
Outer Actions of a Group on a Factor

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First, I will discuss the characteristic square of a factor \mathcal{M} :

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathcal{U}(\mathcal{C}) & \xrightarrow{\partial_\theta} & B_\theta^1(\mathbb{R}, \mathcal{U}(\mathcal{C})) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathcal{U}(\mathcal{M}) & \longrightarrow & \tilde{\mathcal{U}}(\mathcal{M}) & \xrightarrow{\partial_\theta} & Z_\theta^1(\mathbb{R}, \mathcal{U}(\mathcal{C})) \longrightarrow 1 \\
 & & \text{Ad} \downarrow & & \tilde{\text{Ad}} \downarrow & & \downarrow \\
 1 & \longrightarrow & \text{Int}(\mathcal{M}) & \longrightarrow & \text{Cnt}_r(\mathcal{M}) & \xrightarrow{\dot{\partial}_\theta} & H_\theta^1(\mathbb{R}, \mathcal{U}(\mathcal{C})) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

where $\{\mathcal{C}, \mathbb{R}, \theta\}$ is the flow of weights on \mathcal{M} ; $\tilde{\mathcal{U}}(\mathcal{M})$ is the extended unitary group of \mathcal{M} , i.e., the normalizer of \mathcal{M} in the unitary group $\mathcal{U}(\tilde{\mathcal{M}})$ of the core $\tilde{\mathcal{M}}$ of \mathcal{M} . The core $\tilde{\mathcal{M}}$ of \mathcal{M} is the von Neumann algebra generated by the imaginary power $\{\varphi^{it} : t \in \mathbb{R}, \varphi \in \mathfrak{W}_0(\mathcal{M})\}$ of faithful semi-finite normal weights on \mathcal{M} . Scaling $\varphi \mapsto e^{-s}\varphi, s \in \mathbb{R}$, gives rise to the one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ of $\tilde{\mathcal{M}}$ such that

$$\mathcal{M} = \tilde{\mathcal{M}}^\theta \quad \text{and} \quad \mathcal{M}' \cap \tilde{\mathcal{M}} = \mathcal{C}.$$

The normalizer $\tilde{\mathcal{U}}(\mathcal{M})$ of \mathcal{M} in the unitary group $\mathcal{U}(\tilde{\mathcal{M}})$ of $\tilde{\mathcal{M}}$ gives the extended modular automorphism group $\text{Cnt}_r(\mathcal{M})$ as every $u \in \tilde{\mathcal{U}}(\mathcal{M})$ gives an automorphism $\tilde{\text{Ad}}(u)(x) = uxu^*, x \in \mathcal{M}$.

Looking at the middle vertical exact sequence:

$$1 \longrightarrow \mathcal{U}(\mathcal{C}) \longrightarrow \tilde{\mathcal{U}}(\mathcal{M}) \xrightarrow{\widetilde{\text{Ad}}} \text{Cnt}_r(\mathcal{M}) \longrightarrow 1$$

choose a cross-section: $\alpha \in \text{Cnt}_r(\mathcal{M}) \mapsto u(\alpha) \in \tilde{\mathcal{U}}(\mathcal{M})$ such that $\alpha = \widetilde{\text{Ad}}(u(\alpha))$. Then we have:

$$\begin{aligned} \mu(\alpha, \beta) &= u(\alpha)u(\beta)u(\alpha\beta)^* \in \mathcal{U}(\mathcal{C}); \\ \lambda(\alpha, \gamma) &= \gamma(u(\gamma^{-1}\alpha\gamma))u(\alpha)^* \in \mathcal{U}(\mathcal{C}), \quad \alpha, \beta \in \text{Cnt}_r(\mathcal{M}), \gamma \in \text{Aut}(\mathcal{M}). \end{aligned}$$

The pair (λ, μ) is a characteristic cocycle of V.F.R. Jones and gives rise to the characteristic invariant $\Theta(\mathcal{M})$ in the relative cohomology group $A(\text{Aut}(\mathcal{M}) \times \mathbb{R}, \text{Cnt}_r(\mathcal{M}), \mathcal{U}(\mathcal{C}))$, which was named the intrinsic invariant of \mathcal{M} in [4].

If $\alpha : g \in G \mapsto \alpha_g \in \text{Aut}(\mathcal{M})$ is an action of a group G , then the pull-back $\chi(\alpha) = \alpha^*(\Theta(\mathcal{M})) \in A_{\text{mod}(\alpha) \times \theta}(G, N, \mathcal{U}(\mathcal{C}))$ with $N = \alpha^{-1}(\text{Cnt}_r(\mathcal{M}))$ is a cocycle conjugacy invariant. In the case that \mathcal{M} is an approximately finite dimensional factor and G is a countable discrete amenable group, then the triplet $\{\text{mod}(\alpha), \alpha^{-1}(\text{Cnt}_r(\mathcal{M})), \chi(\alpha)\}$ form a complete invariant of the cocycle conjugacy class of α .

