Central sequences in C^*–algebras and strongly purely infinite algebras

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Summary. It is shown that $F(A) := (A' \cap A_\omega)/\mathrm{Ann}(A, A_\omega)$ is a unital C^*-algebra and that $A \mapsto F(A)$ is a stable invariant of separable C^*–algebras A with certain local continuity and permanence properties. Here A_ω means the ultrapower of A.

If A is separable, then $F(A)$ is simple, if and only if, either $A \otimes \mathcal{K} \cong \mathcal{K}$ or A is a simple purely infinite *nuclear* C^*–algebra. In the first case $F(A) \cong \mathbb{C}$, and in the second case $F(A)$ is purely infinite and A absorbs $\mathcal{O}_\infty$ tensorially, i.e. $A \cong A \otimes \mathcal{O}_\infty$.

We show that $F(Q) = \mathbb{C} \cdot 1$ for the Calkin algebra $Q := \mathcal{L}/\mathcal{K}$, in contrast to the separable case.

We introduce a "locally semi-projective" invariant $\mathrm{cov}(B) \in \mathbb{N} \cup \{\infty\}$ of unital C^*–algebras B with $\mathrm{cov}(B) \leq \mathrm{cov}(C)$ if there is a unital *-homomorphism from C into B. If B is nuclear and has no finite-dimensional quotient then $\mathrm{cov}(B) \leq \mathrm{dr}(B)+1$ for the decomposition rank $\mathrm{dr}(B)$ of B. (Thus, $\mathrm{cov}(\mathcal{Z}) = 2$ for the Jian–Su algebra $\mathcal{Z}$.) Separable (not necessarily simple) C^*–algebras A are strongly purely infinite in the sense of [25] if A does not admit a non-trivial lower semi-continuous 2-quasi-trace and $F(A)$ contains a simple C^*–subalgebra B with $\mathrm{cov}(B) < \infty$ and $1 \in B$. In particular, $A \otimes \mathcal{Z}$ is strongly purely infinite if A_+ admits no non-trivial lower semi-continuous 2-quasi-trace.

Properties of $F(A)$ will be used to show that A is tensorially $\mathcal{D}$–absorbing, (i.e. that $A \otimes \mathcal{D} \cong A$ by an isomorphism that is approximately unitarily equivalent to $a \mapsto a \otimes 1$), if A is stable and separable, $\mathcal{D}$ is a unital tensorially self-absorbing algebra, and $\mathcal{D}$ is unitally contained in $F(A)$. It follows that the class of tensorially $\mathcal{D}$–absorbing separable stable C^*–algebras A, is closed under inductive limits and passage to ideals and quotients. The local permanence properties of the functor $A \mapsto F(A)$ imply that this class is also closed under extensions, if and only if, every commutator uvu^*v^* of unitaries $u, v \in \mathcal{U}(\mathcal{D})$ is contained in the connected component $\mathcal{U}_0(\mathcal{D})$ of 1 in $\mathcal{U}(\mathcal{D})$. If this is the case, then the class of (not necessarily stable) $\mathcal{D}$–absorbing separable C^*–algebras is also closed under passage to hereditary C^*–algebras.

1 Introduction: The stable invariant $F(A)$.

The different results of this paper (stated in the summary) will be derived from properties of the relative commutant A^c of a C^*–algebra A in its ultrapower. Our considerations suggest that a study of the ideals and simple C^*–subalgebras of the below defined quotient algebra $F(A)$ of A^c could be useful. There are related open problems: the UCT problem, the classification of (strongly tensorially) self-absorbing C^*–algebras $\mathcal{D}$, permanence properties for *all* $\mathcal{D}$-absorbing algebras, the question which additional properties imply that purely infinite algebras are strongly purely infinite, and to the existence of certain asymptotic algebras suitable for a KK-theoretic formulation of the classification of $\mathcal{D}$-absorbing algebras.

Note that our technics is *not* a sort of non-standard analysis: All appearing algebras are honest C^*–algebras over $\mathbb{C}$ and all considered maps between them are at least completely positive maps. We consider here only $A \mapsto F(A)$ for a fixed free ultra-filter ω on $\mathbb{N}$, because we hope that it is helpful for the reader to get an impression of what we consider as asymptotic analysis of C^*–algebras if $\mathbb{N}$ is replaced e.g. by $\mathbb{R}_+$. There are surprising relations between algebraic properties of $F(A)$ and analytic properties of separable A. See e.g. Lemmas 2.8, 2.11(3), Propositions 1.17, 4.11, Corollary 1.13 (in view of applications), and Theorems 2.12, 3.10, 4.5.

Let ω a *free* ultra-filter on $\mathbb{N}$. We also denote by ω the related *character* on $\ell_\infty := \ell_\infty(\mathbb{N})$ with $\omega(c_0(\mathbb{N})) = \{0\}$. Recall that $\lim_\omega \alpha_n$ means the complex number $\omega(\alpha_1, \alpha_2, \ldots)$ for $(\alpha_1, \alpha_2, \ldots) \in \ell_\infty$. For a C^*–algebra A, we let

$$\begin{aligned} c_\omega(A) &:= \{(a_1, a_2, \ldots) \in \ell_\infty(A)\,;\ \lim_\omega \|a_n\| = 0\}\,, \\ A_\omega &:= \ell_\infty(A)/c_\omega(A) \end{aligned}$$

A_ω will be called the *ultrapower* of A. The natural epimorphism from $\ell_\infty(A)$ onto A_ω will be denoted by π_ω. $(a_1, a_2, \ldots) \in \ell_\infty(A)$ is a *representing sequence* for $b \in A_\omega$ if $\pi_\omega(a_1, a_2, \ldots) = b$. We consider A as a C^*–subalgebra of A_ω by the diagonal embedding

$$a \mapsto \pi_\omega(a, a, \ldots) = (a, a, \ldots) + c_\omega(A).$$

Then $A^c := A' \cap A_\omega$ is the *algebra of (ω-) central sequences in A* (modulo ω-zero sequences). It is easy to see that the (two-sided) *annihilator*

$$\mathrm{Ann}(A) := \mathrm{Ann}(A, A_\omega) := \{b \in A_\omega\,;\ bA = \{0\} = Ab\}$$

of A in A_ω is an ideal of A^c. We let

$$F(A) := A^c/\mathrm{Ann}(A) = (A' \cap A_\omega)/\mathrm{Ann}(A, A_\omega)$$

It turns out that $F(A)$ is unital for σ-unital A, and that $A \mapsto F(A)$ is an invariant of Morita equivalence classes of σ-unital C^*–algebras. We generalize A^c and $F(A)$ for C^*–subalgebras $A \subset \mathcal{M}(B)_\omega$ to get more flexible tools for the proofs of permanence properties:

Definitions 1.1 *Suppose that B is a C^*–algebra, $\mathcal{M}(B)$ its multiplier algebra, and that A is a C^*–subalgebra of $\mathcal{M}(B)_\omega$. We let, for $A \subset \mathcal{M}(B)_\omega$,*

$$
\begin{aligned}
(A,B)^c &:= A' \cap B_\omega \subset A' \cap \mathcal{M}(B)_\omega\,,\\
\mathrm{Ann}(A,B_\omega) &:= \{b \in B_\omega\,;\ Ab + bA = \{0\}\}\\
F(A,B) &:= (A,B)^c/\mathrm{Ann}(A,B_\omega)\,,\\
D_{A,B} &:= \overline{\mathrm{span}(AB_\omega A)} \subset B_\omega \quad \textit{and}\\
\mathcal{N}(D_{A,B}) = \mathcal{N}(D_{A,B},B_\omega) &:= \{b \in B_\omega\,;\ bD_{A,B} + D_{A,B}b \subset D_{A,B}\}\,.
\end{aligned}
$$

We denote by $\rho_{A,B}$ *the natural *-morphism*

$$\rho_{A,B}\colon\ F(A,B) \otimes^{\max} A \to D_{A,B} \subset B_\omega$$

given by $\rho_{A,B}((b + \mathrm{Ann}(A,B_\omega)) \otimes a) := ba$ *for* $b \in (A,B)^c$ *and* $a \in A$.

The Definitions of $F(A,B)$ and of $\rho_{A,B}$ make sense, because (obviously) $\mathrm{Ann}(A,B_\omega)$ is a closed ideal of $(A,B)^c$, $(A,B)^c$ and A commute element-wise and $A \cdot \mathrm{Ann}(A,B_\omega) = \{0\}$.

Then $F(A) = F(A,A)$, $(A,B)^c = (A,\mathcal{M}(B))^c \cap B_\omega$ is a closed ideal of $(A,\mathcal{M}(B))^c$ and $\mathrm{Ann}(A,B_\omega) = \mathrm{Ann}(D_{A,B},B_\omega) = \mathrm{Ann}(A,\mathcal{M}(B)_\omega) \cap B_\omega$. We write $\mathcal{N}(D_{A,B})$ for $\mathcal{N}(D_{A,B},B_\omega)$, D_A, for $D_{A,A}$, $\mathcal{N}(D_A)$ for $\mathcal{N}(D_{A,A})$, ρ_A or ρ for $\rho_{A,A}$, ... and so on.

Let $\mathcal{K}$ denote the compact operators on $\ell_2(\mathbb{N})$. $\mathcal{K}^c = \mathrm{Ann}(\mathcal{K}) + \mathbb{C}\cdot 1$ is huge, but $F(\mathcal{K}) \cong \mathbb{C} = \mathbb{C}_\omega$ (cf. Corollary 1.10). Permanence properties of $F(A)$ have to be considered with some care, because e.g. $F(\mathcal{K} + \mathbb{C}\cdot 1) \cong (\mathcal{K} + \mathbb{C}\cdot 1)^c = \mathrm{Ann}(\mathcal{K}) + \mathbb{C}\cdot 1$.

The below given basic facts on $(A,B)^c$, $\mathrm{Ann}(A,B_\omega)$ and $F(A,B)$ will be proved in Appendix B or are taken from [22, sec. 2.2].

Definition 1.2 *A convex subcone* $\mathcal{V} \subset CP(B,C)$ *of the cone of completely positive (=:c.p.) maps from* B *in* C *is (matricially)* operator-convex *if the c.p. map* $b \mapsto c^*V(r^*br)c$ *is in* $\mathcal{V}$ *for every* $V \in \mathcal{V}$ *and every row* $r \in M_{1,n}(B)$ *and column* $c \in M_{n,1}(C)$.

Examples of operator-convex cones are the cone $CP_{nuc}(B,C)$ of nuclear c.p. maps from B into C and the cone $CP_{fin}(B,C)$ of the c.p. maps of finite rank. If $B \subset \mathcal{M}(C)$ then the cone of approximately inner c.p. maps $V \to B \to C$ is operator-convex.

Proposition 1.3 *Suppose that* $A \subset B_\omega$ *is separable, that* $\mathcal{V} \subset CP(B,B)$ *is an operator-convex cone of completely positive maps from* B *into* B*, that* $J \subset B$ *is a closed ideal, and that* $a \in A' \cap B_\omega$*,* $b, c \in B_\omega$ *are positive contractions with* $ab = ac = bc = 0$ *and* $bAc = \{0\}$.
If $c \in J_\omega \subset B_\omega$*, and if there is a bounded sequence* $S_1, S_2, \ldots \in \mathcal{V}$ *such that* $S_\omega(x) = b^*xb$ *for* $x \in A$*, then there are positive contractions* $e, f, g \in A' \cap B_\omega$ *and a sequence of contractions* $T_1, T_2, \ldots \in \mathcal{V}$ *with*

(1) $ea = a$, $fb = b$, $gc = c$ *and* $ef = eg = fg = 0$
(2) $T_\omega(x) = xf$ *for all* $x \in A$.
(3) $g \in J_\omega$

We use only particular aspects of this Proposition, e.g., where at least one of the a, b, c is zero. Note that the assumption on $(\mathcal{V}, b)$ is trivially satisfied for $\mathcal{V} = CP(B, B)$ or for the operator-convex cone $\mathcal{V}$ of all inner c.p. maps (and the conclusion (2) is then trivial, too). The same happens with the assumptions on (J, c) if we let $J = B$.

