Computations

Thanks to Wigner's computations of Moore cohomology.

If G is a real Lie group.

In principle, one computes:

– $H^*(G, \mathbf{R})$ and $H^*(G_i, \mathbf{R})$ (Lie algebra cohomology)

– $H^*(G, \mathbf{Z})$ and $H^*(G_i, \mathbf{Z})$ (classifying spaces).

In this way we compute Kac cohomology for G low dimensional Lie group.

Finally, Moore cohomology (resolutions...):

Theorem (BSV). *Fore every module E as above we have*

$$H_{Kac}(E) = H^*(G, (F_1 \times F_2)/E)$$

where F_i is the set of mesaurable maps from G/G_i to E.

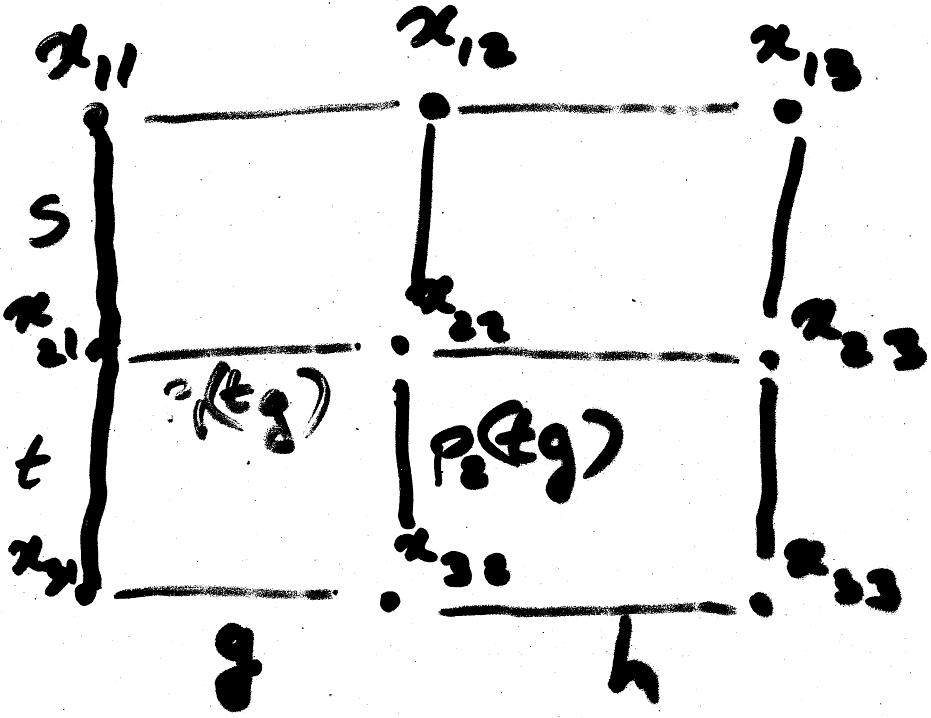
In other words, F_i is obtained by inducing E from G_i to G. Thus $H^*(G, F_i) = H^*(G_i, E)$.

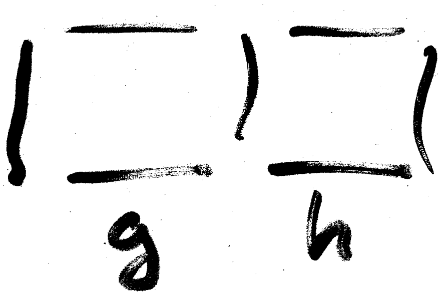
It follows

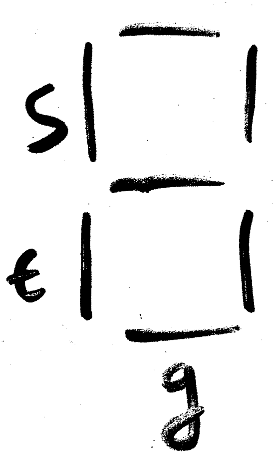
Corollary. *We have a long exact sequence:*

$$\begin{array}{ccccc} \dots & \rightarrow & H^k(G,E) & \rightarrow & H^k(G_1,E) \times H^k(G_2,E) \\ & & & & \downarrow \\ \dots & \leftarrow & H^{k+1}(G,E) & \leftarrow & H^k_{Kac}(E) \end{array}$$

This exact sequence can be directly deduced from the bicomplex.



