat{G}$;
- t = the generator of $\mathbb{Z} \left(\subseteq \widehat{G} \rtimes_{\widehat{\theta}} \mathbb{Z} \right)$.

[Key Relation]

$$t\theta(h)t^{-1} = h, \quad \forall h \in H.$$

$(\because)$

$$\begin{aligned} t \cdot \iota_0(\theta(h)) \cdot t^{-1} &= \widehat{\theta}\left(\iota_0(\theta(h))\right) \\ &= \iota_1(\theta(h)) \\ &= \iota_0(h) \end{aligned}$$

since

$$\iota_k(h) = \iota_{k+1}(\theta(h)), \quad h \in H.$$

3. Reduced HNN Extensions

Given Data:

- v.N.algebra N;
- v.N.subalgebra D;
- family Θ of injective normal $*$-homo.s

$$\theta : D \to N \ (\theta \in \Theta);$$

- $\exists$ f.n.condi.exp.s

$$E_D^N : N \to D,$$
$$E_{\theta(D)}^N : N \to \theta(D) \ (\theta \in \Theta).$$

<u>Reduced HNN Extension:</u>

$$\left(M, E_N^M\right) := \left(N, E_D^N\right) \underset{D}{\bigstar} \left(\Theta, \left\{E_{\theta(D)}^N\right\}_{\theta \in \Theta}\right)$$

is a pair of

- v.N.algebra $M = W^* \langle N, u(\theta) \ (\theta \in \Theta) \rangle$;
- f.n.condi.exp. $E_N^M : M \to N$

with the following two conditions (A), (M).

Condition (A) $=$ Algebraic Property:

$$u(\theta)\theta(d)u(\theta)^* = d, \quad \forall d \in D,\ \forall \theta \in \Theta,$$

that is, the unitary $u(\theta)^*$ implements θ, i.e.,

$$\theta = \mathrm{Ad}\,u(\theta)^*\Big|_D : D \to \theta(D) \subseteq N.$$

$u(\theta)$ is called the **stable unitary** associated with $\theta \in \Theta$.

Reduced Words:

A given word

$$w = u(\theta_0)^{\varepsilon_0} n_1 u(\theta_1)^{\varepsilon_1} n_2 \cdots n_\ell u(\theta_\ell)^{\varepsilon_\ell}$$

in N and the stable unitaries $u(\theta)$, $\theta \in \Theta$, with $\varepsilon_j \in \{\cdot, *\}$ is said to be **reduced form** if, whenever $\theta_{j-1} = \theta_j$, then the following implications hold:

$$\left(\varepsilon_{j-1} = \cdot,\ \varepsilon_j = *\right) \Rightarrow \left(n_j \in \mathrm{Ker} E^N_{\theta_j(D)}\right);$$
$$\left(\varepsilon_{j-1} = *,\ \varepsilon_j = \cdot\right) \Rightarrow \left(n_j \in \mathrm{Ker} E^N_D\right).$$

Example: A word

$$w = u(\theta)^* n_1 u(\theta) n_2 u\left(\theta'\right) n_3 u\left(\theta'\right)^*$$

becomes of reduced form if

$$n_1 \in N^\circ := \mathrm{Ker} E_D^N, \quad n_3 \in N_{\theta'}^\circ := \mathrm{Ker} E_{\theta'(D)}^N,$$

but no need to specify the position of n_2.

<u>Condition (M)</u> $=$ Property of Moments:
For $\forall$ reduced word w in N and the $u(\theta)$'s, one has

$$E_N^M(w) = 0.$$

Operator Algebraic Britton's Lem.

$\left(M, E_N^M\right)$ is characterized by (A) & (M) when $N\,(\subseteq M)$ and the $u(\theta)$'s generate M.

Voiculecu's theory of freeness suggests:

Freeness $\longleftrightarrow$ Reduced Word Thm
for Amalgamated Free Products.

In HNN case,

Condition (M) $\longleftrightarrow$ Reduced Word Thm
for HNN Extensions.