Part (2) shows that (ultapowers of) *operator convex cones* $\mathcal{V} \subset CP(B, B)$ *define in a natural way closed ideals of* $F(A, B)$ (compare the proof of Lemma 2.11).

Definition 1.4 *We call a* C^**–algebra* C σ-sub-Stonean *if for every separable* C^**–subalgebra* $A \subset C$ *and every* $b, c \in C_+$ *with* $bc = 0$ *and* $bAc = \{0\}$ *there are positive contractions* $f, g \in A' \cap C$ *with* $fg = 0$*,* $fb = b$ *and* $gc = c$*.*

Obviously, if C is σ-sub-Stonean, then C is sub-Stonean (which is the case $A = \{0\}$), and $B' \cap C$ is σ-sub-Stonean for every separable C^*–subalgebra B of C (consider $C^*(B, A)$ in place of A in the definition). It is easy to see, that if D is a hereditary C^*–subalgebra of C, then D is σ-sub-Stonean if and only if for every $a \in D_+$ there is a positive contraction $e \in D$ with $ea = e$. In particular, $\mathrm{Ann}(d, C)$ is σ-sub-Stonean for every $d \in C_+$ if C is σ-sub-Stonean. Further, if C is σ-sub-Stonean and $I \lhd C$ is a σ-sub-Stonean closed ideal of C, then C/I is σ-sub-Stonean. (An exercise.)

Definitions 1.5 *We call a closed ideal* I *of a* C^**–algebra* C *a* σ-ideal of C *if for every separable* C^**–subalgebra* $A \subset C$ *and every* $d \in I_+$ *there is a positive contraction* $e \in A' \cap I$ *with* $ed = d$*.*

We say that a short exact sequence of C^**–algebras* $0 \to B \to C \to D \to 0$ *(with epimorphism* $\pi \colon C \to D$*) is* strongly locally semi-split *if for every separable* C^**–subalgebra* $A \subset D$ *there is a *-morphism* ψ *from* $C_0((0,1], A) \cong C_0(0,1] \otimes A$ *into* C *such that* $\pi \circ \psi(f_0 \otimes a) = a$*, where* $f_0(t) = t$ *for* $t \in (0,1]$*.*

Note that $A' \cap I$ is σ-sub-Stonean if C is σ-sub-Stonean, $I \lhd C$ is a σ-ideal and A is a separable.

One can see, that $A' \cap I$ is a σ-ideal of $A' \cap C$ and is a non-degenerate C^*-subalgebra of I.

It is easy to see, that the image $\varphi(I)$ is σ-ideal of $\varphi(C)$ for every morphism $\varphi \colon C \to E$. Furthermore, if $I \subset C \subset E$ and I is a closed σ-ideal of E, then I is also a σ-ideal of C. Clearly, the intersection and sum of two σ-ideals is a σ-ideal.

An elementary consequence of the definitions is given by:

Proposition 1.6 *If* I *is a* σ*-ideal of a* C^**–algebra* C*, then, for every separable* C^**–subalgebra* $A \subset C$*,* $A' \cap I$ *is a non-degenerate* C^**–subalgebra of* I*,* $\pi_I(\mathrm{Ann}(A, I)) = \mathrm{Ann}(\pi_I(A), C/I)$ *and the sequence*

$$0 \to A' \cap I \to A' \cap C \to \pi_I(A)' \cap (C/I) \to 0$$

is exact and strongly locally semi-split.

The epimorphism $A' \cap C \to \pi_I(A)' \cap (C/I)$ is the restriction of the natural epimorphism π_I from C onto C/I.

Proposition 1.3 and the above discussed permanence properties for σ-sub-Stonean algebras and σ-ideals imply:

Corollary 1.7 *Suppose that J is a closed ideal of B and that A is a separable C^*–subalgebra of B_ω.*

Then B_ω, $(A,B)^c$, $\mathrm{Ann}(A,B_\omega)$, and $F(A,B)$ are σ-sub-Stonean.

J_ω, $J_\omega \cap (A,B)^c$ and $\mathrm{Ann}(A,B_\omega)$ are σ-ideals of B_ω respectively of (A,B).

In particular, B_ω, $\mathrm{Ann}(A,B_\omega)$, $\mathrm{Ann}(A)$, $(A,B)^c$, A^c, $F(A,B)$, $F(A)$, $J_\omega \cap (A,B)^c$ and $J_\omega \cap \mathrm{Ann}(A,B_\omega)$ are sub–Stonean.

The permanence properties for σ-ideals imply e.g. that $J_\omega \cap \mathrm{Ann}(A,B_\omega)$ is a σ-ideal in $(A,B)^c$ and $\mathrm{Ann}(A,B_\omega)$.

By Proposition 1.6 the statement that $\mathrm{Ann}(A,B_\omega)$ is a σ-ideal of $(A,B)^c$ implies:

Corollary 1.8 *Suppose that A is a separable C^*–subalgebra of B_ω, and that C is a separable C^*–subalgebra of $F(A,B)$. There is a *-morphism*

$$\lambda\colon\ C_0((0,1],C) \to (A,B)^c = A' \cap B_\omega$$

with $\lambda(f) + \mathrm{Ann}(A,B_\omega) = f(1) \in C \subset F(A,B)$ for $f \in C_0((0,1],C)$.

The following proposition gives some elementary properties of $(A,B)^c$, $\mathrm{Ann}(A,B)$ and $F(A,B)$.

Proposition 1.9 *Suppose that A is a σ-unital C^*-subalgebra of B_ω, and let $D_{A,B}$, $\mathcal{N}(D_{A,B})$, $\mathrm{Ann}(A,B_\omega)$, $(A,B)^c$, $F(A,B)$ and $\rho_{A,B}$ be as in Definitions 1.1. Then*

(1) $\mathrm{Ann}(A,B_\omega)$ *is an ideal of $\mathcal{N}(D_{A,B})$, and*

$$\mathrm{Ann}(A,B_\omega) \subset (A,B)^c \subset \mathcal{N}(D_{A,B})\,.$$

(2) For every countable subset $Y \subset B_\omega$ there exists a positive contraction $e \in (A,B)^c$ with $ey = ye = y$ for all $y \in Y$.

(3) $F(A,B)$ is unital. Moreover, if $a_0 \in A_+$ a strictly positive element of A and $e \in B_\omega$ is a positive contraction, then e satisfies $ea_0 = a_0 = a_0e$, if and only if, $e \in (A,B)^c$ and $e + \mathrm{Ann}(A,B_\omega) = 1$ *in $F(A,B)$.*

*(4) The natural *-morphism $\mathcal{N}(D_{A,B}) \to \mathcal{M}(D_{A,B})$ is an* epimorphism *onto $\mathcal{M}(D_{A,B})$ with kernel $= \mathrm{Ann}(A,B_\omega)$.*

*(5) The epimorphism from $\mathcal{N}(D_{A,B})$ onto $\mathcal{M}(D_{A,B})$ defines a *-isomorphism η from $F(A,B)$ onto $A' \cap \mathcal{M}(D_{A,B})$ with $\rho_{A,B}(g \otimes a) = \eta(g)a$ for $g \in F(A,B)$ and $a \in A$, i.e.*

$$F(A,B) := (A,B)^c/\mathrm{Ann}(A,B_\omega) \cong A' \cap \mathcal{M}(D_{A,B})\,.$$

(6) $(A,B)^c$ is unital, if and only if, B_ω is unital, if and only if, B is unital.

(7) $\mathrm{Ann}(A, B_\omega) = \{0\}$, *if and only if,* B *is unital and* $1_B \in A$.
(8) *Suppose that* $d \in A_+$ *is a full positive contraction, and let* $E := \overline{dAd}$. *Then the natural *-morphism*

$$c \in A' \cap \mathcal{M}(D_{A,B}) \mapsto c \in E' \cap \mathcal{M}(D_{E,B})$$

is bijective and defines an isomorphism ψ *from* $F(A, B)$ *onto* $F(E, B)$ *with*

$$\rho_{A,B}(c \otimes a) = \rho_{E,B}(\psi(c) \otimes a)$$

for $c \in F(A, B)$ *and all* $a \in E$.
(9) *If* $C \subset B$ *is a hereditary* C^**–subalgebra with* $A \subset C_\omega \subset B_\omega$, *then* $(A, B)^c = (A, C)^c + \mathrm{Ann}(A, B_\omega)$ *and* $F(A, B) \cong F(A, C)$.

The proof is given in Appendix B. The only non-trivial parts are (4) and (8). Part (9) and the proof of part (8) show that

$$F(A_1, B_1) \cong F(A_2, B_2)$$

if the pairs (A_1, D_{A_1,B_1}) and (A_2, D_{A_2,B_2}) are Morita equivalent, and A_1, A_2 are both σ-unital.

The proofs of parts (5) and (8) use part (4) and a lemma on Morita equivalence of non-degenerate pairs $A_j \subset D_j$ (cf. Lemma B.1). Part (7) follows from part (6) and Remark 2.7.

Corollary 1.10 *Suppose that* A *is* σ*-unital. Then*

(1) $\mathrm{Ann}(A)$ *is an ideal of* A^c *and* $F(A)$ *is unital.*
(2) A *is unital, if and only if,* $\mathrm{Ann}(A) = \{0\}$, *if and only if,* A^c *is unital.*
(3) $F(E) \cong F(A)$ *if* E *is* σ*-unital and Morita equivalent to* A.
(4) *If* $b \in A_+$ *is a full positive element of* A *and* $E := \overline{bAb}$, $D_E := \overline{bA_\omega b} \subset B_\omega$ *then* $\rho_A \colon F(A) \otimes^{\max} A \to A_\omega$ *induces an* isomorphism ψ *from* $F(A)$ *onto* $E' \cap \mathcal{M}(D_E)$ *with* $\psi(d)c = \rho_A(d \otimes c) = fb$ *for* $n \in \mathbb{N}$, $c \in E$ *and* $d \in F(A)$, *where* $f \in A^c$ *is any element with* $f + \mathrm{Ann}(A) = c$.
(5) *Let* $f \in A^c$ *and* $b \in A_+$ *a* full *element of* A, *then* $\|d\| = \lim_{n\to\infty} \|b^{1/n} f\|$ *for every* $d \in F(A)$ *and* $f \in A^c$ *with* $d = f + \mathrm{Ann}(A)$.
(6) $F(A) = F(A, A + \mathbb{C} \cdot 1) = F(A, \mathcal{M}(A))$ *and*

$$F(A + \mathbb{C} \cdot 1) = (A + \mathbb{C} \cdot 1)^c = A^c + \mathbb{C} \cdot 1 \subset (A + \mathbb{C} \cdot 1)_\omega \cong A_\omega + \mathbb{C} \cdot 1\,.$$

Part (6) follows from part (9) of 1.9.

Remark 1.11 *If* $A \subset B_\omega$ *and* A *is* σ*–unital, then*

$$F(A, B) \cong A' \cap \mathcal{M}(D_{A,B}) = \mathcal{M}(A)' \cap \mathcal{M}(D_{A,B})\,.$$

and

$$\mathcal{Z}(\mathcal{M}(A)) \cup \mathcal{Z}(\mathcal{M}(D_{A,B}) \subset \mathcal{Z}(F(A, B)) \subset F(A, B)\,.$$

□

Proposition 1.12 *Suppose that B is separable, A is a separable C^*–subalgebra of B_ω, and D is a separable C^*–subalgebra of $F(B)$ with $1_{F(B)} \in D$,*

*(1) There is a *-morphism $\psi\colon C_0((0,1],D) \to (A,B)^c$ with $\psi(f_0 \otimes 1)b = b$ for all $b \in A$, i.e. $\psi(C_0((0,1),D)) \subset \mathrm{Ann}(A,B_\omega)$ and*

$$[\psi]\colon\ d \in D \to \psi(f_0 \otimes d) + \mathrm{Ann}(A,B_\omega) \in F(A,B)$$

*is a unital *-monomorphism from D into $F(A,B)$.*

(2) If in addition, $B \subset A$, then $\psi\colon C_0((0,1],D) \to (A,B)^c$ in (2) can be found such that, moreover, $\psi(C_0((0,1),D)) = \psi(C_0((0,1],D)) \cap \mathrm{Ann}(B)$, i.e. that $[\psi](D)$ has trivial intersection with the image of $(A,B)^c \cap \mathrm{Ann}(B)$ in $F(A,B)$.

