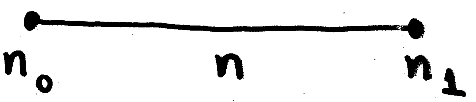
C^*-algebras that absorb the Jiang-Su algebra $\mathcal{Z}$

Interval / dimension drop C^*-alg's:

- $n_0, n_1, n \in \mathbb{N}$ s.t. $n_j \mid n$
- $\varphi_0 : M_{n_0} \to M_n$, $\varphi_1 : M_{n_1} \to M_n$ unital *-hom

$$I(n_0, n, n_1) \overset{\text{DEF}}{=} \{ f \in C([0,1], M_n) \mid f(j) \in \mathrm{Im}(\varphi_j),\ j = 0,1 \}$$



__FACT__ If $\gcd(n_0, n_1) = 1$, then

- $K_0(I(n_0,n,n_1)) \cong \mathbb{Z}$, $K_1(I(n_0,n,n_1)) = 0$
- $I(n_0,n,n_1)$ has no proj. other than: $0, 1$.

__THM__ (Jiang-Su) A, B unital, simple, inf. dim., inductive limits of dim. drop C^*-alg's, then

$$A \cong B \iff \mathrm{Inv}(A) \cong \mathrm{Inv}(B)$$

__THM__ (Jiang-Su) $\exists!$ unital simple inf. dim. C^*-alg $\mathcal{Z}$ s.t. $\mathrm{Inv}(\mathcal{Z}) \cong \mathrm{Inv}(\mathbb{C})$, i.e., $\mathcal{Z}$ has unique trac

$$(K_0(\mathcal{Z}), K_0(\mathcal{Z})^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, 1), \quad K_1(\mathcal{Z}) = 0,$$

Elliott invariant in the unital case:

$$(K_0(A), K_0(A)^+, [1], K_1(A), T(A), T(A) \times K_0(A) \to \mathbb{R})$$

Properties of $\mathcal{Z}$

$\mathcal{Z}$ has no projections other than: $0, 1$

THM (Jiang-Su) $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \cong \bigotimes_1^\infty \mathcal{Z}$

Two properties of a unital C^*-alg A:

(I) $\forall \varphi \in \mathrm{End}_1(A)\ \exists (u_n)_{n=1}^\infty \subseteq \mathcal{U}(A)$: $\mathrm{Ad}\, u_n \to \varphi$

(II) For every inf. dim. unital simple (nuclear) C^*-alg B one has unital embedding $A \hookrightarrow B$

THM (Jiang-Su) $\mathcal{Z}$ has property (I).

EASY FACT If A, B inf. dim. unital simple (nuclear) C^*-alg w/ (I) and (II), then $A \cong B$.

Q : Does $\mathcal{Z}$ have property (II)?

Q : Does $\mathcal{Z}$ embed unitally into any unital (nuclear) C^*-alg with no finite dim. rep's.?

THM (R + Perera) A unital, RR0. TFAE:

(i) A has no finite dim. rep's.

(ii) $\exists \bigotimes_1^\infty (M_2 \oplus M_3) \hookrightarrow A$, unital *-hom.

(iii) $\exists$ (simple, unital, inf. dim. AF-alg) $\hookrightarrow A$, –"–

(iv) $\exists\ \mathcal{Z} \hookrightarrow A$, –"–

(3)

DEF A is __$\mathcal{Z}$-absorbing__ if $A \cong A \otimes \mathcal{Z}$.

FACT $A \otimes \mathcal{Z}$ is $\mathcal{Z}$-absorbing $\forall$ C^*-alg A

THM (Gong-Jiang-Su) A simple unital C^*-alg. Then

$K_0(A)$ weakly unperforated

$\Updownarrow$

$(K_0(A), K_0(A)^+) \cong (K_0(A \otimes \mathcal{Z}), K_0(A \otimes \mathcal{Z})^+)$

$\Updownarrow$

$\mathrm{Inv}(A) \cong \mathrm{Inv}(A \otimes \mathcal{Z})$

__Q__: Which C^*-alg's are $\mathcal{Z}$-absorbing?

__Q__: Does Elliott's invariant classify all simple, separable, nuclear, inf. dim., (unital) $\mathcal{Z}$-absorbing C^*-alg's?

FACT: A separable C^*-alg A is $\mathcal{Z}$-absorbing iff

$$\mathcal{Z} \hookrightarrow \mathcal{M}(A)_\omega \cap A' \qquad \text{(unital embedding)}$$

[For $\omega = \{\text{co-finite sets}\}$: $D_\omega = \ell^\infty(D) / c_0(D)$]

FACT: A = unital C^*-alg. TFAE:

(i) $A_\omega \cap A'$ has no abelian quotient

(ii) $\exists$ $B \subseteq A_\omega \cap A'$ s.t. B unital + separable w/ no abelian quotient

(iii) $\exists$ $\bigotimes_{n=1}^{\infty} B \to A_\omega \cap A'$, —"—

(iv) $A_\omega \cap A'$ has no finite dim. rep's.

