Outer Actions of a Group on a Factor
Abel Symposium, 2004.

Yoshikazu Katayama

Department of Mathematics
Osaka Kyoiku University, Osaka, Japan

and

Masamichi Takesaki

Department of Mathematics
UCLA
and
Department of Mathematical Science
Norwegian University of Science and Technology

First, I will discuss the characteristic square of factor $\mathcal{M}$:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathcal{U}(\mathcal{C}) & \xrightarrow{\partial_\theta} & \mathrm{B}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{U}(\mathcal{M}) & \longrightarrow & \widetilde{\mathcal{U}}(\mathcal{M}) & \xrightarrow{\partial_\theta} & \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))) & \longrightarrow & 1 \\
 & & {\scriptstyle \mathrm{Ad}}\downarrow & & {\scriptstyle \widetilde{\mathrm{Ad}}}\downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Int}(\mathcal{M}) & \longrightarrow & \mathrm{Cnt_r}(\mathcal{M}) & \xrightarrow{\dot{\partial}_\theta} & \mathrm{H}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

where $\{\mathcal{C}, \mathbb{R}, \theta\}$ is the flow of weights on $\mathcal{M}$; $\widetilde{\mathcal{U}}(\mathcal{M})$ is the extended unitary group of $\mathcal{M}$, i.e., the normalizer of $\mathcal{M}$ in the unitary group $\mathcal{U}(\widetilde{\mathcal{M}})$ of the core $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. The core $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is the von Neumann algebra generated by the imaginary power $\{\varphi^{it} : t \in \mathbb{R}, \varphi \in \mathfrak{W}_0(\mathcal{M})\}$ of faithful semi-finite normal weights on $\mathcal{M}$. Scaling $\varphi \mapsto e^{-s}\varphi, s \in \mathbb{R}$, gives rise to the one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ of $\widetilde{\mathcal{M}}$ such that

$$\mathcal{M} = \widetilde{\mathcal{M}}^\theta \quad \text{and} \quad \mathcal{M}' \cap \widetilde{\mathcal{M}} = \mathcal{C}.$$

The normalizer $\widetilde{\mathcal{U}}(\mathcal{M})$ of $\mathcal{M}$ in the unitary group $\mathcal{U}(\widetilde{\mathcal{M}})$ of $\widetilde{\mathcal{M}}$ gives the extended modular automorphism group $\mathrm{Cnt_r}(\mathcal{M})$ as every $u \in \widetilde{\mathcal{U}}(\mathcal{M})$ gives an automorphism $\widetilde{\mathrm{Ad}}(u)(x) = uxu^*, x \in \mathcal{M}$.

Looking at the middle vertical exact sequence:

$$1 \longrightarrow \mathcal{U}(\mathcal{C}) \longrightarrow \widetilde{\mathcal{U}}(\mathcal{M}) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{Cnt_r}(\mathcal{M}) \longrightarrow 1,$$

choose a cross-section: $\alpha \in \mathrm{Cnt_r}(\mathcal{M}) \mapsto u(\alpha) \in \widetilde{\mathcal{U}}(\mathcal{M})$ such that $\alpha = \widetilde{\mathrm{Ad}}(u)(\alpha))$. Then we have:

$$\begin{aligned} &\mu(\alpha, \beta) = u(\alpha)u(\beta)u(\alpha\beta)^* \in \mathcal{U}(\mathcal{C}); \\ &\lambda(\alpha, \gamma) = \gamma(u(\gamma^{-1}\alpha\gamma))u(\alpha)^* \in \mathcal{U}(\mathcal{C}), \end{aligned} \quad \alpha, \beta \in \mathrm{Cnt_r}(\mathcal{M}), \gamma \in \mathrm{Aut}(\mathcal{M}).$$

The pair (λ, μ) is a characteristic cocycle of V.F.R. Jones and gives rise to the characteristic invariant $\Theta(\mathcal{M})$ in the relative cohomology group $\Lambda(\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}, \mathrm{Cnt_r}(\mathcal{M}), \mathcal{U}(\mathcal{C}))$, which was named the intrinsic invariant of $\mathcal{M}$ in [KtST].

If $\alpha : g \in G \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ is an action of a group G, then the pull-back $\chi(\alpha) = \alpha^*(\Theta(\mathcal{M})) \in \Lambda_{\mathrm{mod}(\alpha)\times\theta}(G, N, \mathcal{U}(\mathcal{C}))$ with $N = \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$ is a cocycle conjugacy invariant. In the case that $\mathcal{M}$ is an approximately finite dimensional factor and G is a countable discrete amenable group, then the triplet $\{\mathrm{mod}(\alpha), \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M})), \chi(\alpha)\}$ form a complete invariant of the cocycle conjugacy class of α.

To move on one step further to outer actions, we make first definition.

Definition. A map $\alpha \colon g \in G \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ is called an *outer action* if

$$\alpha_g \circ \alpha_h \equiv \alpha_{gh} \mod \mathrm{Int}(\mathcal{M}), \quad g, h \in G.$$

We usually assume that $\alpha_e = \mathrm{id}$ for the identity $e \in G$. If

$$\alpha_g \notin \mathrm{Int}(\mathcal{M}), \quad g \neq e,$$

then it is called a *free* outer action.