$U(s,g,h) = f_{12}\Big($  $\Big)$

$V(s,t,g) = f_{21}\Big($  $\Big)$

Kac cohomology (homogenious presentation)

(Kac in the finite case, Baaj-S-Vaes in the general case)

Put $Y_{k,\ell} = \{(x_{i,j})_{0\leq i\leq k;\ 0\leq j\leq \ell} \in G^{(k+1)(\ell+1)};$
$$x_{i,j}^{-1}x_{i,j'} \in G_1,\ x_{i,j}^{-1}x_{i',j} \in G_2\ \ \forall i,i',j,j'\}$$

Obvious vertical and horizontal face maps (removing a row or a column).

Every (Polish) abelian group which is a G-module E gives rise to a bi-complex of measurable G-equivariant maps from $Y_{k,\ell}$ $(k,\ell \geq 1)$ to E (up to null ones).

Put $H_{Kac}(E)$ the cohomology of this bicomplex.

One identifies easily:

Proposition. *We have* $H_{ext} \simeq H^2_{Kac}(U(1))$.

It is not too hard, though less direct:

Theorem (BSV). *We have* $H_{pent} \simeq H_{ext} \simeq H^2_{Kac}(U(1))$.

Quantum group extensions

Kac and then Vaes-Vainerman study: locally compact quantum groups arising as extensions of a group (G_1) by a group dual (G_2).

Theorem. *(Kac - G_i finite, Vaes-Vainerman - general)*
These are classified by a group $G = G_1G_2$ and two functions:
$\mathcal{U} : G_2 \times G_1 \times G_1 \to U(1)$ *and* $\mathcal{V} : G_2 \times G_2 \times G_1 \to U(1)$
such that:

$$\mathcal{U}(p_2(sg), h, k)\overline{\mathcal{U}}(s, gh, k)\mathcal{U}(s, g, hk)\overline{\mathcal{U}}(s, g, h) = 1$$

$$\mathcal{V}(t, u, g)\overline{\mathcal{V}}(st, u, g)\mathcal{V}(s, tu, g)\overline{\mathcal{V}}(s, t, p_1(ug)) = 1$$

$$\mathcal{U}(t, g, h)\overline{\mathcal{U}}(st, g, h)\mathcal{U}(s, p_1(tg)p_1(p_2(tg)h))$$
$$\mathcal{V}(p_2(sp_1(tg)), p_2(tg), h)\overline{\mathcal{V}}(s, t, gh)\mathcal{V}(s, t, g) = 1$$

First two conditions: 2-cocycles for the action of G_i on G/G_i.

A cocycle $(\mathcal{U}, \mathcal{V})$ is trivial if there is a measurable map $\mathcal{R} : G_2 \times G_1 \to U(1)$ such that, almost everywhere

$$\mathcal{U}(s, g, h) = \overline{\mathcal{R}}(p_2(sg), h)\mathcal{R}(s, gh)\overline{\mathcal{R}}(s, g) \quad \text{and}$$
$$\mathcal{V}(s, t, g) = \mathcal{R}(t, g)\overline{\mathcal{R}}(st, g)\mathcal{R}(s, p_1(tg))$$

The corresponding cohomology group

$$H_{ext} = \{\text{Cocycles}\}/\{\text{trivial}\}$$

Cocycles

Let G, G_1, G_2 be as above

$v : G \times G \to G \times G$ the corresponding pentagonal transformation

$V : \xi \mapsto d^{1/2}\xi \circ v$ the corresponding multiplicative unitary.