By induction, part (2) of Proposition 1.12 implies:

Corollary 1.13 *If A is separable and $C, B_1, B_2, \ldots$ are separable unital C^*-subalgebras of $F(A)$, then there is a unital a *-morphism*

$$\psi\colon\ C \otimes^{\max} B_1 \otimes^{\max} B_2 \otimes^{\max} \ldots \to F(A)$$

*with $\psi(c \otimes 1 \otimes 1 \otimes \cdots) = c$ for $c \in C$, such that the *-morphisms*

$$b \in B_n \mapsto \psi(1 \otimes \cdots 1 \otimes b \otimes 1 \otimes \cdots) \in F(A)$$

are faithful.

The stable invariant $F(A)$ has the following local continuity property:

Proposition 1.14 *Let $A_1 \subset A_2 \subset \ldots A$ C^*–subalgebras such that $\bigcup_n A_n$ is dense in A and A is separable. Then for every separable unital C^*-subalgebra B of the ultrapower*

$$\prod_\omega \{F(A_1), F(A_2), \ldots\} := \ell_\infty\{F(A_1), F(A_2), \ldots\}/c_\omega\{F(A_1), F(A_2), \ldots\}$$

*there is a unital *-morphism from B into $F(A)$.*

*In particular, for every simple separable unital *-subalgebra D of $F(A)_\omega$, there is a copy of D unitally contained in $F(A)$.*

See cf. Appendix B for the proof.

If J is a closed ideal of B, let $\eta_J\colon B \to \mathcal{M}(J)$ and $\pi_J\colon B \to B/J$ the natural *-morphisms. We denote by $\eta := (\eta_J)_\omega\colon B_\omega \to \mathcal{M}(J)_\omega$ and $\pi := (\pi_J)_\omega\colon B_\omega \to (B/J)_\omega$ the ultrapowers of η_J and π_J.

Recall that $(X,J)^c := X' \cap J_\omega = (X,\mathcal{M}(J))^c \cap J_\omega$ for C^*–subalgebras $X \subset \mathcal{M}(J)_\omega$.

Remark 1.15 *Suppose that J is a closed ideal of B and that $A \subset B_\omega$ is a separable C^*–subalgebra. Let $\eta := (\eta_J)_\omega$ and $\pi := (\pi_J)_\omega$ as above.*

(1)

$$\mathrm{Ann}(A,J_\omega) := \mathrm{Ann}(A,B_\omega)\cap J_\omega = \mathrm{Ann}(\eta(A),J_\omega)\,,$$
$$(A,J)^c := A'\cap J_\omega = (A,B)^c\cap J_\omega = \eta(A)'\cap J_\omega = (\eta(A),J)^c$$
$$(\pi_J)_\omega(\mathrm{Ann}(A,B_\omega)) = \mathrm{Ann}(\pi(A),(B/J)_\omega)\,.$$
and
$$(\pi_J)_\omega((A,B)^c) = (\pi(A),B/J)^c\,.$$

In particular, $F(A,J) := (\eta(A),J)^c/\mathrm{Ann}(\eta(A),J_\omega)$, *is isomorphic to the ideal* $(((A,B)^c\cap J_\omega)+\mathrm{Ann}(A,B_\omega))/\mathrm{Ann}(A,B_\omega)$ *of* $F(A,B) = (A,B)^c/\mathrm{Ann}(A,B_\omega)$.

(2) The sequences

$$0\to (A,J)^c\to (A,B)^c\to (\pi(A),B/J)^c\to 0$$
$$0\to \mathrm{Ann}(A,J_\omega)\to (A,J)^c\to F(A,J)\to 0$$
$$0\to F(A,J)\to F(A,B)\to F(\pi(A),B/J)\to 0.$$

are short-exact and strongly locally semisplit in the sense of Definition 1.5.

*(3) If J is a closed ideal of $A=B$, then the natural *-morphism $F(A)\to F(A/J)$ is an epimorphism with kernel $F(A,J)$, if J is a closed ideal of A, and there is a unital *-morphism $F(A)\to F(J)\cong (J,A)^c/\mathrm{Ann}(J,A_\omega)$ with kernel $(\mathrm{Ann}(J,A_\omega)\cap A^c)/\mathrm{Ann}(A)$.*

One gets the first two lines of part (1) by straight calculations. Then parts (1)-(3) follow from Proposition 1.6, Corollary 1.7 and Proposition 1.9(9).

Corollary 1.16 *Suppose that $J\triangleleft B$ is an* essential *ideal of B and that A is a separable C^*–subalgebra of B_ω. Then $F(A,J)$ is an essential ideal of $F(A,B)$.*

Proposition 1.17 *Suppose that B is a separable unital C^*-algebra, J a closed ideal of B, and $1_B\in A\subset B_\omega$ is a separable C^*–subalgebra. If D_1 is a unital separable C^*–subalgebra of $F((\pi_J)_\omega(A),B/J) = ((\pi_J)_\omega(A),B/J)^c$ and D_0 is a unital separable C^*–subalgebra of $F(J)$ then there is a unital *-morphism $h\colon\ \mathcal{E}(D_0,D_1)\to F(A,B)=(A,B)^c$.*

Here

$$\mathcal{E}(D_0,D_1) := \{f\in C([0,1],\, D_0\otimes^{\max} D_1)\,;\ f(0)\in D_0\otimes 1,\ f(1)\in 1\otimes D_1\,\}.$$

The proof in Appendix B gives h with the additional property $\pi(h(f)) = f(1)$ for $f\in\mathcal{E}(D_0,D_1)$ and the natural *-morphisms $\pi\colon\ F(A,B)\to F((\pi_J)_\omega(A),B/J)$.

Note that $\mathcal{E}(D_0,D_2)$ is unitally contained in $\mathcal{E}(D_0,D_1)$ if $D_2\subset D_1$. There is a unital *-morphism $D_1\to\mathcal{E}(D_1,\mathcal{O}_2)$ if D_1 is simple and nuclear, because then $D_1\otimes\mathcal{O}_2\cong\mathcal{O}_2$ and all unital *-monomorphisms of separable unital exact C^*–algebras into $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \ldots$ are homotopic (by basics of classification). We apply Proposition 1.17 to the extensions

$$0 \to J \otimes \mathcal{K} \to (A \otimes \mathcal{K}) + \mathcal{O}_2 \to ((A/J) \otimes \mathcal{K}) + \mathcal{O}_2 \to 0$$

and

$$0 \to (A/J) \otimes \mathcal{K} \to ((A/J) \otimes \mathcal{K}) + \mathcal{O}_2 \to \mathcal{O}_2 \to 0\,,$$

where $\mathcal{O}_2 \subset \mathcal{M}(\mathcal{K})$ unitally, and we use the unital *-morphism from $F((A \otimes \mathcal{K}) + \mathcal{O}_2)$ into $F(A \otimes \mathcal{K}) \cong F(A)$. We obtain finally:

Corollary 1.18 *Let A a separable C^*-algebra and J a closed ideal of A. If D_1 is a unital separable C^*–subalgebra of $F(A/J)$ and D_0 is a unital separable C^*–subalgebra of $F(J)$ then there exists a unital *-morphism $h\colon\ \mathcal{E}(D_0, \mathcal{E}(D_1, \mathcal{O}_2)) \to F(A)$.*

*If, moreover, D_1 is simple and nuclear, then there exists a unital *-morphism from $\mathcal{E}(D_0, D_1)$ into $F(A)$.*

2 The case of simple $F(A)$.

We study here some particular ideals of $F(A)$ for separable A. The less trivial basic facts are given in Lemmas 2.8, 2.9 and 2.11(3,4). We apply them in the particular case where A is simple, to get the main result of this section: Theorem 2.12. Then we have a look to non-separable A, and we pose some related problems.

The following Lemma seems to be wrong for *non-separable* A, because it might be that $F(\mathcal{L}(\ell_2)) = \mathcal{L}(\ell_2)^c \cong \mathbb{C}$, cf. Question 2.22.

Lemma 2.1 *Suppose that A is separable. If $F(A)$ is simple, then A is simple.*

More precisely, if J is a non-trivial closed ideal of A, then

(1) J_ω is a closed ideal of A_ω with $A \cap J_\omega = J$, and
(2) $F(A, J) = (A^c \cap J_\omega)/(\mathrm{Ann}(A) \cap J_\omega)$ is a non-trivial closed ideal of $F(A)$.

$F(A, J)$ is an essential ideal of $F(A)$ if J is essential, cf. Corollary 1.16.

If even A^c is simple, then A is simple *and* unital, because then $F(A) = A^c/\{0\}$ is simple, and $\mathrm{Ann}(A) = \{0\}$ implies that A is unital (for σ-unital A, cf. Corollary 1.10).

Proof. (1): It is clear that J_ω is a closed ideal of A_ω. If $a \in A \cap J_\omega$ then there is a bounded sequence $b_1, b_2, \ldots \in J$ with $\lim_\omega \|a - b_n\| = 0$. Thus, there is a sub-sequence $c_k := b_{n_k} \in J$ with $\lim_{k\to\infty} c_k = a$, i.e. $a \in J$.

(2): $A^c \cap J_\omega$ is a closed ideal of A^c by (1).

Since A is separable, J is separable and contains a strictly positive contraction $b \in J_+$ for J, moreover, there are $b_1, b_2, \ldots \in C^*(b)_+$ with $\|b_n\| = 1$,

$b_nb_{n+1} = b_n$, $\|b - b_nb\| < 1/n$ and $\lim_{n\to\infty} \|b_na - ab_n\| = 0$ for all $a \in A$ (cf. the proof of [29, thm. 3.12.14]).

Thus $c := \pi_\omega(b_1, b_2, \ldots)$ is in $A^c \cap J_\omega$ and $cb = b \neq 0$. Thus $c \notin \mathrm{Ann}(A)$, i.e. $A^c \cap J_\omega \not\subset \mathrm{Ann}(A)$ and $F(A, J) = A^c \cap J_\omega/(\mathrm{Ann}(A) \cap J_\omega)$ is a non-zero closed ideal of $F(A)$.

Let a_0 is a strictly positive contraction in A_+. $F(A, J) \neq F(A)$, because otherwise there is a positive contraction $e \in A^c \cap J_\omega$ with $e + \mathrm{Ann}(A) = 1 \in F(A)$ and $a_0 = \rho(1 \otimes a_0) = ea_0 \in J_\omega$, i.e. $a_0 \in J$ by (1), which contradicts the non-triviality of J. □

A modification of the proof of Lemma 2.1(2) shows:

Remark 2.2 *If $\{0\} \neq J \neq A$, then $I := A^c \cap \mathrm{Ann}(J, J_\omega)$ is a closed ideal of A^c that is not contained in* $\mathrm{Ann}(A)$.
Note that $b \in I/(I \cap \mathrm{Ann}(A)) \subset F(A)$ if and only if $\rho_A(b \otimes J) = 0$.

Lemma 2.3 *If A is antiliminal then for every positive $b \in A_\omega$ with $\|b\| = 1$ there exists a *-monomorphism ψ from $C_0((0, 1], \mathcal{K})$ into A_ω with $b\psi(c) = \psi(c)$ for every $c \in C_0((0, 1], \mathcal{K})$.*

Recall that "antiliminal" means that $\{0\}$ is the only Abelian hereditary C^*–subalgebra of A.

Proof. Let $(b_1, b_2, \ldots) \in \ell_\infty(A)_+$ a representing sequence for b with $\|b_n\| = 1$, let $d_n := (b_n - (n-1)/n)_+ \neq 0$ and let D_n denote the closure of d_nAd_n. Then $bc = c$ for all elements c in $\prod_\omega\{D_n\,;\, n \in \mathbb{N}\} \subset A_\omega$.