CAUTION. One should not confuse this with the concept of *free actions*.

Consider the quotient group $\mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$ and fix a cross-section: $g \in \mathrm{Out}(\mathcal{M}) \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ of the quotient map $\pi : \alpha \in \mathrm{Aut}(\mathcal{M}) \mapsto [\alpha] \in \mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$ and also choose a Borel cross-section $\alpha \in \mathrm{Cnt_r}(\mathcal{M}) \mapsto u(\alpha) \in \widetilde{\mathcal{U}}(\mathcal{M})$ in such a way that $u(\alpha) \in \mathcal{U}(\mathcal{M})$ for every $\alpha \in \mathrm{Int}(\mathcal{M})$. Then we have for $g, h, k \in \mathrm{Out}(\mathcal{M})$

$$\begin{aligned} u(g, h &= u(\alpha_g \circ \alpha_h \circ \alpha_{gh}^{-1}) \in \mathcal{U}(\mathcal{M}), ; \\ c(g, h, k) &= \alpha_g(u(h, k))u(g, hk)\{u(g, h)u(gh, k)\}^* \in \mathbb{T}. \end{aligned}$$

The three variable function c is indeed a cocycle $c \in \mathrm{Z}^3(\mathrm{Out}(\mathcal{M}), \mathbb{T})$. The cohomology class $[c] \in \mathrm{H}^3(\mathrm{Out}(\mathcal{M}), \mathbb{T})$ is called the *intrinsic obstruction* and denoted by $\mathrm{Ob}(\mathcal{M})$. If α is an outer action of G on $\mathcal{M}$, then the pull back $\mathrm{Ob}(\alpha) = \alpha^*(\mathrm{Ob}(\mathcal{M}))$ is an invariant of the outer conjugacy class of α. If $\mathcal{M}$ is a factor of type II_1, then one can work directly on the obstruction, employing the Brower group trick. But in the case of type III, this direct method does not work. For example, the group $\mathrm{Cnt_r}(\mathcal{M})$ is not stable under the tensor product, while $\mathrm{Int}(\mathcal{M})$ is stable. To deal with this problem, we will do the following:

To each factor $\mathcal{M}$, we associate an invariant $\mathrm{Ob_m}(\mathcal{M})$ to be called the *intrinsic modular obstruction* as a cohomological invariant which lives in the "third" cohomology group:

$$\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(\mathrm{Out}(\mathcal{M}) \times \mathbb{R}, \mathrm{H}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})), \mathcal{U}(\mathcal{C}))$$

where $\{\mathcal{C}, \mathbb{R}, \theta\}$ is the flow of weights on $\mathcal{M}$. If α is an outer action of a countable discrete group G on $\mathcal{M}$, then its modulus $\mathrm{mod}(\alpha) \in \mathrm{Hom}(G, \mathrm{Aut}_\theta(\mathcal{C}))$, $N = \alpha^{-1}(\mathrm{Cnt_r}(M))$ and the pull back

$$\mathrm{Ob_m}(\alpha) = \alpha^*(\mathrm{Ob_m}(\mathcal{M})) \in \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C})))$$

to be called the *modular obstruction* of α are invariants of the outer conjugacy class of the outer action α.

We prove that if the factor $\mathcal{M}$ is approximately finite dimensional and G is amenable, then the invariants uniquely determine the outer conjugacy class of α and the every invariant occurs as the invariant of an outer action α of G on $\mathcal{M}$. If we have enough time, we will discuss the case that $\mathcal{M}$ is a factor of type III_λ, $0 < \lambda \leq 1$. In this case the modular obstruction group $\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C}))$ and the modular obstruction $\mathrm{Ob}_{\mathrm{m}}(\alpha)$ take simpler forms. But this does not mean that our work is easier. The difficulties in this case can be seen in the fact that $\mathrm{Aut}(\mathcal{M})$ does not act on the discrete core, a fact that is overlooked sometimes. Also some examples will be discussed if we have enought time.

References

[Cnn] A. Connes, *Periodic automorphisms of the hyperfinite factor of type* II_1, Acta Math. Szeged, **39** (1977), 39-66.

[FT] A.J. Falcone and M. Takesaki, *Non-commutative flow of weights on a von Neumann algebra*, J. Functional Analysis.

[J] V.F.R. Jones, *Actions of finite groups on the hyperfinite type* III *factor*, Amer. Math. Soc. Memoirs, **237** (1980).

[KtST] Y. Katayama, C.E. Sutherland and M. Takesaki, *The characteristic square of a factor and the cocycle conjugacy of discrete amenable group actions on factors*, Invent. Math., **132** (1998), 331-380.

[KtT] Y. Katayama and M. Takesaki, *Outer actions of a countable discrete amenable group on approximately finite dimensional factors* I*, General Theory*, von Neumann-Stone AMS Contemporary Mathematics Operator Algebra Book.

[Tk1] M. Takesaki, *Theory of Operator Algebras* I, Springer - Verlag, Heidelberg, New York, Hong Kong, Tokyo,, 1979.

[Tk2] *Theory of Operator Algebras* II, Springer - Verlag, Heidelberg, New York, Hong Kong, Tokyo, 2002.

[Tk3] *Theory of Operator Algebras* III, Springer - Verlag, Heidelberg, New York, Hong Kong, Tokyo, 2002.