4. Construction

Assume $\Theta = \{\theta\}$ for simplicity !

- Δ_2 := the diagonals in $M_2(\mathbf{C})$.

- The inclusion map

$$\iota_1 : D \otimes \Delta_2 \hookrightarrow N \otimes M_2(\mathbf{C})\,;$$
$$\begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \mapsto \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}.$$

- The f.n.condi.exp.:

$$E_1 : N \otimes M_2(\mathbf{C}) \to D \otimes \Delta_2;$$
$$E_1 := \begin{bmatrix} E_D^N & \\ & E_D^N \end{bmatrix}.$$

- The distinguished embedding map:

$$\iota_{\Theta} : D \otimes \Delta_2 \hookrightarrow N \otimes M_2(\mathbf{C}) ;$$
$$\begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \mapsto \begin{bmatrix} d_1 & \\ & \theta(d_2) \end{bmatrix} .$$

- The f.n.condi.exp.:

$$E_{\Theta} : N \otimes M_2(\mathbf{C}) \to D \otimes \Delta_2 ;$$
$$E_{\Theta} := \begin{bmatrix} E_D^N & \\ & E_{\theta(D)}^N \end{bmatrix} .$$

Denote

$$N^{(2)} := N \otimes M_2(\mathbf{C})$$
$$\Delta_2^D := D \otimes \Delta_2$$

Consider

$$(\mathcal{N}, \mathcal{E}) = \left(N^{(2)}, E_\ominus : \iota_\ominus\right) \underset{\Delta_2^D}{\bigstar} \left(N^{(2)}, E_1 : \iota_1\right).$$

$\lambda_\ominus =$ Embedding of $N^{(2)}$ onto 1st free-part;
$\lambda_1 =$ Embedding $N^{(2)}$ onto 2nd free-part;
$\lambda =$ Embedding of Δ_2^D into $\mathcal{N}$.

Fact. $\lambda_\ominus \circ \iota_\ominus = \lambda = \lambda_1 \circ \iota_1$.

$$\begin{aligned}
\lambda_{\ominus}\left(\begin{bmatrix} d_1 & \\ & \theta\left(d_2\right)\end{bmatrix}\right) &= \lambda_{\ominus} \circ \iota_{\ominus}\left(\begin{bmatrix} d_1 & \\ & d_2\end{bmatrix}\right) \\
&= \lambda\left(\begin{bmatrix} d_1 & \\ & d_2\end{bmatrix}\right) \\
&= \lambda_1 \circ \iota_1\left(\begin{bmatrix} d_1 & \\ & d_2\end{bmatrix}\right) \\
&= \lambda_1\left(\begin{bmatrix} d_1 & \\ & d_2\end{bmatrix}\right).
\end{aligned}$$

- $p := \lambda\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right) = \lambda_{\ominus}\left(\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right).$

- $\pi(n) := \lambda_{\ominus}\left(\begin{bmatrix} n & \\ & 0 \end{bmatrix}\right), \ \forall n \in N$

$$\rightsquigarrow \quad \pi : N \to p\left(\lambda_{\ominus}\left(N^{(2)}\right)\right)p \subseteq p\mathcal{N}p.$$

- $u(\theta) := \lambda_1\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\lambda_{\ominus}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$

$\rightsquigarrow$ (i) a unitary in $p\mathcal{N}p$,
(ii) a reduced word of length 2.

- $M = \pi(N) \vee \{u(\theta)\}$.

- $E^M_{\pi(N)}$ denotes the restriction:

  $\left(\text{The Cond.Exp.: } \mathcal{N} \to \lambda_\Theta\left(N^{(2)}\right)\right)\Big|_M$.

Since

$$p \in \lambda_\Theta\left(N^{(2)}\right), \quad p\lambda_\Theta\left(N^{(2)}\right)p = \pi(N),$$

$E^M_{\pi(N)}$ becomes a condi.exp.: $M \to \pi(N)$.