<u>THM</u> (Gong-Jiang-Su) Any simple $\mathcal{Z}$-absorbing C^*-alg is either stably finite or purely infinite.

<u>THM</u> (Toms-Winter) Any approximately divisible separable C^*-alg is $\mathcal{Z}$-absorbing.

→ Surprisingly many C^*-alg's are $\mathcal{Z}$-absorbing, eg

- irr. rotation C^*-alg's
- simple non-type I AF-alg's
- Cuntz algebras $\mathcal{O}_n$
- Kirchberg alg's.

→ Not all simple, sep., nuclear C^*-alg's are $\mathcal{Z}$-absorbing:

- Villadsen's example of simple AH-alg w/ non-weakly unperforated K_0-group
- Example of simple, unital, nuclear, sep. C^*-alg w/ finite and infinite proj. [R]
- Toms' example of simple unital ASH-alg A s.t. $A \not\cong A \otimes \mathcal{Z}$ but $\mathrm{Inv}(A) \cong \mathrm{Inv}(A \otimes \mathcal{Z})$

<u>THM</u> (Toms-Winter) The class of separable $\mathcal{Z}$-absorbing C^*-alg's is closed under:

- stable isomorphism
- inductive limits
- extensions

Main results

THM Any simple unital finite $\mathcal{Z}$-absorbing C^*-alg has stable rank one.

THM Let A be an exact simple finite unital $\mathcal{Z}$-abs. C^*-alg. TFAE:

(i) $RR(A) = 0$

(ii) $K_0(A)$ is uniformly dense in $\mathrm{Aff}\,T(A)$

(iii) $K_0(A)$ is weakly divisible, and projections in A separate traces on A

DEF (G, G^+) is <u>weakly divisible</u> if $\forall g \in G^+$ $\forall n \in \mathbb{N}$ $\exists h_1, h_2 \in G^+$: $g = nh_1 + (n+1)h_2$.

COR Let A be an exact simple unital $\mathcal{Z}$-abs. C^*-alg w/ unique trace τ. Then

$$RR(A) = 0 \iff \tau_* K_0(A) \text{ dense in } \mathbb{R}$$

DEF An <u>extended trace</u> on a C^*-alg A is a function $\tau : A^+ \to [0, \infty]$, which is additive and homogeneous with the trace property: $\tau(x^*x) = \tau(xx^*)$ $\forall x \in A$.

τ is <u>non-trivial</u> if $\exists a \in A^+$: $0 < \tau(a) < \infty$.

DEF A is <u>traceless</u> if A admits no non-trivial extended trace.

THM If A is exact and $\mathcal{Z}$-abs. (possibly non-simple), then A is purely infinite iff A is traceless.

THM If A is separable, nuclear and $\mathcal{Z}$-abs. (again possibly non-simple), then $A \cong A \otimes \mathcal{O}_\infty$ iff A traceless.

THM Blackadar's fundamental comparison properties hold for any (possibly non-simple) $\mathcal{Z}$-absorbing C^*-alg.

A recent classification result by N. Brown:

Let A, B be simple unital $\mathcal{Z}$-absorbing C^*-alg's with unique trace. Suppose that A, B are inductive limits of type I C^*-alg's and that $\tau_* K_0(A)$, $\tau_* K_0(B)$ are dense in $\mathbb{R}$. Then

$$A \cong B \iff \mathrm{Inv}(A) \cong \mathrm{Inv}(B)$$

$\Rightarrow$ Conditions above imply that A, B are TAF-alg's (as considered by H. Lin).

On the proof of the stable rank one result

Lemma If A is $\mathcal{Z}$-absorbing, then $\exists\, \sigma_n : A \otimes \mathcal{Z} \to A$ isomorphisms, s.t.

$$\|\sigma_n(a \otimes 1) - a\| \to 0, \quad \forall a \in A.$$

Let A be unital, simple, finite, $\mathcal{Z}$-abs.
Let $a \in A$ be non-invertible.

Enough to show: $a \otimes 1 \in \overline{GL(A \otimes \mathcal{Z})}$

[If $b \in GL(A \otimes \mathcal{Z})$ and $\|a \otimes 1 - b\| < \varepsilon$, then $\sigma_n(b) \in GL(A)$, $\|a - \sigma_n(b)\| < \varepsilon$ for n large enough.]

$$E \overset{\text{DEF}}{=} \{ f \in C([0,1], M_{6^\infty}) \mid f(0) \in M_{2^\infty},\ f(1) \in M_{3^\infty} \}$$

2^∞ $\qquad$ 6^∞ $\qquad$ 3^∞

Lemma. $E \hookrightarrow \mathcal{Z}$ unitally.

