Outer Actions of a Group on a Factor

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First, I will discuss the characteristic square of a factor $\mathcal{M}$:

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[d] & \mathcal{U}(\mathcal{C}) \arrow[r, "\partial_\theta"] \arrow[d] & \mathrm{B}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{U}(\mathcal{M}) \arrow[r] \arrow[d, "\mathrm{Ad}"'] & \widetilde{\mathcal{U}}(\mathcal{M}) \arrow[r, "\partial_\theta"] \arrow[d, "\widetilde{\mathrm{Ad}}"'] & \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Int}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Cnt_r}(\mathcal{M}) \arrow[r, "\dot{\partial}_\theta"] \arrow[d] & \mathrm{H}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
$$

where $\{\mathcal{C}, \mathbb{R}, \theta\}$ is the flow of weights on $\mathcal{M}$; $\widetilde{\mathcal{U}}(\mathcal{M})$ is the extended unitary group of $\mathcal{M}$, i.e., the normalizer of $\mathcal{M}$ in the unitary group $\mathcal{U}(\widetilde{\mathcal{M}})$ of the core $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. The core $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is the von Neumann algebra generated by the imaginary power $\{\varphi^{it} : t \in \mathbb{R}, \varphi \in \mathfrak{W}_0(\mathcal{M})\}$ of faithful semi-finite normal weights on $\mathcal{M}$. Scaling $\varphi \mapsto e^{-s}\varphi, s \in \mathbb{R}$, gives rise to the one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ of $\widetilde{\mathcal{M}}$ such that

$$\mathcal{M} = \widetilde{\mathcal{M}}^\theta \quad \text{and} \quad \mathcal{M}' \cap \widetilde{\mathcal{M}} = \mathcal{C}.$$

The normalizer $\widetilde{\mathcal{U}}(\mathcal{M})$ of $\mathcal{M}$ in the unitary group $\mathcal{U}(\widetilde{\mathcal{M}})$ of $\widetilde{\mathcal{M}}$ gives the extended modular automorphism group $\mathrm{Cnt_r}(\mathcal{M})$ as every $u \in \widetilde{\mathcal{U}}(\mathcal{M})$ gives an automorphism $\widetilde{\mathrm{Ad}}(u)(x) = uxu^*, x \in \mathcal{M}$.

Looking at the middle vertical exact sequence:

$$1 \longrightarrow \mathcal{U}(\mathcal{C}) \longrightarrow \widetilde{\mathcal{U}}(\mathcal{M}) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{Cnt_r}(\mathcal{M}) \longrightarrow 1$$

choose a cross-section: $\alpha \in \mathrm{Cnt_r}(\mathcal{M}) \mapsto u(\alpha) \in \widetilde{\mathcal{U}}(\mathcal{M})$ such that $\alpha = \widetilde{\mathrm{Ad}}(u)(\alpha))$. Then we have:

$$\mu(\alpha,\beta) = u(\alpha)u(\beta)u(\alpha\beta)^* \in \mathcal{U}(\mathcal{C});$$
$$\lambda(\alpha,\gamma) = \gamma(u(\gamma^{-1}\alpha\gamma))u(\alpha)^* \in \mathcal{U}(\mathcal{C}), \quad \alpha,\beta \in \mathrm{Cnt_r}(\mathcal{M}), \gamma \in \mathrm{Aut}(\mathcal{M}).$$

The pair (λ,μ) is a characteristic cocycle of V.F.R. Jones and gives rise to the characteristic invariant $\Theta(\mathcal{M})$ in the relative cohomology group $\Lambda(\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}, \mathrm{Cnt_r}(\mathcal{M}), \mathcal{U}(\mathcal{C}))$, which was named the intrinsic invariant of $\mathcal{M}$ in [4].

If $\alpha : g \in G \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ is an action of a group G, then the pull-back $\chi(\alpha) = \alpha^*(\Theta(\mathcal{M})) \in \Lambda_{\mathrm{mod}(\alpha)\times\theta}(G, N, \mathcal{U}(\mathcal{C}))$ with $N = \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$ is a cocycle conjugacy invariant. In the case that $\mathcal{M}$ is an approximately finite dimensional factor and G is a countable discrete amenable group, then the triplet $\{\mathrm{mod}(\alpha), \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M})), \chi(\alpha)\}$ form a complete invariant of the cocycle conjugacy class of α.