Since $C_0((0, 1], \mathcal{K}) \subset \prod_\omega\{C_0((0, 1], M_n)\,;\, n \in \mathbb{N}\}$, is suffices to find faithful *-morphisms $\psi_n\colon\ C_0((0, 1], M_n) \to D_n$. By the Glimm halving lemma (cf. [29, lem. 6.7.1]) there is a non-zero *-morphism $h_n\colon\ C_0((0, 1], M_n) \to D_n$ because D_n is antiliminal as well (because it is a non-zero hereditary C^*–subalgebra of A). Let E_n the hereditary C^*-subalgebra of $D_n \subset A$ generated by $h_n(f_0 \otimes e_{1,1})$. If M is a maximal Abelian C^*-subalgebra of E_n with $h_n(f_0 \otimes e_{1,1}) \in M$, then M can not contain a minimal idempotent p, because otherwise $pAp = \mathbb{C}p$ which contradicts that A is antiliminal. It follows that h_n can be replaced by a *-*mono*morphism $\psi_n\colon\ C_0((0, 1], M_n) \to D_n$. □

Remark 2.4 *Let A a σ-unital C^*–algebra.*
The closed ideal J_A of A_ω generated by A is simple, if and only if, either A is simple and purely infinite or A is isomorphic to the compact operators $\mathcal{K}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$.
If $A \ncong \mathcal{K}(\mathcal{H})$ and J_A is simple, then A_ω and A are simple and purely infinite.
If $A \cong \mathcal{K}(\mathcal{H})$, then $J_A \cong \mathcal{K}(\mathcal{H}_\omega)$.
(If $A \cong \mathcal{K}(\mathcal{H})$ and is $\sigma - unital$, then $\mathcal{H}$ is separable, and $\mathrm{Dim}(\mathcal{H}) = \infty$ *if and only if $J_A \neq A_\omega$.)*

Proof. If $A \cong \mathcal{K}(\mathcal{H})$ then $J_A \cong \mathcal{K}(\mathcal{H}_\omega)$, because there is a natural *-monomorphism λ from $\mathcal{K}(\mathcal{H})_\omega$ into $\mathcal{L}(\mathcal{H}_\omega)$ that satisfies

$$\lambda\left(\left(\langle ., x_n \rangle y_n\right)_\omega\right) = \langle ., x_\omega \rangle y_\omega \, .$$

$\lambda(\mathcal{K}(\mathcal{H})) = P\mathcal{K}(\mathcal{H}_\omega)P$ is the hereditary C^*–subalgebra of $\mathcal{K}(\mathcal{H}_\omega)$ that is defined by the orthogonal projection P from $\mathcal{H}_\omega$ onto $\mathcal{H} \subset \mathcal{H}_\omega$. Since $\lambda(\mathcal{K}(\ell_2)_\omega) \not\subset \mathcal{K}((\ell_2)_\omega)$, it holds $A_\omega = \mathcal{K}(\mathcal{H})_\omega \subset \mathcal{K}(\mathcal{H}_\omega)$ if and only if $\mathcal{H}_\omega = \mathcal{H}$ if and only if and $\mathrm{Dim}(\mathcal{H}) = n < \infty$.

It is easy to see (with help of representing sequences in case $\|b\| = \|c\| = 1$) that for every $b, c \in (A_\omega)_+$ there is a contraction $d \in (A_\omega)_+$ with $\|c\| d^* b d = \|b\| c$ if A is simple and purely infinite. Thus A_ω is simple and purely infinite, and $J_A = A_\omega$.

Conversely, suppose that J_A is simple. This implies that A must be simple, because otherwise $J_A \cap I_\omega \supset I$ is a non-trivial closed ideal of J_A if I is a non-trivial closed ideal of A. Suppose that $A \ncong \mathcal{K}(\mathcal{H})$ (for any Hilbert space $\mathcal{H}$), i.e. that A is antiliminal. Let $b, c \in (J_A)_+$ with $\|b\| = \|c\|$. Since A is antiliminal, by Lemma 2.3 there exists a *-monomorphism $\psi\colon\ C_0((0,1], \mathcal{K}) \hookrightarrow A_\omega$ with $b\psi(f) = \psi(f)$ for every $f \in C_0((0,1], \mathcal{K})$. Let D denote the hereditary C^*–subalgebra of A_ω generated by the image of ψ. D is non-zero, stable and satisfies $bg = g = gb$ for all $g \in D$. In particular, $D \subset J_A$. Since J_A is simple and D is stable, there is $d \in J_A$ with $d^*d = c$ and $dd^* \in D$. Thus $d^*bd = d^*d = c$. It follows that A is purely infinite, because we can take $b, c \in A$ and find a representing sequence $(d_1, d_2, \ldots) \in \ell_\infty(A)$ for d with $d^*bd = c$ in A_ω. □

Lemma 2.5 *Suppose that B is a separable C^*–subalgebra of A_ω.*

If λ is a pure state on B, then there exists a sequence of pure states $\mu_1, \mu_2, \ldots$ on A such that λ is the restriction of the state $\mu_\omega\colon\ A_\omega \to \mathbb{C} \cong \mathbb{C}_\omega$ to B.

If $(\mu_1, \mu_2, \ldots)$ is any sequence of pure states on A, then there are positive contractions $g_n \in A_+$ such that $\mu_n(g_n) = 1$, and $gbg = \mu_\omega(b)g^2$ for all $b \in B$, where $g := \pi_\omega(g_1, g_2, \ldots)$.

Note that $\|g\| = 1$.

Proof. If C is a C^*-algebra and λ is a pure state on C, then for every *separable* C^*-subalgebra $B \subset C$ there is $c \in C_+$ with $\lambda(c) = \|c\| = 1$ and $\lim_{n\to\infty} \|c^n b c^n - \lambda(b) c^{2n}\| = 0$ for every $b \in B$ (cf. [6, lem. 2.14]).

Clearly, in the case $B = C$, the limes property of $c \in B$ implies that $\nu = \lambda$ for all $\nu \in B^*$ with $\nu(c) = \|\nu\| = 1$. (In fact, the latter property of c equivalent to the limes property of c.)

We find a sequence $c_1, c_2, \ldots \in A_+$ with $\|c_n\| = 1$ and $\pi_\omega(c_1, c_2, \ldots) = c$. Let $\mu_1, \mu_2, \ldots$ pure states on A with $\mu_n(c_n) = 1$. Then $\mu_\omega(c) = 1 = \|\mu_\omega\|$. Thus $\mu_\omega | B = \mu$.

Suppose that $(\mu_1, \mu_2, \ldots)$ is any sequence of pure states on A. Let $b^{(1)}, b^{(2)}, \ldots \in B$ a dense sequence in the unit ball of B. There are representing sequences $s_k = (b_1^{(k)}, b_2^{(k)}, \ldots) \in \ell_\infty(A)$ with $\|b_n^{(k)}\| \leq 1$ and $\pi_\omega(s_k) = b^{(k)}$. By the above mentioned result of [6, lem. 2.14], there are $c_n \in A_+$ and $p_n \in \mathbb{N}$ with $\|c_n\| = 1 = \mu_n(c_n)$ and $\|c_n^{p_n} b_n^{(k)} c_n^{p_n} - \mu_n(b_n^{(k)}) c_n^{2p_n}\| < 2^{-n}$ for $k \leq n = 1, 2, \ldots$. Let $g_n := c_n^{p_n}$ for $n \in \mathbb{N}$. Then $g := \pi_\omega(g_1, g_2, \ldots) \in A_\omega$ satisfies $0 \leq g$, $\|g\| = 1 = \mu_\omega(g)$ and $gb^{(k)}g = \mu_\omega(b^{(k)})g^2$. □

Remark 2.6 *Let $g \in A_\omega$ with $0 \leq g$, $\|g\| = 1$ and $gbg = \mu(b)g^2$ for $b \in B \subset A_\omega$, then $g \in (B, A)^c$ if and only if μ is a character on B.(Left to the reader.)*

Remark 2.7 . *Lemma 2.5 implies that*

$$\mathrm{Ann}(\mathrm{Ann}(B, A_\omega), A_\omega) = D_{B,A} := \overline{b_0 A_\omega b_0}$$

if B is σ-unital and $b_0 \in B$ is a strictly positive contraction for B.

Proof. If $a \in (A_\omega)_+$ is *not* in $D_{B,A}$, then $\inf_n \|(1 - b_0^{1/n}) a (1 - b_0^{1/n})\| > 0$. Thus, there is a pure state μ on $C^*(b_0, a)$ with $\mu(b_0) = 0$ and $\mu(a) > 0$. By Lemma 2.5 there exists $g \in (A_\omega)_+$ with $\|g\| = 1$ and $gcg = \mu(c)g^2$ for $c \in C^*(b_0, a)$. Hence, $g \in \mathrm{Ann}(B, A_\omega)$ and $ag = g \neq 0$. □

The *socle* of a C^*–algebra A is the (algebraic) ideal generated by the projections $p \in A$ with $pAp = \mathbb{C} \cdot p$. If A is simple, then $\mathrm{socle}(A) \neq \{0\}$ if and only if $A \cong \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Lemma 2.8 $\mathrm{socle}(F(A)) = \{0\}$ *if A is separable and* $\mathrm{socle}(A) = \{0\}$.

Proof. Let $p^2 = p^* = p \in F(A)$ a non-zero projection. We show that $pF(A)p \neq \mathbb{C} \cdot p$ if $\mathrm{socle}(A) = \{0\}$. The idea of the proof goes as follows: Let $s \in A^c$ a self-adjoint contraction with $s + \mathrm{Ann}(A) = 1 - 2p$, and $d := s_+$, $q := s_-$. We show below that there exist positive contractions $g, h \in A_\omega$ such that $dg = dh = hg = 0$, $gAh = \{0\}$, $Ah \neq \{0\}$ and $Ag \neq \{0\}$.

If we have found $g, h \in (A_\omega)_+$ with this properties, then Proposition 1.3 (with $J = A = B$, $\mathcal{V} = CP(A, A)$ and a, b, c replaced by d, g, h) yields: There are positive contractions $e, f \in A^c$ with $de = df = ef = 0$, $eg = g$, $fh = h$. It follows that $e' := e + \mathrm{Ann}(A)$ and $f' := f + \mathrm{Ann}(A)$ satisfy $e'p = e'$, $f'p = f'$ and $e'f' = 0$ in $F(A)$. Since $Ag \neq \{0\}$ and $Ah \neq \{0\}$, we get $e, f \notin \mathrm{Ann}(A)$ and $e' \neq 0 \neq f'$. Hence, $pF(A)p \neq \mathbb{C} \cdot p$.

The rest of the proof is concerned with the construction of the positive contractions $g, h \in A_\omega$ (as stipulated):

Step 1 (Construction of $a_0 \in A$, $E \subset A_\omega$, μ, ν):

Consider the *-morphism $\psi \colon a \in A \mapsto \rho(p \otimes a) = qa \in A_\omega$. ψ is non-zero because $p \neq 0$ and $\rho((\cdot) \otimes A)$ is separating for $F(A)$. Let J denote the kernel of ψ, and let $a_1 \in J_+$ and $a_2 \in (A/J)_+$ strictly positive contractions with $\|a_1\| = \|a_2\| = 1$. There is a positive contraction $a_3 \in A_+$ with $a_3 + J = a_2$. $a_0 := (1 - a_1)^{1/2} a_3 (1 - a_1)^{1/2} + a_1$ is a strictly positive contraction of A with

$$\|qa_0\| = \|\rho(p \otimes a_0)\| = \|a_0 + J\| = \|a_2\| = 1.$$

Since q and a_0 are commuting positive contractions with $\|qa_0\| = 0$ there is a character χ on $C^*(a_0, q)$ with $\chi(qa_0) = \chi(q) = \chi(a_0) = 1$. By Lemma 2.5 there is a sequence $(\mu_1, \mu_2, \ldots)$ of pure states on A such that χ is the restriction of $\mu := (\mu_1, \mu_2, \ldots)_\omega$ to $C^*(a_0, q)$. The state $\mu\colon A_\omega \to \mathbb{C}$ is supported on the closure E of $a_0 q A_\omega a_0 q$. In particular $\mu(\mathrm{Ann}(E, A_\omega)) = \{0\}$. It follows $\mu(d) = 0$ because $dq = 0$.

E is contained in the closure D_A of $a_0 A_\omega a_0$. Thus, for every non-zero element $y \in E_+$ holds $Ay \neq \{0\}$.

