Free Entropy Dimension

and

L^2 cohomology

based on:

D. Sh. "Some estimates for the non-microstates free entropy dim..." math.OA/0308093 appeared in IMRN

D. Sh. + A. Connes "L^2 homology for von Neumann alg" math.OA/0309343

D. Sh. + I. Mineyev "Non-microstats free entropy dimension for groups" math.OA/0312242

Th (with I. Mineyev). Let Γ be a finitely gen. discrete group.

$X_1, \dots, X_n \in \mathbb{C}\Gamma$, $X_j = X_j^*$ generators.

Then

$$\underbrace{\beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1}_{\text{L^2 Betti nos}} = \underbrace{\delta^*(X_1, \dots, X_n)}_{\text{Voiculescu's non-microstates free entropy dimension}}$$

L^2 Betti nos (Atiyah, Cheeger-Gromov, Lück ....)

Voiculescu's non-microstates free entropy dimension

geometry ? free prob. theory

L^2 Betti numbers for groups.

Γ discrete group acting freely and properly on a Riemannian manifold M

M

$$\begin{array}{ccc} \Omega^k_{(2)}(M) & \xrightarrow{d} & \Omega^{k+1}_{(2)}(M) \\ \cup & & \cup \\ \Omega^k_{(c)}(M) & \xrightarrow{d} & \Omega^{k+1}_{(c)}(M) \end{array}$$

k- forms with compact support

k- forms which are L^2

$$H^k_{(2)}(M) = \frac{\ker d}{\overline{\operatorname{im} d}^{L^2}}$$

Murray von Neumann dim

$$\beta^k_{(2)}(M;\Gamma) = \dim_{L(\Gamma)} H^k_{(2)}(M)$$

- Can replace M by e.g. a "co-finite" CW complex with a Γ-action

- M contractible (K-connected)

$\Rightarrow$ $\beta^k_{(2)}(M;\Gamma)$ are indep of M ($k \leq K$)

- M contractible

$$\Rightarrow \overline{\mathrm{im}(d : \Omega^{k-1}_{(2)} \to \Omega^k_{(2)})}$$
$$\shortparallel$$
$$\overline{\mathrm{im}(d : \Omega^{k-1}_{(c)} \to \Omega^k_{(c)})}$$
$$\shortparallel$$
$$\overline{\ker(d : \Omega^k_{(c)} \to \Omega^{k+1}_{(c)})}$$

So $H^k_{(2)} = \dfrac{\ker(d : \Omega^k_{(2)} \to \Omega^{k+1}_{(2)})}{\overline{\ker(d : \Omega^k_{(c)} \to \Omega^{k+1}_{(c)})}}$

L^2 cohomology

L^2 solutions to ... equations

vs

C_c solutions to

$$\ker\left(d : \Omega^k_{(2)} \to \Omega^{k+1}_{(2)} \right)$$

vs

$$\overline{\ker\left(d : \Omega^k_{(c)} \to \Omega^{k+1}_{(c)} \right)}$$

<u>Example</u> $\Gamma = \langle g_1, \dots, g_n \rangle$

$$\beta_1^{(2)} - \beta_0^{(2)} + 1 = n - \dim_{L(\Gamma)} \overline{\{ (f_1, \dots, f_n) \in C_c(\Gamma)^n \text{ s.t. } \sum (\rho_{g_i} - id) f_i = 0 \}}^{\ell^2}$$

Algebras (joint with A. Connes)

(A,τ) dense subalg of (M,τ) τ trace

modules over Γ

$\downarrow$

bimodules over A

L^2 vs C_c sol'ns to equations from group (co) homology

$$\ell^2(\Gamma) \longleftrightarrow L^2(M) \bar{\otimes} L^2(M)^o$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad HS(L^2(M))$$

$$C_c(\Gamma) \subseteq \ell^2 \longleftrightarrow \mathcal{F}R \subseteq HS$$

group homology $\longleftrightarrow$ Hochschild homology

$X_1, \dots, X_n$ generate A

$$\beta_1 - \beta_0 + 1 = n - \dim \overline{\{(T_1, \dots, T_n) \in \mathcal{F}R^n : \sum [T_i, X_i] = 0\}}^{HS}$$