To move on one step further to outer actions, we first make the definition.

Definition 1. *A map* $\alpha\colon g \in G \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ *is called an* outer action *if*

$$\alpha_g \circ \alpha_h \equiv \alpha_{gh} \mod \mathrm{Int}(\mathcal{M}), \quad g,h \in G.$$

We usually assume that $\alpha_e = \mathrm{id}$ *for the identity* $e \in G$*. If*

$$\alpha_g \notin \mathrm{Int}(\mathcal{M}), \quad g \neq e,$$

then it is called a free *outer action.*

Remark 2. One should not confuse this with the concept of *free actions*.

Consider the quotient group $\mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$ and fix a cross-section: $g \in \mathrm{Out}(\mathcal{M}) \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ of the quotient map $\pi : \alpha \in \mathrm{Aut}(\mathcal{M}) \mapsto [\alpha] \in \mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$ and also choose a Borel cross-section $\alpha \in \mathrm{Cnt_r}(\mathcal{M}) \mapsto u(\alpha) \in \widetilde{\mathcal{U}}(\mathcal{M})$ in such a way that $u(\alpha) \in \mathcal{U}(\mathcal{M})$ for every $\alpha \in \mathrm{Int}(\mathcal{M})$. Then we have for $g,h,k \in \mathrm{Out}(\mathcal{M})$

$$u(g,h = u(\alpha_g \circ \alpha_h \circ \alpha^{-1}gh) \in \mathcal{U}(\mathcal{M}), ;$$
$$c(g,h,k) = \alpha_g(u(h,k))u(g,hk)\{u(g,h)u(gh,k)\}^* \in \mathbb{T}.$$

The three variable function c is indeed a cocycle $c \in \mathrm{Z}^3(\mathrm{Out}(\mathcal{M}),\mathbb{T})$. The cohomology class $[c] \in \mathrm{H}^3(\mathrm{Out}(\mathcal{M}),\mathbb{T})$ is called the *intrinsic obstruction* and denoted by $\mathrm{Ob}(\mathcal{M})$. If α is an outer action of G on $\mathcal{M}$, then the pull back $\mathrm{Ob}(\alpha) = \alpha^*(\mathrm{Ob}(\mathcal{M}))$ is an invariant of the outer conjugacy class of α. If $\mathcal{M}$ is a factor of type II_1, then one can work directly on the obstruction, employing the Brower group trick. But in the case of type III, this direct method does not work. For example, the group $\mathrm{Cnt_r}(\mathcal{M})$ is not stable under the tensor product, while $\mathrm{Int}(\mathcal{M})$ is stable. To deal with this problem, we will do the following:

To each factor $\mathcal{M}$, we associate an invariant $\mathrm{Ob_m}(\mathcal{M})$ to be called the *intrinsic modular obstruction* as a cohomological invariant which lives in the "third" cohomology group:

$$\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(\mathrm{Out}(\mathcal{M}) \times \mathbb{R}, \mathrm{H}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})), \mathcal{U}(\mathcal{C}))$$

where $\{\mathcal{C}, \mathbb{R}, \theta\}$ is the flow of weights on $\mathcal{M}$. If α is an outer action of a countable discrete group G on $\mathcal{M}$, then the triple consisting of its modulus $\mathrm{mod}(\alpha) \in \mathrm{Hom}(G, \mathrm{Aut}_\theta(\mathcal{C}))$ together with $N = \alpha^{-1}(\mathrm{Cnt_r}(M))$ and the pull back

$$\mathrm{Ob_m}(\alpha) = \alpha^*(\mathrm{Ob_m}(\mathcal{M})) \in \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C})))$$

is called the *modular obstruction* of α and is an invariant of the outer conjugacy class of the outer action α.

We have proved that if the factor $\mathcal{M}$ is approximately finite dimensional and G is amenable, then this invariant uniquely determines the outer conjugacy class of α, and then every value of the triple occurs as the invariant of an outer action α of G on $\mathcal{M}$. In the case that M is a factor of type III_λ, $0 < \lambda \leq 1$, the modular obstruction group $\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C}))$ and the modular obstruction $\mathrm{Ob_m}(\alpha)$ take simpler forms. But this does not mean that our work is easier. The difficulties in this case can be seen in the fact that $\mathrm{Aut}(\mathcal{M})$ does not act on the discrete core, a fact that is overlooked sometimes.

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