Verify Condition (A):

$$
\begin{aligned}
&u(\theta)\pi\left(\theta(d)\right)u(\theta)^* \\
&= \lambda_1\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \times \\
&\qquad \lambda_\ominus\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \lambda_\ominus\left(\begin{bmatrix} \theta(d) & \\ & 0 \end{bmatrix}\right) \lambda_\ominus\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \\
&\qquad\qquad \times \lambda_1\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \\
&= \lambda_1\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \lambda_\ominus\left(\begin{bmatrix} 0 & \\ & \theta(d) \end{bmatrix}\right) \lambda_1\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \\
&= \lambda_1\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \lambda_1\left(\begin{bmatrix} 0 & \\ & d \end{bmatrix}\right) \lambda_1\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \\
&= \lambda_1\left(\begin{bmatrix} d & \\ & 0 \end{bmatrix}\right) \\
&= \lambda_\ominus\left(\begin{bmatrix} d & \\ & 0 \end{bmatrix}\right) = \pi(d).
\end{aligned}
$$

Condition (M) follows from the fact:

$u(\theta)$ is a (special) reduced word in $\mathcal{N}$
of length 2

in essence !

Prop.
With $n = \pi(n)$, $\left(M, E_N^M\right)$ satisfies (A) & (M).

Hence, this is a realization of the HNN extension with base algebra N and stable unitary $u(\theta)$ inside a corner algebra of $\mathcal{N}$.

5. Modular Theory

Thm. $\forall$ f.n.s.weight ψ on D,

$$\sigma_t^{\psi\circ E_D^N\circ E_N^M}(u(\theta))$$
$$= u(\theta)\left[D\psi\circ\theta^{-1}\circ E_{\theta(D)}^N : D\psi\circ E_D^N\right]_t .$$

Cor. If τ is a trace on N s.t.

$$\tau\circ E_D^N = \tau, \quad \tau\circ E_{\theta(D)}^N = \tau,$$
$$\left(\tau\big|_{\theta(D)}\right) = \left(\tau\big|_D\right)\circ\theta,$$

then $\tau\circ E_N^M$ becomes again a trace.

Continuous Cores:

$$M \overset{E^M_N}{\supseteq} N \overset{E^N_D}{\supseteq} D \rightsquigarrow \widetilde{M} \overset{\widehat{E^M_N}}{\supseteq} \widetilde{N} \overset{\widehat{E^N_D}}{\supseteq} \widetilde{D}.$$

$$\theta : D \to N \rightsquigarrow \tilde{\theta} : \widetilde{D} \to \widetilde{N},$$

$$E^N_{\theta(D)} : N \to \theta(D) \rightsquigarrow \widehat{E^N_{\theta(D)}} : \widetilde{D} \to \theta\widetilde{(D)} = \tilde{\theta}\left(\widetilde{D}\right).$$

Thm.

$$\left(\widetilde{M}, \widehat{E^M_N}\right) \cong \left(\widetilde{N}, \widehat{E^N_D}\right) \underset{\widetilde{D}}{\bigstar} \left(\widetilde{\Theta}, \left\{\widehat{E^N_{\theta(D)}}\right\}_{\theta\in\Theta}\right),$$

where

$$\widetilde{\Theta} := \left\{\tilde{\theta} : \theta \in \Theta\right\}.$$

($\because$) Verify (A) & (M).

6. Factoriality

Let

$$\left(M, E_N^M\right) := \left(N, E_D^N\right) \underset{D}{\bigstar} \left(\theta, E_{\theta(D)}^N\right).$$

Prop.

Suppose

- $\exists$ f.n.states φ, φ_θ on D;
- $\exists$ unitaries $v \in N_{\varphi \circ E_D^N}$, $v_\theta \in N_{\varphi_\theta \circ \theta^{-1} \circ E_{\theta(D)}^N}$

s.t.

$$E_D^N(v^n) = E_{\theta(D)}^N(v^n) = 0, \quad E_{\theta(D)}^N(v_\theta^n) = 0$$

as long as $n \neq 0$.

Then,

$$\{v, v_\theta\}' \cap M^\omega \subseteq N^\omega,$$

and in particular,

$$\{v, v_\theta\}' \cap M \subseteq N.$$

Thm.