$M \bar{\otimes} M^o$

$T_i : L^2(M) \to L^2(M)$

Free entropy dim

"log. volume of $t^{1/2}$-nghd"

$$\delta_0(X_1, \ldots, X_n) = n - \liminf_{t \to 0} \frac{\chi(X_1^t, \ldots, X_n^t)}{\log t^{1/2}}$$

$$X_j^t = X_j + \sqrt{t}\, S_j$$

$(S_1, \ldots, S_n)$ free semicirc. free fm $X_1, \ldots, X_n$

L'Hopital's rule + some foul computatn

$$\delta^*(X_1, \ldots, X_n) = n - \liminf_{t \to 0} t\, \Phi^*(X_1^t \ldots X_n^t$$

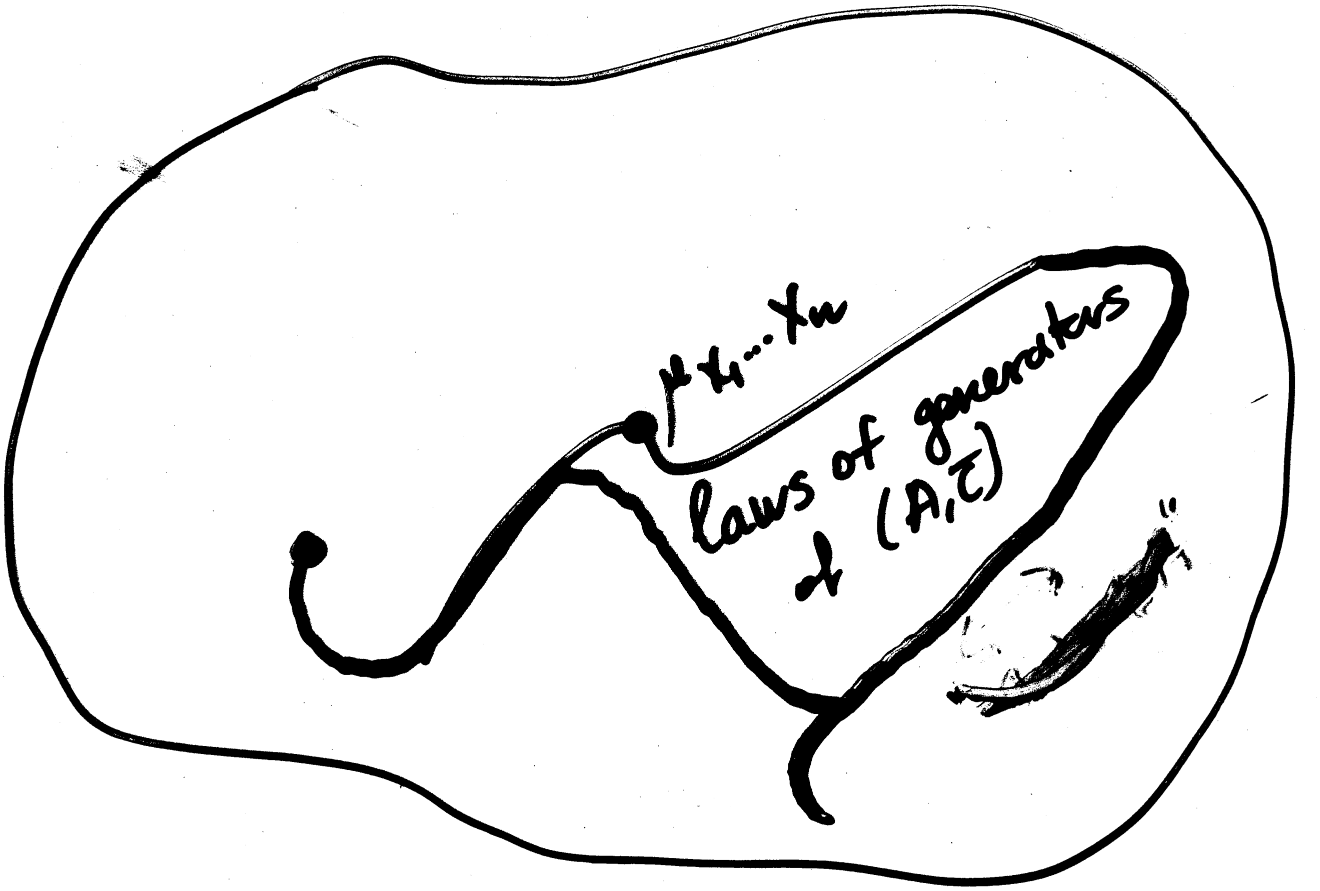
Free Entropy dim.