Let $G = \{u_1 = 1, u_2, \ldots\}$ a countable dense subgroup of the unitary group of $\widetilde{A} = A + \mathbb{C} \cdot 1$, and $\nu(b) := \sum_{n=1}^{\infty} 2^{-n} \mu(u_n^* b u_n)$ for $b \in E$. Since a_0 is strictly positive in A and q commutes with A, we get that $AE + EA \subset E$, ν is a state on E with $\mu \leq 2\nu$. Clearly $b \in L_\nu := \{x \in E\,; \nu(x^*x) = 0\}$ if and only if $bu \in L_\mu := \{x \in E\,; \mu(x^*x) = 0\}$ for all $u \in G$. Thus, $L_\nu G \subset L_\nu$ and $L_\nu A \subset L_\nu$.

Step 2 (ν is *not* faithful on E):
We find a representing sequence $(c_1, c_2, \ldots) \in \ell_\infty(A)_+$ with $\pi_\omega(c_1, c_2, \ldots) = a_0 q$ and $\|c_n\| = 1$. Let C_n a maximal Abelian C^*–subalgebra of $D_n := (c_n - 1/2)_+ A (c_n - 1/2)_+$ that contains $(c_n - 1/2)_+$. C_n does not contains a *minimal idempotent* $r \neq 0$, because otherwise r must satisfy $rAr = \mathbb{C} \cdot r$, i.e. $r \in \mathrm{socle}(A) = \{0\}$. Hence, the primitive ideal space of C_n is a perfect locally compact metric space and there is $f_n \in (C_n)_+$ with $\mathrm{Spec}(f_n) = [0, 1]$, i.e. C_n contains a copy of $C_0((0, 1])$ up to isomorphisms.

The corresponding monomorphic image C of $C_0((0, 1])_\omega$ in A_ω satisfies $wb = bw = b$ for $b \in C$ and $w := 2(a_0 q - (a_0 q - 1/2)_+)$, thus $C \subset E$.

Let $x \in (0, 1)$ and $f_{x,n}(t)$ the continuous function in $t \in [0, 1]$ with $f_{x,n}(x) = 1$, $f_{x,n}(t) = 0$ for $t \in [0, x - \min(1/n, x)] \cup [x + \min(1/n, 1 - x), 1]$ and $f_{x,n}$ is linear on $[x - \min(1/n, x), x]$ and $[x, x + \min(1/n, 1 - x)]$.

$\delta_x := \pi_\omega(f_{x,1}, f_{x,2}, \ldots)$ is a positive contraction in $C_0((0, 1])_\omega \cong C$ with $\|\delta_x\| = 1$ and $\delta_x \delta_y = 0$ for $x \neq y$.

It follows that $E \supset C$ contains uncountably many pair-wise orthogonal non-zero positive contractions, because $\{\delta_x\}_{x \in (0,1)}$ is a family of pair-wise orthogonal positive elements in $C_0((0, 1])_\omega \cong C$ with $\|\delta_x\| = 1$. Hence ν can not be faithful on E, i.e. $D := L_\nu^* \cap L_\nu = L_\nu^* L_\nu$ is a non-zero hereditary C^*-subalgebra of E.

Step 3 (Construction of $g, h \in E_+$):
Let $D := L_\nu^* \cap L_\nu$ and let $h \in D_+$ with $\|h\| = 1$. Then $dh = 0$, $Ah \neq 0$, $AD + DA \subset D$ and $\mu(a^* h^2 a) \leq 2\nu(a^* h^2 a) = 0$ for all $a \in A + \mathbb{C} \cdot 1$, because $L_\nu A \subset L_\nu \subset E \subset \mathrm{Ann}(d, A_\omega) \cap D_A$. By Lemma 2.5 there is $g \in (A_\omega)_+$ with $\|g\| = 1$ such that $gyg = \mu(y) g^2$ for all $y \in C^*(A, q, d, h)$, because $\mu_1, \mu_2, \ldots$ are pure states on A. It follows $gd^2 g = \mu(d^2) g^2 = 0$, $gh^2 g = \mu(h^2) g^2 = 0$, $ga^* h^2 a g = \mu(a^* h^2 a) g^2 = 0$, and $ga_0 g = \mu(a_0) g^2 = g^2 \neq 0$, i.e. $g, h \in A_\omega$ are as required. $\square$

Lemma 2.9 *Suppose that A is separable.*

(1) $\mathrm{Prim}(A)$ *is quasi-compact, if and only if, for every non-invertible* $e \in F(A)_+$*, there exists non-zero* $d \in F(A)_+$ *with* $de = 0$*.*
(2) If $\mathrm{Prim}(A)$ *is quasi-compact, then every maximal family of mutually orthogonal positive contractions in* $F(A)$ *is either finite and has invertible sum or is not countable.*

Part(2) applies to *simple* C^*–algebras A, because $\mathrm{Prim}(A)$ is a singleton if and only if A is simple. The Bourbaki terminology "quasi-compact" is used for non-Hausdorff T_0 spaces.

Proof. (1): Recall that there is a one-to-one isomorphism from the lattice of closed ideals of A onto the lattice of open subsets of $\mathrm{Prim}(A)$. Since A is separable, $\mathrm{Prim}(A)$ is second countable. Thus, if $\mathrm{Prim}(A)$ is not quasi-compact, then there is an increasing sequence $J_1 \subset J_2 \subset \cdots$ of closed ideals of A with $J_n \neq J_{n+1}$ and $\bigcup_n J_n$ dense in A. For each $n \in \mathbb{N}$ there exists a positive contraction $c_n \in J_n$ with $\|c_n + J_{n-1}\| = 1$ such that $c_n + J_{n-1}$ is a strictly positive element of J_n/J_{n-1} (where we let $J_0 := \{0\}$). Then $a_n := \sum_{1\leq k\leq n} 2^{-k} c_k$ is a strictly positive contraction in J_n, and $b_0 := \sum_{1\leq n<\infty} 2^{-n} c_n$ is a strictly positive contraction in A (with norm $\geq 1/2$). Let $f_0 := 0$. By induction, we find positive contractions $f_n \in (A, J_n)^c = A' \cap (J_n)_\omega \subset A^c$ with $a_n f_n = a_n$ and $f_{n-1} f_n = f_{n-1}$ (cf. Remark 1.15(1)). Let $f := \sum_{1\leq n<\infty} 2^{-n} f_n \in (A^c)_+$ and let $e := f + \mathrm{Ann}(A) \in F(A)$. Then $fb_0 = b_0 f$ is positive, $a_n = f_n a_n \leq 2^n f_n b_0 \leq 4^n f b_0$ for $n \in \mathbb{N}$, and $\|f - f_{k+1} f\| \leq 2^{-k}$. If $d \in (A^c)_+$ satisfies $df \in \mathrm{Ann}(A)$, then $dfb_0 = 0$ and $da_n = 0$ for all $n \in \mathbb{N}$. Hence, $db_0 = 0$ and $d \in \mathrm{Ann}(A)$ whenever $d \in (A^c)_+$ and $df \in \mathrm{Ann}(A)$. Let $e := f + \mathrm{Ann}(A)$. Then $g = 0$, if $g = d + \mathrm{Ann}(A) \in F(A)_+$ with $d \in A^c_+$ and $ge = 0 \in F(A)$.

The image $F(A, J_n)$ of $(A, J_n)^c = (J_n)_\omega \cap A^c$ in $F(A)$ is a *non-trivial* ideal of $F(A)$ by Lemma 2.1(2), because J_n is a non-trivial ideal of A. Since $f_n \in (J_n)_\omega \cap A^c$, the element $e_n := f_n + \mathrm{Ann}(A) \in F(A)_+$ is not invertible in $F(A)$. It follows that $e := f + \mathrm{Ann}(A)$ is not invertible, because $\|e_{k+1} e - e\| \leq 2^{-k}$.

Conversely, suppose that $\mathrm{Prim}(A)$ is quasi-compact and that $e \in F(A)_+$ is not invertible. We can suppose that $\|e\| = 1$. Then there is a contraction $f \in A^c_+$ with $e = f + \mathrm{Ann}(A)$. Let $a_0 \in A_+$ a strictly positive contraction with $\|a_0\| = 1$, and let J_n denote the closure of $\mathrm{span}(A(a_0 - 1/n)_+ A)$. Then J_n is an increasing sequence of closed ideals of A with $\bigcup_n J_n$ dense in A, i.e. the corresponding increasing sequence of open subsets of $\mathrm{Prim}(A)$ covers $\mathrm{Prim}(A)$. Since $\mathrm{Prim}(A)$ is quasi-compact, there is $n \in \mathbb{N}$ such that $J_n = A$, i.e. that $b := (a_0 - 1/n)_+$ is a *full* positive contraction in A_+. Let $c := 2n((a_0 - 1/(2n))_+ - (a_0 - 1/n)_+)$, then $bc = b$ Note that $C^*(b, c, f) \subset A_\omega$ is an Abelian C^*–algebra.

By Corollary 1.10, $\rho\colon\ F(A) \otimes^{\max} A \to A_\omega$ induces an *isomorphism* ψ from $F(A)$ onto $(bAb)' \cap \mathcal{M}(D_b)$ with $\psi(e)(b^n) = \rho(e \otimes b^n) = fb^n$ for $n \in \mathbb{N}$, where $D_b := \overline{bA_\omega b}$ denotes the hereditary C^*–subalgebra of A_ω generated by b.

$(b - nfb)_+ \neq 0$ for each $n \in \mathbb{N}$, because, otherwise, there is $n \in \mathbb{N}$ with $b^2 \leq n^2 (fb)^2$ and

$$\|\psi(e)(bxb)\|^2 = \|\rho(e \otimes bxb)\|^2 = \|bx^*bf^2bxb\| \geq n^{-2}\|bxb\|^2$$

for all $x \in A_\omega$, which contradicts that $\psi(e) \in \mathcal{M}(D_b)_+$ is *not* invertible. Thus, for every $n \in \mathbb{N}$ there exists a character χ_n on $C^*(b,c,f)$ with $\chi_n(b - nfb) > 0$, i.e. $\chi_n(b) > 0$ and $\chi_n(f) < 1/n$. Since $cb = b$, it follows $\chi_n(c) = 1$. The set of characters χ on $C^*(b,c,f)$ with $\chi(c) = 1$ is compact in the space of characters on $C^*(b,c,f)$. Let χ a character on $C^*(b,c,f)$ that is a cluster point of the sequence $\chi_1, \chi_2, \ldots$, then $\chi(f) = 0$ and $\chi(c) = 1$.

By Lemma 2.5 there exists $g \in (A_\omega)_+$ with $\|g\| = 1$, $gfg = \chi(f)g^2 = 0$ and $gcg = \chi(c)g^2 = g^2$. Then $fg = 0$ and $cg = g$. By Proposition 1.3 (with $J = A = B$, $\mathcal{V} = CP(A,A)$, and $f, 0, g$ in place of a, b, c) there there are positive contractions $h, k \in A^c$ with $kf = f$, $hg = g$ and $hk = 0$. Since $hc^2 \geq cgc \neq 0$ and $hf = hkf$, we get $h \notin \mathrm{Ann}(A)$ and $hf = 0$. Let $d := h + \mathrm{Ann}(A) \in F(A)_+$, then $d \neq 0$ and $de = 0$.

(2): If $e_1, e_2, \ldots \in F(A)$ is a sequence of pairwise orthogonal positive contractions, and $e := \sum 2^{-n} e_n$. If e is invertible, then $e_n = 0$ for $n \leq n_0$. If e is not invertible, then there exists non-zero $d \in (F(A))_+$ with $ed = 0$ by (1). Thus $e_n d = 0$ for all $n \in \mathbb{N}$. □

The following proposition characterizes $A \cong \mathcal{K}(\mathcal{H})$ by properties of $F(A)$ if A is simple and separable.

Proposition 2.10 *Suppose that A is separable and simple. The following are equivalent:*

(1) $A \otimes \mathcal{K} \cong \mathcal{K}$.
(2) $F(A) \cong \mathbb{C}$.
(3) $\mathrm{socle}(F(A)) \neq \{0\}$.
(4) $F(A)$ *is separable.*
(5) $F(A)$ *is simple and stably finite.*
(6) $F(A)$ *has a faithful finite quasi-trace.*
(7) Every commutative C^*–subalgebra of $F(A)$ admits a faithful state.
(8) Every family of mutually orthogonal positive contractions is at most countable.