Suppose

- $\exists$ f.n.states φ, φ_θ on D;
- $\exists$ unitaries $v \in N_{\varphi \circ E_D^N}$, $v_\theta \in N_{\varphi_\theta \circ \theta^{-1} \circ E_{\theta(D)}^N}$

s.t.

$$E_D^N(v^n) = E_{\theta(D)}^N(v^n) = 0, \quad E_{\theta(D)}^N(v_\theta^n) = 0$$

as long as $n \neq 0$.

Then,

$$\mathcal{Z}(M) = \left\{x \in D \cap \theta(D) \cap N' : \theta(x) = x\right\};$$
$$\mathcal{Z}\left(\widetilde{M}\right) = \left\{x \in \widetilde{D} \cap \tilde{\theta}\left(\widetilde{D}\right) \cap \widetilde{N}' : \tilde{\theta}(x) = x\right\};$$
$$M' \cap M^\omega = \left\{x \in D^\omega \cap \theta^\omega\left(D^\omega\right) \cap N' : \theta^\omega(x) = x\right\}.$$

7-1. Concrete Setting 1

Let $\theta \notin \mathbb{Q}$.

$$A_\theta := C^*(u, v) \text{ with } uv = e^{2\pi i\theta} vu.$$
$$R := A_\theta'' \text{ w.r.t. the canonical trace } \tau.$$
$$D := \{u\}'' (\subseteq R).$$
$$\theta := \text{the } *\text{-iso.: } D \to R \text{ with } \theta(u) = v.$$
$$M := R \underset{D}{\bigstar} \theta \text{ w.r.t. the } \tau\text{-cond.exp.s.}$$

Conclusions:

- M is a type II_1 factor.
- M has no central sequence, i.e., M is a full factor.

7-2. Concrete Setting 2

Given Data:

- v.N.algebras $N_i \supseteq D_i$ $(i = 1, 2, 3)$;
- bijective normal $*$-homo.

$$\theta_{21} : D_1 \to D_2 \ (\theta \in \Theta);$$

- auto. $\theta_3 \in \mathrm{Aut}\,(D_3)$.
- $\exists$ f.n.cond.exp.s $E_{D_i}^{N_i} : N_i \to D_i$ $(i = 1, 2, 3)$.

Assumption: $\forall i = 1, 2$;
$\exists$ f.n.states φ_i, φ_2 on D_i with the property:
$\exists$ unitaries $v_i \in (N_i)_{\varphi_i \circ E_{D_i}^{N_i}}$ s.t.

$$\varphi_i \circ E_{D_i}^{N_i}(v_i^n) = 0, \quad \forall n \neq 0.$$

Set

$$N := N_1 \otimes N_2 \otimes N_3 \supseteq D := D_1 \otimes \mathbf{C}1 \otimes D_3;$$

$$\theta : D \to N \quad \text{defined by}$$

$$\theta\,(d_1 \otimes 1 \otimes d_3) := 1 \otimes \theta_{21}\,(d_1) \otimes \theta_3\,(d_3)\,;$$

$$E_D^N := E_{D_1}^{N_1} \otimes \left(\varphi_2 \circ E_{D_2}^{N_2}\right) \otimes E_{D_3}^{N_3};$$

$$E_{\theta(D)}^N := \left(\varphi_1 \circ E_{D_1}^{N_1}\right) \otimes E_{D_2}^{N_2} \otimes E_{D_3}^{N_3}.$$

Conclusions:

$$\left(M, E_N^M\right) := \left(N, E_D^N\right) \underset{D}{\bigstar} \left(\theta, E_{\theta(D)}^N\right)$$

satisfies

$$\mathcal{Z}(M) = D_3^{\theta_3} \cap \mathcal{Z}\left(N_3\right);$$

$$\mathcal{Z}\left(\widetilde{M}\right) = \left(\mathbf{C}1 \otimes \mathbf{C}1 \otimes \left(\widetilde{D_3}^{\widetilde{\theta_3}} \cap \mathcal{Z}\left(\widetilde{N_3}\right)\right)\right) \cap \mathcal{Z}\left(\widetilde{N}\right);$$

$$M_\omega \subseteq M' \cap M^\omega = (D_3^\omega)^{\theta_3^\omega} \cap N_3'.$$

Remark: The right-hand sides as well as the T-set $T(M)$ can be computed explicitly in many concrete cases.