$X_1, \ldots, X_n \in (M, \tau) \qquad X_j = X_j^*$

$\updownarrow$

law $\mu_{X_1, \ldots, X_n} : \mathbb{C}\langle t_1, \ldots, t_n \rangle \to \mathbb{C}$

$\mu_{X_1 \ldots X_n}(t_{i_1} \ldots t_{i_k}) = \tau(X_{i_1} \ldots X_{i_k})$



all laws

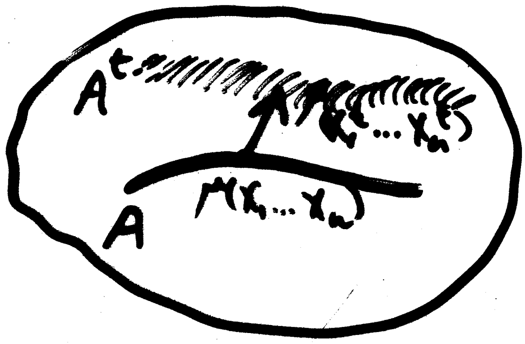
free BM in direction $\left(\sum_j a_j^{(i)} \otimes b_j^{(i)}\right)_{i=1}^{n}$

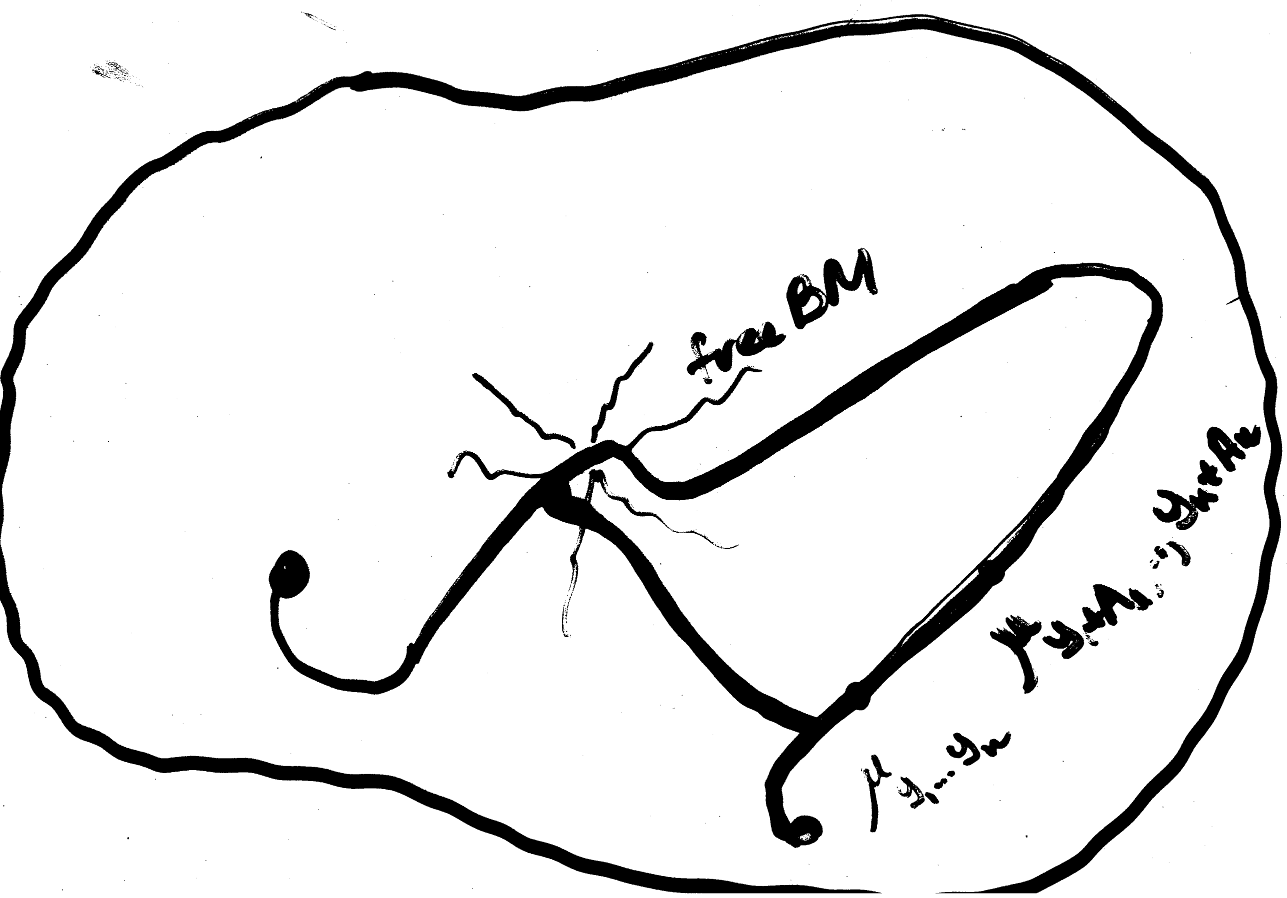
$$X_i \longmapsto X_i^t = X_i + \sqrt{t} \sum_j a_j^{(i)} S_i b_j^{(i)}$$