Note that (2) also implies that A is simple (for separable A, by Lemma 2.1), thus $F(A) \cong \mathbb{C}$ *if and only if* $A \otimes \mathcal{K} \cong \mathcal{K}$ (for separable A). Clearly one can restrict in (7) to maximal Abelian C^*–subalgebras.

Proof. The implications (2) ⇒ (4) ⇒ (7) ⇒ (8), (2) ⇒ (5), (6) ⇒ (7), and (2) ⇒ (3) are obvious.

(5) ⇒ (6) follows from [9], because $F(A)$ is unital and finite dimension-functions on simple unital C^*–algebras B integrate to faithful finite quasi-traces on B.

(1)⇒(2): $F(A) \cong F(A \otimes \mathcal{K})$ and $F(\mathcal{K}) \cong F(\mathbb{C}) = \mathbb{C}$ by Corollary 1.10.

(3)⇒(1): $\mathrm{socle}(A) \neq \{0\}$ follows from (3) by Lemma 2.8. Thus $A \cong \mathcal{K}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$, i.e. $A \otimes \mathcal{K} \cong \mathcal{K}$.

(8)⇒(3): Let $C \subset F(A)$ a maximal commutative C^*–subalgebra of $F(A)$. Every family $X \subset C_+$ of mutually orthogonal positive contractions in C is contained in a maximal family $Y \subset F(A)_+$ of mutually orthogonal positive contractions in $F(A)$. By (8) and Lemma 2.9(2), $Y \supset X$ is finite. Thus, the primitive ideal space $\widehat{C}$ of C can contain only a finite number of points, i.e. C is of finite dimension. If $p \neq 0$ is a minimal idempotent of C, then $p^* = p = p^2$ and $pF(A)p \cong \mathbb{C} \cdot p$ (by maximality of C). □

We call a completely positive map $T\colon A \to A_\omega$ *ω-nuclear* if there is a bounded sequence of nuclear c.p. maps $T_n\colon A \to A$ such that $T = T_\omega|A$. Here $T_\omega\colon A_\omega \to A_\omega$ means the ultrapower $(T_1, T_2, \ldots)_\omega$ of the bounded sequence of maps $(T_n\colon A \to A)_n$ given by $T_\omega(\pi_\omega(a_1, a_2, \ldots)) := \pi_\omega(T_1(a_1), T_2(a_2), \ldots)$.

Lemma 2.11 *Suppose that A is a separable C^*–algebra.*

(1) The set $\mathcal{C}_{\omega\mathrm{nuc}}$ of ω-nuclear completely positive maps $V\colon A \to A_\omega$ is a point-norm closed (matricially) operator-convex cone (cf. Definition 1.2).
(2) Let κ denote the set of positive elements $b \in F(A)_+$ with the property that the c.p. map

$$a \in A \mapsto \rho_A(b \otimes a) \in A_\omega$$

is ω-nuclear. Then κ is the positive part of a closed ideal J_{nuc} of $F(A)$.
(3) J_{nuc} is an essential ideal of $F(A)$. In particular, J_{nuc} is non-zero for every separable C^*–algebra $A \neq \{0\}$.
(4) $J_{\mathrm{nuc}} = F(A)$ if and only if A is nuclear.

Proof. (1): Obviously, $\mathcal{V}_\omega \subset CP(A_\omega, B_\omega)$ is operator-convex in the sense of Definition 1.2, if $\mathcal{V} \subset CP(A, B)$ is an operator-convex cone and if $\mathcal{V}_\omega$ denotes the set of ultapowers of bounded sequences $T_1, T_2, \ldots$ in $\mathcal{V}$. That is, $\mathcal{V}_\omega$ is a convex subcone of $CP(A_\omega, B_\omega)$ and $bT(a^*(\cdot)a)b^* \in \mathcal{V}_\omega$ for $T = (T_1, T_2, \ldots)_\omega \in \mathcal{V}_\omega$ and rows $a \in M_{1,n}(A_\omega)$, $b \in M_{1,n}(B_\omega)$. We get an operator-convex subcone $\mathcal{V}_\omega|A$ of $CP(A, B_\omega)$, if we restrict the elements of $\mathcal{V}_\omega$ to $A \subset A_\omega$.

We can apply this construction to $B := A$ and the operator-convex cone $\mathcal{V} := CP_{\mathrm{nuc}}(A, A)$ and get $\mathcal{V}_\omega|A = \mathcal{C}_{\omega\mathrm{nuc}}$.

By Lemma A.5, every ω-nuclear c.p. map $V\colon A \to A_\omega$ can be represented by a sequence of $T_1, T_2, \ldots$ of nuclear c.p. maps from A into A such that $\|T_n\| \leq \|V\|$ and $T_\omega|A = V$, because $CP_{\mathrm{nuc}}(A, A)$ is operator-convex.

If $V_1, V_2, \ldots$ is a sequence in $\mathcal{C}_{\omega\mathrm{nuc}}$ that converges to a map $W\colon A \to A_\omega$ in point-norm topology, then $\gamma := \sup_n \|V_n\| < \infty$, by the uniform boundedness theorem. Thus, we find nuclear c.p. maps $T_k^{(n)}$ from A to A with $\|T_k^{(n)}\| \leq \gamma$ such that $V_n = T_\omega^{(n)}|A$, (cf. Lemma A.5). By Lemma A.3 there are $S_1, S_2, \ldots \in CP_{\mathrm{nuc}}(A, A)$ with $W = S_\omega|A$. Thus $W \in \mathcal{C}_{\omega\mathrm{nuc}}$.

(2): Let κ denote the set of positive elements $b \in F(A)$ such that $a \in A \mapsto \rho(b \otimes a)$ is in $\mathcal{C}_{\omega\mathrm{nuc}}$. Then κ is a closed convex sub-cone of $F(A)_+$ by part (1). If $b \in \kappa$, $c \in F(A)$ and $d \in A^c$ with $c = d + \mathrm{Ann}(A)$ then $a \mapsto \rho(c^*bc \otimes a) = d^*\rho(b \otimes a)d$ is in $\mathcal{C}_{\omega\mathrm{nuc}}$ by (1). Thus $c^*\kappa c \subset \kappa$ for all $c \in F(A)$.

It follows that κ is the positive part of the closed ideal $J_{\mathrm{nuc}} := \overline{\kappa - \kappa + i(\kappa - \kappa)}$ of $F(A)$ (by [29, prop. 1.3.8 and 1.4.5]).

(3): Let $e \in F(A)$ a positive contraction with $\|e\| = 1$. There is a positive contraction $f \in A^c$ with $e = f + \mathrm{Ann}(A)$ and $f \notin \mathrm{Ann}(A)$. Further let $a_0 \in A_+$ a strictly positive element with $\|a_0\| = 1$. Then $\rho_A(e \otimes a_0) = fa_0 \neq 0$, because $\mathrm{Ann}(A) = \mathrm{Ann}(a_0, A_\omega)$. Thus, there is a character χ on $C^*(a_0, f)$ with $\chi(a_0 f) = \|a_0 f\| \neq 0$. We extend χ to a pure state μ on $C^*(A, f)$.

By Lemma 2.5 there exist pure states $\mu_1, \mu_2, \ldots$ on A and $g_1, g_2, \ldots \in A_+$ such that $\|g_n\| = 1$ and $\mu = \mu_\omega | C^*(A, f)$ and $V_g(y) := gyg = \mu(y) g^2$ for all $y \in C^*(A, f)$ for $g := \pi_\omega(g_1, g_2, \ldots)$. In particular, $\|g\| = 1$, $g \geq 0$, $gfa_0g = \mu(fa_0)g^2 = \|fa_0\| g^2 \neq 0$. Thus $V_g|A = S_\omega|A$ for the sequence of nuclear c.p. contractions $S_1, S_2, \ldots \in CP_{\mathrm{nuc}}(A, A)$ with $S_n(a) := \mu_n(a) g_n^* g_n$.

By Proposition 1.3 (with $A = B = J$, $\mathcal{V} := CP_{\mathrm{nuc}}(A, A)$ and a, b, c, e, f, g replaced here by $0, g, 0, 0, h, 0$), there are nuclear c.p. contractions $T_1, T_2, \ldots \in CP_{\mathrm{nuc}}(A, A)$ and a positive contraction h in A^c such that $gh = g$ and $y \in A \to yh \in A_\omega$ is the restriction of T_ω to A.

Let $k := h + \mathrm{Ann}(A)$. Then $\rho_A(k \otimes y) = hy$ for $y \in A$ and $ke = (kf) + \mathrm{Ann}(A)$. It follows $k \in J_{\mathrm{nuc}}$ and $g\rho_A(ke \otimes a_0)g = ghfa_0g = ghfa_0g = gfa_0g \neq 0$, i.e., $ke \neq 0$. Hence, J_{nuc} is an *essential* ideal of $F(A)$.

(4): If A is nuclear, then $a \in A \to a = \rho(1 \otimes a) \in A_\omega$ is the restriction of $(\mathrm{id}_A)_\omega$ to A and id_A is nuclear. Thus $1 \in J_{\mathrm{nuc}}$, i.e. $F(A) = J_{\mathrm{nuc}}$.

Conversely, if $1 \in J_{\mathrm{nuc}}$, then there exists a sequence $(V_1, V_2, \ldots)$ of nuclear c.p. maps $V_n \colon A \to A$ such that the inclusion map $a \in A \mapsto a = \rho(1 \otimes a) \in A_\omega$ is the restriction of V_ω to A. This means that id_A can be approximated in point-norm by (convex combinations of) the nuclear c.p. maps V_n, $n = 1, 2, \ldots$. Hence, A is nuclear. □

Theorem 2.12 *Suppose that A is a separable C^*–algebra and let $F(A) := A^c/\mathrm{Ann}(A)$.*

(1) $F(A) \cong \mathbb{C}$ if and only if $A \otimes \mathcal{K} \cong \mathcal{K}$.

(2) If $F(A)$ is simple and $F(A) \not\cong \mathbb{C}$, then A is simple, purely infinite and nuclear.

(3) If A is simple, purely infinite and nuclear, then $F(A)$ and A_ω are simple and purely infinite, and $A \cong A \otimes \mathcal{O}_\infty$.

Proof. Recall that $F(A)$ is unital by Corollary 1.10 , that A is simple *and* purely infinite iff, A_ω is simple by Remark 2.4, and that A is unital iff $\mathrm{Ann}(A) = \{0\}$ iff $A^c = F(A)$ by Corollary 1.10 .

(1): $F(A) = \mathbb{C} \cdot 1$ implies that A is simple (cf. Lemma 2.1). Thus $F(A) \cong \mathbb{C}$, if and only if, $A \otimes \mathcal{K} \cong \mathcal{K}$ (cf. Proposition 2.10).

(2): If $F(A)$ is simple, then A is simple by Lemma 2.1. Thus Proposition 2.10 applies to A: $F(A)$ is simple and *stably finite* if and only if $F(A) = \mathbb{C} \cdot 1$. We get that $F(A)$ is *not* stably finite if $F(A)$ is simple and $F(A) \not\cong \mathbb{C}$. I.e., there is $n \in \mathbb{N}$ such that $F(A) \otimes M_n$ contains a copy of $\mathcal{O}_\infty$ unitally (because $F(A)$ is unital and *simple*).

It follows that A is purely infinite, indeed:
A is simple by Lemma 2.1, and is antiliminal by Proposition 2.10. Let $h\colon C_0((0,1],M_n) \cong M_n \otimes C_0((0,1]) \to A$ a *-morphism, $a := h(1_n \otimes f_0) \in A_+$ and let D the hereditary C^*–subalgebra of A generated by a. (Here $f_0(t) := t$ for $t \in [0,1]$.) Consider the natural embedding of $\mathcal{O}_\infty \otimes^{\min} C^*(a)$ into $(F(A) \otimes M_n) \otimes^{max} C^*(a) \cong F(A) \otimes^{max} (M_n \otimes C^*(a))$ given by $\mathcal{O}_\infty \subset F(A) \otimes M_n$ and compose with $\rho\colon F(A) \otimes^{max} A \to A_\omega$. Then we get a *-monomorphism $k\colon \mathcal{O}_\infty \otimes C^*(a) \to A_\omega$ with $k(1 \otimes a) = a$. Hence, a is *properly infinite* in A_ω, i.e. for every $\varepsilon > 0$ there exist $d_1, d_2 \in A_\omega$ with $d_i^* a d_j = \delta_{i,j}(a-\varepsilon)_+$ (cf. [24, prop. 3.3]). It implies that a is also properly infinite in A itself (use representing sequences for d_1 and d_2). Since every non-zero hereditary C^*–subalgebra of the antiliminal C^*–algebra A contains a non-zero n-homogenous element (cf. [29, lem. 6.7.1]), it follows that every non-zero element of A is properly infinite by [24, lem. 3.8]. Thus A is purely infinite by [24, lem. 4.2, prop. 5.4].