Let $u : G \times G \to U(1) = \{z \in \mathbf{C};\ |z| = 1\}$ be a measurable function.

Definition. We say u is a cocycle if the unitary

$$W_u : \xi \mapsto u d^{1/2} \xi \circ v$$

is multiplicative.

A cocycle of the form $(x, y) \mapsto \big((z \times z) \circ v\big)(z \times z)^{-1}$ is said to be trivial where $z : G \to U(1) = \{z \in \mathbf{C};\ |z| = 1\}$ is a measurable function and $z \times z$ is the function $(x, y) \mapsto z(x)z(y)$.

The corresponding cohomology group

$$H_{pent} = \{\text{Cocycles}\}/\{\text{trivial}\}$$

Questions

- What is the relevent cohomology theory?
- How to compute it ?

Regularity

Associated with a locally compact quantum group $S \bowtie \widehat{S}$: the C^*-algebra generated by the group C^*-algebra and its dual.

In our case $S \bowtie \widehat{S} = C_0(G) \rtimes (G_2 \times G_1)$.

The *regular* case $G = G_1 G_2$: then $S \bowtie \widehat{S} = \mathcal{K}$. We then have Takesaki-Takai duality.

The *semi-regular* case $G_1 G_2$ open in G: then $S \bowtie \widehat{S} \supset \mathcal{K}$.

A non semi-regular case: R a locally compact ring with R^* not open but of full measure (adelic type). As an example, $G = ax + b$, $a \in R^*$, $b \in R$, $G_1 = a(x+1) - 1$ and $G_2 = ax$.

Construction of R

Let $\mathcal{P}$ be a set of prime numbers and R is the restricted product

$$R = \prod_{p \in \mathcal{P}}{}' \mathbf{Q}_p.$$

If P is infinite, R^* is not open in R; if $\sum_{\mathcal{P}} p^{-1} < \infty$, then R^* has full measure.

5. $G = ax + b,\ a \in \mathbf{R}^*,\ b \in \mathbf{R}$, $G_1 = a(x+1) - 1$ and $G_2 = ax$.

Taking the 'bad' orbit' $\{h \in G;\ h(-1) = 0\}$, this gives a pentagonal transformation of $\mathbf{R} \times \mathbf{R}$:

$$v(x,y) = (xy + x + y, y + \frac{y}{x}).$$

Note that $v^{-1}(x,y) = \left(\frac{x}{y+1}, \frac{xy}{x+y+1}\right)$. Thus, both v and v^{-1} leave $\mathbf{R}_+^* \times \mathbf{R}_+^*$ invariant and also $\mathbf{Q}_+^* \times \mathbf{Q}_+^*$.

The restrictions of v to $\mathbf{R}_+^* \times \mathbf{R}_+^*$ and $\mathbf{Q}_+^* \times \mathbf{Q}_+^*$ are *very singular* pentagonal transformations (and give rise to singular multiplicative unitaries).

The associated algebras are *triangular*!

Examples

1. Semi-direct products.

2. G_2 a finite group; $G = \mathcal{S}_{G_2}$ and $G_1 = fix(e)$.

3. Iwasawa decomposition:
G semi-simple Lie group $(SL(n, \mathbf{R}))$;
$G_1 = K$ (=$S0(n)$ maximal compact subgroup);
$G_2 = AN$ (upper triangular matrices with positive diagonal entries).

4. Decompositions of the type LU.

a) $G = GL(n)$;
G_1= upper triangular matrices with positive diagonal entries;
G_2= lower triangular matrices with diagonal entries of modulus one.

b) $G = GL(2n, \mathbf{R})$;
G_1 is the set of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1_n \end{pmatrix}$;

G_2 is the set of matrices of the form $\begin{pmatrix} 1_n & 0 \\ a & b \end{pmatrix}$.