$$T_i = \sum_j a_j^{(i)} \otimes b_j^{(i)} \in L^2(M) \bar{\otimes} L^2(M)$$

$$\text{"} T_{\mu_{X_1 \ldots X_n}} (\text{laws}) \cong \left(L^2(M) \bar{\otimes} L^2(M)\right)^n \text{"}$$

$$\mu^t = \mu_{X_1^t \ldots X_n^t}$$





$(A \otimes A)^n$

Fact $t > 0 \Rightarrow$ "$\forall$" direction $(T_1, .., T_n) = \vec{T}$

$\exists \; \vec{\xi}^t_{\vec{T}} \in L^2(M_t)^n$ [redacted] s.t.

FBM(μ^t) at time s in directn $\vec{T}$

$$\sim \mu^t\left(X + s\,\xi^{t\,(1)}_{\vec{T}}, \ldots, X + s\,\xi^{t\,(n)}_{\vec{T}}\right)$$

[redacted] A^t

So FBM in dir $\vec{T}$

is "tangent" to A^t

$$\|\xi^t_{\vec{T}}\|_2 \le \frac{1}{\sqrt{t}} \cdot C$$

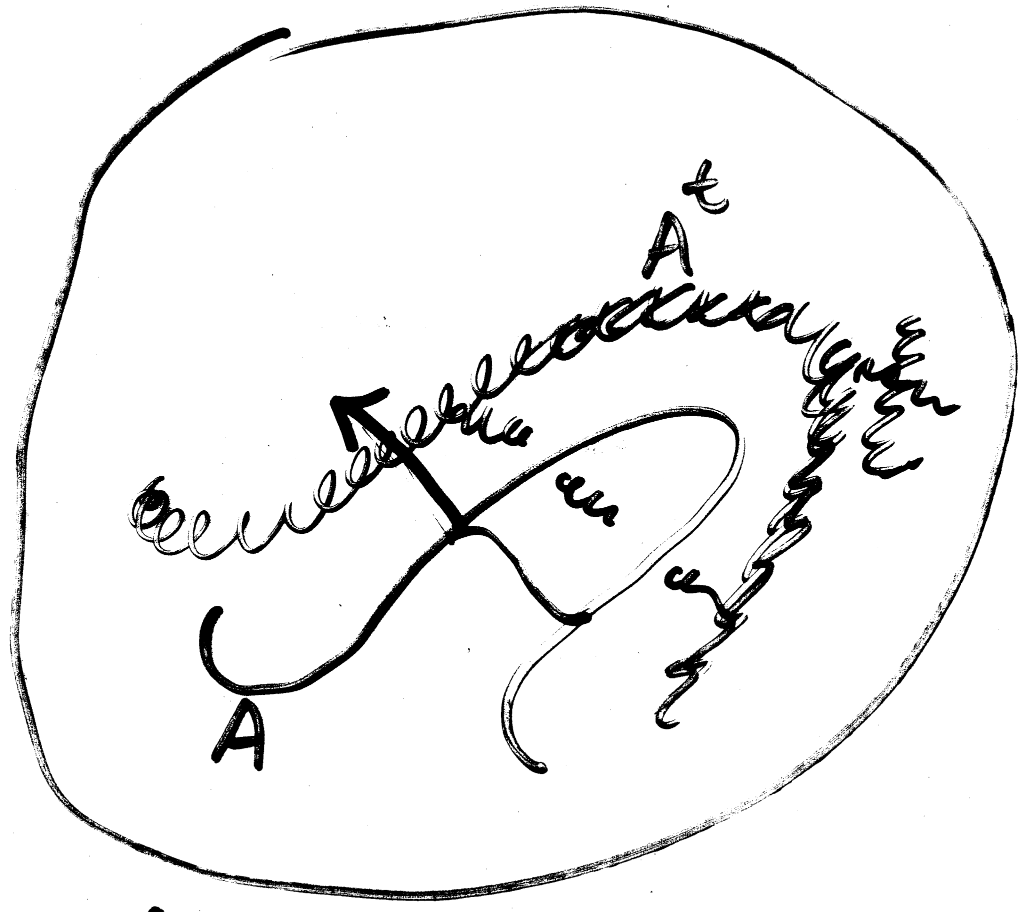
norm in tangent space to A^t

"$P(e_j)$ onto $\perp$ tang. space to A"

$$\delta^*(X_1, .., X_n) = n - \liminf_{t \downarrow 0} \left(\varepsilon \sum_j \|\xi^{t\,(j)}_{e_j}\|_2^2 \right)$$