If $F(A)$ is simple then $F(A) = J_{\mathrm{nuc}}$ by Lemma 2.11(4). Hence, A is nuclear by Lemma 2.11(5).

(3): If A is simple, purely infinite and separable, then A_ω is simple and purely infinite by Remark 2.4.

For the rest of the proof it suffices to consider the case, where A is unital, because, if A is not unital, then there is a non-zero projection $p \in A$ such that $A \cong pAp \otimes \mathcal{K}$ (Zhang dichotomy, [33]). Thus $F(A) \cong F(pAp) = (pAp)^c \subset (pAp)_\omega$ by Corollary 1.10 .

If A is simple, purely infinite, separable, unital and nuclear, then, $F(A) = A^c \neq \mathbb{C}1$ by Proposition 2.10. Moreover, for $b \in A^c$ with $0 \le b \le 1$ and $\|b\| = 1$, there is an isometry $S \in A_\omega$ with $S^*bS = 1$ and $S^*aS = a$ for all $a \in A$. It follows $SS^* \in A^c$ and $S \in A^c$. Thus $F(A)$ is simple and purely infinite.

To get S, recall that the unital nuclear c.p. map $f \to f(1)$ from $C(\mathrm{Spec}(b), A) \cong C^*(b,1) \otimes A \cong C^*(b,A)$ into $A \subset A_\omega$ is approximately one-step inner (in A_ω) by [25, thm. 7.21]. Then use Proposition A.4 to pass from the approximate solutions of $x^*x - 1 = 0$, $x^*bx - 1 = 0$ and $x^*a_nx - a_n = 0$ for a dense sequence $(a_1, a_2, \ldots)$ in A_+ to the precise solution S.

It remains to show that $A \otimes \mathcal{O}_\infty \cong A$ if $F(A)$ is simple and purely infinite (and A is unital):

Then $F(A) = A^c$ contains a copy of $\mathcal{O}_\infty$ unitally. Thus $A^c \subset A_\omega$ contains a copy of $\mathcal{O}_\infty$ unitally. If the contractions $(u_1, u_2, \ldots)$ and $(v_1, v_2, \ldots)$ are representing sequences in $\ell_\infty(A)$ for s_1 and s_2 in $C^*(s_1, s_2, \ldots) = \mathcal{O}_\infty \subset A^c$ then $\lim_n \|d_n^* a d_n - a \otimes 1_2\| = 0$ for suitably chosen row-matrix $d_n := (u_{k_n}, v_{k_n}) \in M_{1,2}(A)$ and all $a \in A$. It follows $A \cong A \otimes \mathcal{O}_\infty$ by [25, prop. 8.4]. □

A variation of the proof of the implication "$F(A)$ simple and not stably stably finite" $\Rightarrow$ "A purely infinite" shows also:

Remark 2.13 *Suppose that A is simple, separable and is not stably projection-less, and that $F(A)$ is not stably finite. Then A is purely infinite. (Here we do not assume that $F(A)$ is simple!)*

Proof. We can suppose that that A is unital, because A is stably isomorphic to a unital C^*-algebra B and $F(A) \cong F(B) = B^c$. On the the other hand, there is $n \in \mathbb{N}$ such that there is a *-monomorphism ψ from the Toeplitz algebra $\mathcal{T} = C^*(t; t^*t = 1)$ into $M_n(A^c) \subset M_n(A_\omega) \cong (M_n(A))_\omega$.

Since $\mathcal{T}$ is (weakly) semi-projective, there is a also *-monomorphism $\varphi\colon \mathcal{T} \to M_n(A)$. In particular, $\mathcal{K} \otimes A$ contains an infinite projection q, and A is antiliminal.

Let $0 \neq a \in A_+$. Since A is antiliminal, there is a non-zero *-morphism $h\colon C_0(0,1] \otimes M_n \to \overline{aAa}$ by the Glimm halving lemma [29, lem. 6.7.1]. Let $d := h(f_0 \otimes e_{1,1}) \in A_+$ and $D := \overline{dAd}$, and recall that $\mathcal{K} \subset \psi(\mathcal{T}) \subset M_n(F(A))$. Thus,

$$\mathcal{K} \otimes A \cong (id_n \otimes \rho)(\mathcal{K} \otimes D) \subset M_n \otimes D_\omega \cong M_n(D)_\omega \,.$$

Since $\mathcal{K} \otimes A$ contains an infinite projection q, $M_n(D)_\omega$ contains an infinite projection p. Since the defining relations for infinite projections are semi-projective, we get that $M_n(D) \cong \overline{h(f_0 \otimes 1_n)Ah(f_0 \otimes 1_n)}$ and $\overline{aAa}$ contain infinite projections. $\square$

Remark 2.13 suggests the question:

Question 2.14 *Suppose that A is simple, separable and stably projection-less. Is 1 finite in $F(A)$?*

Remark 2.15 *Let $\mathcal{A}$ denote the simple purely infinite reduced free product C^*-algebra of two matrix-algebras with respect to non-central states as considered in [14]. Then $F(\mathcal{A})$ is finite and is not* not *simple.*

Proof. $\mathcal{A}$ is simple, purely infinite, unital and exact, but $F(A) = \mathcal{A}^c$ does not contain a non-unitary isometry (because $\mathcal{A}^c$ does not contain non-trivial projections by [14]). $F(\mathcal{A})$ is not simple by Theorem 2.12, because $\mathcal{A}$ is not nuclear. $\square$

Proposition 2.10 implies that $C^*_{\mathrm{red}}(F_2)^c = F(C^*_{\mathrm{red}}(F_2))$ is a *non-separable* algebra, moreover, its maximal Abelian C^*–subalgebras have perfect maximal ideal spaces and are not separable. $F(C^*_{\mathrm{red}}(F_2))$ is stably finite by Remark 2.13. The natural *-morphism from $C^*_{\mathrm{red}}(F_2)^c$ to the commutant $\cong \mathbb{C}$ of $C^*_{\mathrm{red}}(F_2)$ in the von-Neumann ultrapower $VN(F_2)^\omega$ defines a character on $C^*_{\mathrm{red}}(F_2)^c$. Thus $C^*_{\mathrm{red}}(F_2)^c$ is not simple (that also follows from Theorem 2.12).

Question 2.16 *Is $C^*_{\mathrm{red}}(F_2)^c$ non-Abelian? Is its essential ideal J_{nuc} simple?*

Remark 2.17 *Every separable nuclear C^*-algebra is in the UCT–class, if and only if, $[1] = 0$ in $K_0(F(D))$ for every simple p.i.s.u.n. algebra D with $K_*(D) = 0$.*

Proof. For simple p.i.s.u.n. algebras D holds that $D \cong \mathcal{O}_2$ if $[1] = 0$ in $K_0(F(D))$, i.e. if $\mathcal{O}_2$ is unitally contained in the simple purely infinite algebra $F(D) = D^c$ (cf. [23], or end of [20], or [21], or Section 4 for different proofs).

On the other hand: For every separable C^*-algebra A there are a separable commutative C^*-algebra C and a semisplit extension

$$0 \to SA \otimes \mathcal{K} \to \mathcal{E} \to C \otimes \mathcal{K} \to 0$$

with $K_*(\mathcal{E}) = 0$ (cf. [3, sec. 23]). $\mathcal{E}$ is in the UCT-class iff A is in the UCT-class. $\mathcal{E}$ is nuclear (respectively exact) iff A is nuclear (respectively exact).

For every nuclear (respectively exact) separable C^*–algebra $\mathcal{E}$ there is a KK-equivalent nuclear (respectively exact) separable unital C^*-algebra B, which contains a copy of $\mathcal{O}_2$ unitally and is KK-equivalent to $\mathcal{E}$ (the reader can find a suitable C^*–subalgebra of a corner of $(\mathcal{E}+\mathbb{C}\cdot 1)\otimes\mathcal{O}_\infty$). Let $h_0\colon B \hookrightarrow \mathcal{O}_2 \subset B$ an unital embedding of B into $\mathcal{O}_2$, and let $h := \mathrm{id}_B \oplus h_0 \in \mathrm{End}(B)$ (Cuntz addition). Then $h\colon B \to B$ satisfies $[h]_{KK} = [id_B]_{KK}$, and it is easy to see that the inductive limit

$$D := \mathrm{indlim}_n(h_n\colon B \to B)$$

with $h_n := h$ is simple, p.i. and nuclear (respectively exact). Since the unitary group of $\mathcal{O}_2$ is a contractible space, one can construct explicitly a unital *-morphism $k\colon D \to C_b([1,\infty), B)/C_0([1,\infty), B)$ that has a u.c.p. lift $V\colon D \to C_b([1,\infty), B)$ and that is an “inverse” of the unital embedding $h\colon B \to B \subset D \subset C_b([1,\infty), D)/C_0([1,\infty), D)$ with respect to an “unsuspended” and cp-liftable variant of E-theory. (This is the *crucial point* of the proof, because one has to overcome the discontinuity of the KK-functor with respect to inductive limits, cf. [21, chp. 11] for more details.)

It follows, that $B \to D$ define a KK-equivalences. Thus, a separable nuclear C^*–algebra A is in the UCT-class, if and only if, the above constructed (simple) p.i.s.u.n. algebra D with $K_*(D) = K_*(B) = K_*(\mathcal{E}) = 0$ is isomorphic to $\mathcal{O}_2$, and this is the case, if and only if, $[1] = 0$ in $K_0(F(D))$. □

Similar arguments show:

Remark 2.18 $K_0(D \otimes D) = 0$ *for all (simple) p.i.s.u.n. algebras D with $K_*(D) = 0$, if and only if, the Künneth theorem on tensor products (KTP) for the calculation of $K_*(B_1 \otimes B_2)$ holds for every pair (B_1, B_2) of* nuclear *C^*-algebras.*

There are separable purely infinite unital non-separable C^*–algebras A with $A^c \cong \mathbb{C}$ (e.g. the Calkin algebra by Corollary 2.21). This comes from the following Lemma and from Voiculescu’s description of the neutral element of $\mathrm{Ext}(B)$ for separable B (cf. proof of Proposition 2.20).

Lemma 2.19 *Let B a separable unital C^*-algebra. There exist a unital C^*-algebra D, a unital *-monomorphism $\eta\colon B \to D$ and a projection $p \in D$ such that*

$$\|(1-p)\eta(b)p\| = \|p\eta(b) - \eta(b)p\| = \mathrm{dist}(b, \mathbb{C}\cdot 1)$$

for every $b \in B$.