Enough to show: $a \otimes 1 \in \overline{GL(A \otimes E)}$

$B = M_{6^\infty}$, $B_0 = M_{2^\infty}$, $B_1 = M_{3^\infty}$

$$A \otimes E = \{ f \in C([0,1], A \otimes B) \mid f(0) \in A \otimes B_0,\ f(1) \in A \otimes B_1 \}$$

$A \otimes B_0$ $\qquad$ $A \otimes B$ $\qquad$ $A \otimes B_1$

⑧

$$D_0 \overset{\text{DEF}}{=} \{ f \in C([0,1/2], A\otimes B) \mid f(0) \in A\otimes B_0 \}$$

$$D_1 \overset{\text{DEF}}{=} \{ f \in C([1/2,1], A\otimes B) \mid f(1) \in A\otimes B_1 \}$$

$$\begin{tikzcd} & A\otimes E \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi_1"] \arrow[dd, "ev_{1/2}"] & \\ D_0 \arrow[dr, "ev_{1/2}"'] & & D_1 \arrow[dl, "ev_{1/2}"] \\ & A\otimes B & \end{tikzcd}$$

$$\varphi_0(f) = f|_{[0,1/2]}, \quad \varphi_1(f) = f|_{[1/2,1]}.$$

$$D_0 \cong D_0 \otimes B_0, \quad D_1 \cong D_1 \otimes B_1$$

- $\varphi_0(a\otimes 1) \in \overline{GL(D_0)}$, $\varphi_1(a\otimes 1) \in \overline{GL(D_1)}$
- D_0, D_1 strongly K_1-surjective
- $A\otimes B$ strongly K_1-injective
- $\operatorname{Im}(K_1(D_0) \to K_1(A\otimes B)) + \operatorname{Im}(K_1(D_1) \to K_1(A\otimes B))$
  $= K_1(A\otimes B)$

$\Rightarrow$ $a\otimes 1 \in \overline{GL(A\otimes E)}$

$\therefore$ $sr(A) = 1$

Blackadar's fundamental comparison property

The Cuntz semigp $W(A)$

A C^*-alg. $M_\infty(A)^+ = \bigcup_{n=1}^{\infty} M_n(A)^+$

$a \in M_n(A)^+$, $b \in M_m(A)^+$

$a \precsim b \overset{\text{DEF}}{\iff} \exists\, x_k \in M_{m,n}(A) : x_k^* b x_k \to a$

$a \sim b \overset{\text{DEF}}{\iff} a \precsim b \wedge b \precsim a$

$a \oplus b \overset{\text{DEF}}{=} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+m}(A)^+$

eq. class containing a

$W(A) \overset{\text{DEF}}{=} M_\infty(A)^+/\sim \; = \{ \langle a \rangle \mid a \in M_\infty(A)^+ \}$

$\langle a \rangle \leq \langle b \rangle \overset{\text{DEF}}{\iff} a \precsim b$

$\langle a \rangle + \langle b \rangle \overset{\text{DEF}}{=} \langle a \oplus b \rangle$

$\Rightarrow$ $(W(A), +, \leq)$ ordered abelian semigroup

<u>DEF</u> $(W, +, \leq)$ is <u>almost unperforated</u> if

$\forall x, y \in W \ \forall n, m \in \mathbb{N} : nx \leq my \wedge n > m$

$\Rightarrow x \leq y$

<u>DEF</u> $S(W,x)$ = set of states on W normalized at x. I.e., $f \in S(W,x)$ iff $f: W \to [0,\infty]$ is order preserving and additive, and $f(x) = 1$.

<u>DEF</u> $x \propto y \overset{\text{DEF}}{\iff} \exists n \in \mathbb{N} : x \leq ny$

<u>PROP</u> $(W, +, \leq)$ almost unperforated iff

$\forall x,y \in W$: $x \propto y \ \wedge \ f(x) < f(y) \ \forall f \in S(W,y)$

$\Rightarrow x \leq y$

PROP (G, G^+) ordered abelian group. Then G^+ almost unperforated iff

$\forall g \in G \ \forall n \in \mathbb{N}$: $ng, (n+1)g \in G^+ \Rightarrow g \in G^+$.

<u>THM</u> If A is $\mathcal{Z}$-absorbing, then $W(A)$ is almost unperforated.

Enough to show: $a, b \in A^+$, $n \in \mathbb{N}$, then

$(n+1)\langle a \rangle \leq n \langle b \rangle$ in $W(A)$

$\Rightarrow \langle a \otimes 1 \rangle \leq \langle b \otimes 1 \rangle$ in $W(A \otimes \mathcal{Z})$

<u>Lemma</u> $\exists e_n \in \mathcal{Z}^+ : n \langle e_n \rangle \leq \langle 1 \rangle \leq (n+1) \langle e_n \rangle$

$\langle a \otimes 1 \rangle \leq (n+1) \langle a \otimes e_n \rangle \leq n \langle b \otimes e_n \rangle \leq \langle b \otimes 1 \rangle$.