$e_j = (0, .., 1, 0 ... 0)$
$\quad\quad\quad\quad\quad\;\; j$

"dim of tangent space to A"



$\vec{T}$ normal

if "$\|\vec{T}\|_{A_t}$" blows up laws as $\frac{1}{\sqrt{t}}$ as $t \to 0$.
(sometimes)

Singular metric

non-singular at each A^t

Magic facts

$(T_1, \dots, T_n) \in (L^2 \otimes L^2)^n \cong HS^n$

s.t. $T_j = [X_j, D]$, for some $D \in B(L^2)$

"dX_j"

$\| \xi^t_{\vec{T}} \|_2 = O(1)$

as $t \to 0$

($\vec{T} \in$ "tangent space to A")

$\vec{T} \perp \vec{S}$

if $\vec{S} \in \{(Q_1, \dots, Q_n) \in \mathbb{R}^n$

s.t. $\sum [Q_i, X_j] = 0\}$

exact L^2 1-cocycles

$(T_1, \dots, T_n) \mapsto \sum \mathrm{Tr}(T_j \otimes [X_j, D])$

is a "closed 1-form"

$\sum_i \mathrm{Tr}(T_i [X_i, D])$

#2

• $(T_1, \ldots, T_n) \in \overline{\{(Q_1, \ldots, Q_n) \in \mathcal{J}\mathbb{R}^n : \sum [Q_j, X_j] = 0\}}$

$$\Downarrow$$

$$\| \xi^*_{\vec{T}} \|_2^2 \geq \| \vec{T} \|_2^2 / \sqrt{\varepsilon}^2$$

(hence $\vec{T} \perp$ "tangent space to A")

Combined, these facts imply:

$$\dim_{M\bar{\otimes}M^o} \overline{\{([D, X_1], \ldots, [D, X_n]) : D \in B(L^2)\} \cap HS^n}^{HS} \leq$$

$$\leq \delta^*(X_1, \ldots, X_n)$$

$$\leq n - \dim_{M\bar{\otimes}M^o} \overline{\{(Q_1, \ldots, Q_n) \in \mathcal{J}\mathbb{R}^n : \sum [Q_j, X_j] = 0\}}$$

$$= \beta_1 - \beta_0 + 1$$

$(L^2 \otimes L^2)^n$

$([D,X_1], \dots, [D,X_n])$ closable $1 \otimes \mathrm{dom}\, D$

δ^* $D \in \cancel{X}(L^2)$

$[D, X_j] \in HS \ \forall j$

$\oplus$

$(Q_1, \dots, Q_n) \in \nexists R^n$

$\sum [Q_j, X_j] = 0$

$\oplus$

$\mathcal{X}$?

$(L^2 \bar{\otimes} L^2)^n$ ||

$\vec{T} : \dfrac{\|\xi_T^t\|_2^2}{1/t} \to 0$

"tangent space to A"

$\oplus$

"normal to A"

$t \|\xi_T^t\|_2^2 \not\to 0$

#1 $\leq$

#2 $\leq$

$n - (\beta_1 - \beta_0 + 1)$ $\quad \vec{T} \to$ w-lim $t \, \xi_{\vec{T}}^t$

M bi-linear

#2 if $X_1, \dots, X_n \in \mathbb{C}\Gamma$ generators, $\mathcal{X} = 0$.

3 based on following:

$\mathcal{G}$ = Cayley graph of Γ

CW complex — vertices, edges

ω = 1-form on $\mathcal{G}$, $d\omega = 0$

$\omega \in \ell^2(\text{edges})$

$$\Downarrow$$

$$\omega = df, \quad f \in \mathcal{F}(\Gamma)$$

$\forall \varepsilon \quad \exists \tilde{f} \in \ell^\infty(\Gamma)$,

$$\|\omega - d\tilde{f}\|_2 < \varepsilon.$$

$$\tilde{f}(g) = \begin{cases} R, & f(g) > R \\ f(g), & f(g) \in [-R, R] \\ -R, & f(g) < -R \end{cases}$$

R suff large.