To move on one step further to outer actions, we first make the definition.

Definition 1. A map $\alpha : g \in G \mapsto \alpha_g \in \text{Aut}(\mathcal{M})$ is called an outer action if

$$\alpha_g \circ \alpha_h \equiv \alpha_{gh} \pmod{\text{Int}(\mathcal{M})}, \quad g, h \in G.$$

We usually assume that $\alpha_e = \text{id}$ for the identity $e \in G$. If

$$\alpha_g \notin \text{Int}(\mathcal{M}), \quad g \neq e,$$

then it is called a free outer action.

Remark 2. One should not confuse this with the concept of free actions.

Consider the quotient group $\text{Out}(\mathcal{M}) = \text{Aut}(\mathcal{M})/\text{Int}(\mathcal{M})$ and fix a cross-section: $g \in \text{Out}(\mathcal{M}) \mapsto \alpha_g \in \text{Aut}(\mathcal{M})$ of the quotient map $\pi : \alpha \in \text{Aut}(\mathcal{M}) \mapsto [\alpha] \in \text{Out}(\mathcal{M}) = \text{Aut}(\mathcal{M})/\text{Int}(\mathcal{M})$ and also choose a Borel cross-section $\alpha \in \text{Cnt}_r(\mathcal{M}) \mapsto u(\alpha) \in \tilde{\mathcal{U}}(\mathcal{M})$ in such a way that $u(\alpha) \in \mathcal{U}(\mathcal{M})$ for every $\alpha \in \text{Int}(\mathcal{M})$. Then we have for $g, h, k \in \text{Out}(\mathcal{M})$

$$\begin{aligned} u(g, h) &= u(\alpha_g \circ \alpha_h \circ \alpha^{-1}gh) \in \mathcal{U}(\mathcal{M}), ; \\ c(g, h, k) &= \alpha_g(u(h, k))u(g, hk)\{u(g, h)u(gh, k)\}^* \in \mathbb{T}. \end{aligned}$$

The three variable function c is indeed a cocycle $c \in Z^3(\text{Out}(\mathcal{M}), \mathbb{T})$. The cohomology class $[c] \in H^3(\text{Out}(\mathcal{M}), \mathbb{T})$ is called the *intrinsic obstruction* and denoted by $\text{Ob}(\mathcal{M})$. If α is an outer action of G on \mathcal{M} , then the pull back $\text{Ob}(\alpha) = \alpha^*(\text{Ob}(\mathcal{M}))$ is an invariant of the outer conjugacy class of α . If \mathcal{M} is a factor of type II_1 , then one can work directly on the obstruction, employing

the Brower group trick. But in the case of type III, this direct method does not work. For example, the group $\text{Cnt}_r(\mathcal{M})$ is not stable under the tensor product, while $\text{Int}(\mathcal{M})$ is stable. To deal with this problem, we will do the following:

To each factor \mathcal{M} , we associate an invariant $\text{Ob}_m(\mathcal{M})$ to be called the *intrinsic modular obstruction* as a cohomological invariant which lives in the “third” cohomology group:

$$H_{\alpha,s}^{\text{out}}(\text{Out}(\mathcal{M}) \times \mathbb{R}, H_{\theta}^1(\mathbb{R}, \mathcal{U}(\mathcal{C})), \mathcal{U}(\mathcal{C}))$$

where $\{\mathcal{C}, \mathbb{R}, \theta\}$ is the flow of weights on \mathcal{M} . If α is an outer action of a countable discrete group G on \mathcal{M} , then the triple consisting of its modulus $\text{mod}(\alpha) \in \text{Hom}(G, \text{Aut}_{\theta}(\mathcal{C}))$ together with $N = \alpha^{-1}(\text{Cnt}_r(\mathcal{M}))$ and the pull back

$$\text{Ob}_m(\alpha) = \alpha^*(\text{Ob}_m(\mathcal{M})) \in H_{\alpha,s}^{\text{out}}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C}))$$

is called the *modular obstruction* of α and is an invariant of the outer conjugacy class of the outer action α .

We have proved that if the factor \mathcal{M} is approximately finite dimensional and G is amenable, then this invariant uniquely determines the outer conjugacy class of α , and then every value of the triple occurs as the invariant of an outer action α of G on \mathcal{M} . In the case that \mathcal{M} is a factor of type III_{λ} , $0 < \lambda \leq 1$, the modular obstruction group $H_{\alpha,s}^{\text{out}}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C}))$ and the modular obstruction $\text{Ob}_m(\alpha)$ take simpler forms. But this does not mean that our work is easier. The difficulties in this case can be seen in the fact that $\text{Aut}(\mathcal{M})$ does not act on the discrete core, a fact that is overlooked sometimes.

References

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