π => measure on G

$$V = \int^{\oplus} e_g \otimes 1_{H_g} \, d\mu(g)$$

* e_g ae unitary.

* $g \longmapsto e_g$ homeomorphism.
  G weak

* $V_{12} V_{13} V_{23} = V_{23} V_{12}$ $\longrightarrow$

$$V (e_g \otimes e_g) = (1 \otimes e_g) V$$

ae $\rightsquigarrow$ e.

Write

$$v(x,y) = (x \cdot y\,,\, x \# y)$$

<u>Condition</u>

$$\phi(x,y) = (x \cdot y, y)$$

$$\eta(x,y) = (x, x \# y)$$

measure class isomorphism

Theorem *(Baaj-S) Every sufficiently regular pentagonal transformation is of that form.*

The associated multiplicative unitary comes from a locally compact quantum group.

Theorem *(Baaj-S-Vaes)*
a) The associated full and reduced C^*-algebras are the full and reduced crossed products $C_0(G/G_1)\rtimes G_1$.
b) The associated full and reduced dual C^*-algebras are the full and reduced crossed products $C_0(G/G_2)\rtimes G_2$.

(The dual locally compact quantum group is obtained by exchanging the roles of G_1 and G_2.)

Matched pairs of groups

Example G.I. Kac (Takeuchi, Majid, Baaj-S, Vaes-Vainerman...) quantum group: matched pair or bicrossed product:

$$G \simeq G_1 G_2 :$$

- G is a locally compact group.
- G_1, G_2 are closed subgroups.
- The map $(x_1, x_2) \mapsto x_1 x_2$ (from $G_1 \times G_2$ to G) is a measure class isomorphism.

The associated pentagonal transformation

Put $X = G$ and

$$v(x,y) = (x p_1(p_2(x)^{-1}y), p_2(x)^{-1}y)$$

where $p_i : G \to G_i$ satisfy $x = p_1(x)p_2(x)$ (almost everywhere).

[One can also take a $G_2 \times G_1$ invariant subset of G + multiplicity.]

Pentagonal transformations

Definition. Let $(X, \mathcal{B}, \mu)$ be a measure space. A *pentagonal transformation* is a measure-class isomorphism $v : X \times X \to X \times X$ satisfying

$$v_{23}v_{13}v_{12} = v_{12}v_{23}.$$

Put then

$$V\xi = d^{1/2}\xi \circ v$$

where d is a Radon-Nykodym derivative: this is a multiplicative unitary.

V commutative mult. unit.

$$S = C^*\{(\omega \otimes \mathrm{id})(V) ; \omega \in \mathcal{L}(H)_*\}$$

$$\begin{array}{c} \| \\ L(\omega) \end{array}$$

Abelian C^*-algebra

$$S = C_0(G)$$

$$\begin{array}{ccc} C_0(G) & \xleftrightarrow{\pi} & \mathcal{L}(H) \\ \psi & & \\ l(\omega) & \longmapsto & L(\omega) \end{array}$$

$$\rho(g) \in \mathcal{L}(H)$$

$$\omega(\rho(g)) = l(\omega)(g)$$

Multiplicative unitaries: (Baaj-S)

Definition. A multiplicative unitary on a Hilbert space H is a unitary $V \in \mathcal{L}(H \otimes H)$ such that

$$V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

Such a unitary is naturally associated with a *'locally compact quantum group'* (Kac, ... long history ..., Kustermans-Vaes) and in fact completely describes the associated quantum group.

Furthermore: **Any sufficiently 'good' (regular (Baaj-S) or semi-regular (Baaj), manageable (Woronowicz)) multiplicative unitary gives rise to,**
- group von Neumann algebra+dual
- full and reduced C^*-algebras (+dual)
- comultiplications...

Pentagonal transformations and Kac cohomology

S. Baaj, GS: Transformations pentagonales. *C.R.A.S. Note* 1998.

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