7-3. Concrete Setting 3

Let

$$N := \left(\bigotimes_{\mathbb{Z}} \left(Q, \varphi_Q \right) \right) \rtimes_\gamma \mathbb{Z}$$

be the crossed-product by the non-commutative Bernoulli shift γ.

Choose a diffuse abelian subalgebra:

$$D := \bigotimes_{\mathbb{Z}} D_0 \subseteq \bigotimes_{\mathbb{Z}} Q_{\varphi_Q}.$$

Then, $\exists\, \theta : \left(D, \left(\otimes_{\mathbb{Z}} \varphi_Q \right) \big|_D \right) \cong \left(\lambda^\gamma (\mathbb{Z})'', \tau_{\mathbb{Z}} \right)$.

Set

$$E_D^N := E_D^{\otimes_{\mathbb{Z}} Q} \circ E_{\otimes_{\mathbb{Z}} Q}^N;$$

$$E_{\theta(D)}^N := \left(\left(\bigotimes_{\mathbb{Z}} \varphi_Q \right) \otimes \mathrm{id} \right) \Big|_N .$$

(The cond.exp.s are chosen w.r.t. $\otimes_{\mathbb{Z}} \varphi_Q$.)

Assumption:

$\exists$ unitary $v \in Q_{\varphi_Q}$ with $\varphi_Q(v) = 0$.

N.B. The classical Bernoulli shift with the equal prob.vec.:

$$\prod_{\mathbb{Z}} \left(\{1, 2, \ldots, n\}, \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) \right)$$

— O.K.

Conclusion:

$$\left(M, E_N^M \right) := \left(N, E_D^N \right) \underset{D}{\bigstar} \left(\theta, E_{\theta(D)}^N \right).$$

is a full factor of type II_1 or type III_λ with $\lambda \neq 0$.

Remark: If D_0 is a Cartan in Q, then so is D in N. However, $\lambda^\gamma(\mathbb{Z})''$ is known to be a singular MASA in N. Therefore, θ cannot be extended to a global auto. on N !

7-4. Analog of H-N-N's Thm

Prop. Each finite v.N.algebra P has a full type II_1 factor $\widetilde{P}$ generated by two Haar unitaries, into which P can be embedded.

(1) The construction of $\widetilde{P}$ is essentially same as in the original H-N-N's Thm.
(2) Our main results is needed to show that the resulting $\widetilde{P}$ becomes a full type II_1 factor.

8. Ascending HNN Etensions

This part is a work still in progress.

Given Data:

- v.N.algebra N;
- endomorphism $\rho : N \to N$;
- $\exists$ conditional expectation

$$E_\rho : N \to \rho(N).$$

Ascending HNN Extensions:

$$\left(M, E_N^M\right) = (N, \mathrm{id}_N) \underset{N}{\bigstar} (\rho, E_\rho)$$

is called an ascending HNN extension.

Let

$$M \overset{E}{\supseteq} N;$$

$$\langle M, N\rangle := \text{the basic ext.},$$

$$\widehat{E} := \text{the dual op.-valued weight: } \langle M, N\rangle \to M,$$

$$\text{i.e., } \widehat{E}\,(m_1 e_N m_2) := m_1 m_2, \ \forall m_1, m_2 \in M.$$

Def. (Relative Følner-type Condition)
$\forall \varepsilon > 0$, $\forall$ finite $\mathcal{F}\,(\subseteq \mathcal{U}(M))$, $\exists$ proj. $f \in \langle M, N\rangle$ s.t.

$$\widehat{E}(f) \in M,$$

$$\|ufu^* - f\|_{L^2(\langle M,N\rangle)} < \varepsilon \cdot \|f\|_{L^2(\langle M,N\rangle)}, \ \forall u \in \mathcal{F}.$$

Prop. Let

$$\left(M, E_N^M\right) = (N, \mathrm{id}_N) \mathop{\bigstar}_{N} (\rho, E_\rho)$$

be an ascending HNN extension.

Suppose $\exists$ f.n.state φ on N s.t.

$$\varphi = \varphi \circ \rho^{-1} \circ E_\rho.$$

Then, $M \supseteq N$ satisfies the relative Følner-type condition.

Remark: The Følner proj. $f \in \langle M, N \rangle$ can be chosen so that $\widehat{E}(f) \in \mathbf{C}1$ in the above prop.

Let $\widehat{\rho}$ be

the non-commutative Bernoulli shift
or
the free Bernoulli shift on

$$\left(\widehat{N}, \widehat{\varphi}\right) := \bigotimes_{\mathbb{Z}} (P, \varphi_P) \text{ or } \bigstar_{\mathbb{Z}} (P, \varphi_P).$$

Let ρ be the restriction of $\widehat{\rho}$ to

$N :=$ the "half" of the algebra $\widehat{N}$,

i.e.,

$$(N, \varphi) := \bigotimes_{\mathbb{N} \sqcup \{0\}} (P, \varphi_P) \text{ or } \bigstar_{\mathbb{N} \sqcup \{0\}} (P, \varphi_P).$$

Let

$E_\rho :=$ the $\widehat{\varphi}$-condi.exp. : $\widehat{N} \to N$.

Prop. In the current setup, let

$$\left(M, E_N^M\right) = (N, \mathrm{id}_N) \mathop{\bigstar}_{N} (\rho, E_\rho)\,.$$

Then,

$$\left(M,\ \varphi \circ E_N^M\right) \cong \left(\widehat{N} \rtimes_{\widehat{\rho}} \mathbb{Z},\ \widehat{\varphi} \circ E_{\widehat{N}}^{\widehat{N} \rtimes_{\widehat{\rho}} \mathbb{Z}}\right).$$

These prop.s explain the relationship between Monod-Popa's and Pestov's counter-examples to Eymard's question on co-amenability in operator algebras.

Monod and Popa's type example obtained by replacing "Bernoulli" shift by free shift:

$$\mathbb{F}_2 = \mathbb{F}(\mathbb{Z}) \rtimes \mathbb{Z} \supseteq \mathbb{F}(\mathbb{Z}) \supseteq \mathbb{F}(\mathbb{Z}_{\geq 0});$$

$$\mathbb{F}(\mathbb{Z}) \rtimes \mathbb{Z} = \text{HNN}\left(\mathbb{F}(\mathbb{Z}_{\geq 0}), \text{restricted free shift}\right).$$

Pestov's one:

$$\begin{aligned} \mathbb{F}_2 =& \mathbb{F}(\mathbb{Z}) \rtimes \mathbb{Z} \\ & \supseteq \mathbb{F}(\mathbb{Z}) = \Gamma_0 \rtimes \mathbb{F}(\mathbb{Z}_{\leq -1}) \\ & \qquad \supseteq \Gamma_0 \left(\supseteq \mathbb{F}\left(\mathbb{Z}_{\geq 0}\right)\right); \end{aligned}$$

$\Gamma_0 :=$ the smallest normal subgroup $\supseteq \mathbb{F}\left(\mathbb{Z}_{\geq 0}\right)$, i.e.,

$$\Gamma_0 = \bigvee_{g \in \mathbb{F}(\mathbb{Z}_{\leq -1})} g\mathbb{F}\left(\mathbb{Z}_{\geq 0}\right) g^{-1}.$$

Addenda: C^*-Setting

(1) Except for the factoriality result, almost all the results (of course, the construction) are still varild even in the C^*-setting.

(2) The embedding and exactness results are derived from the corresponding results (due to Blanchard-Dykema/Dykema-Shlyakhtenko) on amalgamated free products thanks to the construction !