Proof. Let $D := B * E$ the unital full free C^*–algebra product of B and of $E := C^*(1, p = p^2 = p^*) \cong \mathbb{C} \oplus \mathbb{C}$. Then $\eta\colon\ b \mapsto b * 1$ and $\theta\colon\ e \to 1 * e$ are unital *-monomorphisms from B (respectively from E) into D. We identify $e \in E$ with $\theta(e)$. Note that, for all $b \in B$,

$$\max(\|(1-p)\eta(b)p\|,\ \|p\eta(b)(1-p)\|) = \|p\eta(b) - \eta(b)p\| \le \mathrm{dist}(b, \mathbb{C}\cdot 1).$$

Let $b \in B \setminus \mathbb{C}\cdot 1$, i.e. $\mathrm{dist}(b, \mathbb{C}\cdot 1) > 0$. Since $|z| \le \|b - z1\| + \|b\|$, there exists $z_0 \in \mathbb{C}$ with $|z_0| \le 2\|b\|$ such that $\|b - z_0 1\| = \mathrm{dist}(b, \mathbb{C}\cdot 1)$. $\mathrm{dist}(b, \mathbb{C}\cdot 1)$ is the norm of $b + \mathbb{C}\cdot 1$ in $B/(\mathbb{C}\cdot 1)$. Thus, there exists a linear functional φ on B with $\varphi(1) = 0$, $\|\varphi\| = 1$ and $\varphi(b - z_0 1) = \|b - z_0 1\|$. If we use the polar-decomposition $\varphi = |\varphi|(u\cdot)$ of φ in $B^* = (B^{**})_*$, cf. [29, prop. 3.6.7], we can see that there are a unital *-representation $\lambda\colon\ B \to \mathcal{L}(\mathcal{H})$ and vectors $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ such that $\varphi(c) = \langle \lambda(c)x, y\rangle$ for all $c \in B$. It follows $x \perp y$ and $\lambda(b - z_0 1)x = \|b - z_0 1\|y$. Let $q \in \mathcal{L}(\mathcal{H})$ denote the orthogonal projection onto $\mathbb{C}x$. Then $(1-q)\lambda(b)qx = \|b - z_0 1\|y$. Thus

$$\mathrm{dist}(b, \mathbb{C}\cdot 1) \le \|(1-q)\lambda(b)q\| \le \|(1-p)\eta(b)p\|$$

because there is a unital *-morphism $\kappa\colon\ D \to \mathcal{L}(\mathcal{H})$ with $\kappa(p) = q$ and $\kappa(\eta(b)) = \lambda(b)$. □

Proposition 2.20 *For every separable unital C^*–subalgebra B of the Calkin algebra $Q := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ (on $\mathcal{H} \cong \ell_2(\mathbb{N})$) there is a projection $P \in Q$ with $\|Pb - bP\| = \mathrm{dist}(b, \mathbb{C}\cdot 1)$ for all $b \in B$.*

Proof. Let D, $\eta\colon\ B \to D$ and $p \in D$ as Lemma 2.19. D can be unitally and faithfully represented on $\mathcal{H} := \ell_2(\mathbb{N})$ such that $D \cap \mathcal{K} = \{0\}$. Let $s_1, s_2 \in \mathcal{L}(\mathcal{H})$ two isometries with $s_1 s_1 * + s_2 s_2^* = 1$, $\pi\colon\ t \in \mathcal{L}(\mathcal{H}) \mapsto t + \mathcal{K} \in Q$ denotes the quotient map. There is a unitary $U \in Q$ with $U^* b U = \pi(s_1) b \pi(s_1)^* + \pi(s_2 \eta(b) s_2^*)$ for $b \in B$, by the generalized Weyl–von-Neumann theorem of Voiculescu, cf. [2]. Thus $P := U\pi(s_2 p s_2^*)U^*$ is a projection in Q that satisfies $\|Pb - bP\| = \mathrm{dist}(b, \mathbb{C}\cdot 1)$ for all $b \in B$. □

Proposition 2.20 implies:

Corollary 2.21 *Let $Q := \mathcal{L}(\ell_2)/\mathcal{K}(\ell_2)$. Then $Q^c = \mathbb{C}\cdot 1$.*

Proof. Let $b = \pi_\omega(b_1, b_2, \ldots) \in Q_\omega$ for $(b_1, b_2, \ldots) \in \ell_\infty(Q)$, B the unital C^*–subalgebra generated by $b_1, b_2, \ldots$ and $P \in Q$ as in Proposition 2.20. Then

$$Pb - bP = \pi_\omega(Pb_1 - b_1 P, Pb_2 - b_2 P, \ldots)$$

and $\|Pb - bP\| = \lim_\omega \mathrm{dist}(b_n, \mathbb{C}\cdot 1)$. It follows $b \in \mathbb{C}\cdot 1 \cong (\mathbb{C}\cdot 1)_\omega$ if $Pb = bP$. □

Question 2.22 *Is* $\mathcal{L}(\ell_2)^c = \mathbb{C}\cdot 1$*?*

The question leads to a study of the positive elements in $\mathrm{Ann}(\mathcal{K},\mathcal{K}_\omega)$: Note that $\mathcal{L}(\ell_2)^c \subset (\mathcal{K}(\ell_2)+\mathbb{C}\cdot 1)_\omega$ by Cor. 2.21, and that $F(\mathcal{K}+\mathbb{C}\cdot 1) = (\mathcal{K}+\mathbb{C}\cdot 1)^c = \mathrm{Ann}(\mathcal{K},\mathcal{K}_\omega) + \mathbb{C}\cdot 1$, because $F(\mathcal{K}) \cong F(\mathbb{C}) \cong \mathbb{C}$.

Remark 2.23 *If A is a simple C^*-algebra, then for every $g,h \in (A_\omega)_+$ with $\|g\| = \|h\| = 1$ there is $z \in A_\omega$ with $\|z\| = 1$ and $zz^*g = zz^*$, $z^*zh = z^*z$. In particular,* $\mathrm{Ann}(A)$ *does not contain a non-zero closed ideal* J of A_ω *if A is simple.*

3 The invariant $\mathrm{cov}(F(A))$ and applications.

Here we consider the case where A is separable and $F(A)$ contains a full simple C^*–algebra B of dimension $\mathrm{Dim}(B) > 1$. We show below that (in this case) A is strongly purely infinite if A is weakly purely infinite. Other considerations of this section are concerned with a (sufficient) condition on $F(A)$ under which A is weakly purely infinite if every (extended) lower semi-continuous 2-quasi-trace on A_+ is trivial (i.e. takes only the values 0 and ∞, cf. Proposition 3.7). The main result of this section is Theorem 3.10.

We say that $X \subset B_+$ is *full* if the ideal of B generated by X is dense in B, $b \in B_+$ is full if $X := \{b\}$ is full, and a *-morphism $h\colon C \to B$ is full if $h(C_+)$ is full in B.

Recall that a positive contraction $b \in B_+$ is *k-homogenous* if there is a *-morphism $h\colon C_0((0,1])\otimes M_k \to B$ such that $h(f_0\otimes 1_k) = b$. (Here $f_0(t) := t$ for $t \in (0,1]$, and 0 is k-homogenous for every $k \in \mathbb{N}$ by definition.)

Definition 3.1 *We define $\mathrm{cov}(B,m) \in \mathbb{N}\cup\{+\infty\}$ for a unital C^*–algebra B (and $m > 1$) as the minimum of the set of $n \in \mathbb{N}$ with the property that there are $a_1,\dots,a_n \in B_+$ and $d_1,\dots,d_n \in B$ with $\sum_j d_j^* a_j d_j = 1$ and that a_j is the sum $a_j = \sum_{i=1}^{l_j} a_{j,i}$ of mutually orthogonal $k_{j,i}$-homogenous elements $a_{j,i} \in B_+$ with $k_{j,i} \geq m$ for $j = 1,\dots,n$ and $i = 1,\dots,l_j$. (The minimum of an empty subset of $\mathbb{N}$ is considered as $+\infty$.) In other words:*
*$\mathrm{cov}(B,m) \leq n < \infty$, if and only if, there are finite-dimensional C^*–algebras $F_1,\dots,F_n$, *-morphisms $h_j\colon C_0((0,1])\otimes F_j \to B$ and $d_1,\dots,d_j$ such that every irreducible representation of F_j is of dimension $\geq m$ and $1 = \sum_j d_j^* h_j(f_0\otimes 1) d_j$ for $j = 1,\dots,n$.*

We define:
$$\mathrm{cov}(B) := \sup_m \mathrm{cov}(B,m).$$

One can replace the F_j in the definition of $\mathrm{cov}(B,m)$ by those unital C^*-subalgebras $G_j \subset F_j$ which have, moreover, only irreducible representations $D\colon G_j \to \mathcal{L}(\mathcal{H})$ of dimension $m \leq \mathrm{Dim}(\mathcal{H}) < 2m$ and a center $\mathcal{Z}(G_j)$ of dimension $< m$.

It is useful to note that the definition of $\mathrm{cov}(B,m)$ can be described by weakly semi-projective relations (e.g. for the study of cov of ultrapowers or of continuity properties of $B \mapsto \mathrm{cov}(B)$, Remark 3.3 and below):

Remark 3.2 *We can suppose that the $d_1,\ldots,d_n$ and $h_j\colon C_0((0,1]F_j) \to B$ of Definition 3.1 satisfy in addition the weakly semi-projective relations*

$$d_1^* d_1 + \ldots + d_n^* d_n = 1 \qquad \text{and} \qquad h_j(f_0 \otimes 1)d_j = d_j$$

for $j = 1,\ldots,n := \mathrm{cov}(B,m)$.

It follows:

$\mathrm{cov}(B) \le n$, if and only if, there is a unital *-morphism from the "locally" weakly semi-projective C^*–algebra $\mathcal{A}_n := \mathcal{A}_{n,1} * \mathcal{A}_{n,2} * \cdots$ into B.

Here $\mathcal{A}_{n,k}$ denotes the (weakly semi-projective) universal unital C^*–algebra generated by n copies $h_j(C_k) \subset B$ of $C_k := C_0((0,1],(M_2 \oplus M_3)^{\otimes k})$ and elements $d_1,\ldots,d_n$ with relations $d_1^*d_1 + \ldots + d_n^*d_n = 1$ and $h_j(f_0 \otimes 1)d_j = d_j$ for $j = 1,\ldots,n$, and $\mathcal{A}_{n,1} * \mathcal{A}_{n,2} * \mathcal{A}_{n,3} * \cdots$ means the unital universal (="full") free product of unital C^*–algebras.

Proof. To get weakly semi-projective relations, let $k_j\colon C_0((0,1]) \otimes F_j \to B$ and $e_1,\ldots,e_n$ such that $1 = \sum_j e_j^* k_j(f_0 \otimes 1)e_j$ (where F_j is finite-dimensional and every irreducible representation of F_j is of dimension $\ge m$ for $j = 1,\ldots,n$). Then $1/2 < g := \sum_j e_j^* k_j((f_0 - \delta)_+ \otimes 1)e_j \le 1$ for suitable $\delta \in (0,1)$. Let $d_j := k_j((f_0 - \delta)_+ \otimes 1)^{1/2} e_j g^{-1/2}$ then $d_1^*d_1 + \ldots + d_n^*d_n = 1$. There is a *-morphism $\psi\colon C_0(0,1] \to C_0(0,1]$ with $\psi(f_0) = g_\delta$ where $g_\delta(t) := \min(t/\delta, 1)$. Let $h_j := k_j \circ (\psi \otimes \mathrm{id}_{F_j})$, then $h_j(f_0 \otimes 1)d_j = d_j$.

The new relations are away from the old relations, but they do the same job (for the definition of $\mathrm{cov}(B,n)$) and they are *weakly semi-projective* in the category of unital C^*–algebras:

The relation $\sum d_j^* d_j = 1$ is semi-projective in the category of unital C^*–algebras and the defining relations of $C_0((0,1],F_j)$ are even projective in the category of all C^*–algebras (cf. [27, thm. 10.2.1], [28], or the elementary proof in [6, sec. 2.3]). Let $d_1,\ldots,d_n$ contractions with $\sum_j d_j^* d_j = 1$, and $h_j\colon C_0((0,1],F_j) \to B$ *-morphisms with $\|h_j(f_0 \otimes 1_{F_j})d_j - d_j\| < \delta^2/n$ for some $\delta \in (0,1/2)$. Then

$$\|1 - \delta^{-1} \sum d_j^* c_j d_j\| < \delta < 1/2$$

for $c_j := h_j(f_0 \otimes 1)$, because $\|d_j\| \le 1$ and $\delta - \max(0, t - (1-\delta)) \le 1 - t$ for $0 < \delta < 1$, $t \in [0,1]$ (i.e. because $1 - \delta^{-1}(c_j - (1-\delta))_+ \le \delta^{-1}(1 - c_j)$).

Let $g_0 := \delta^{-1}(f_0 - (f_0 - (1-\delta)))_+$, $w := (\sum_j \delta^{-1} d_j^* c_j d_j)^{-1/2}$, $d_j' := \delta^{-1/2} c_j^{1/2} d_j w$ and define *-morphisms $h_j'\colon C_0((0,1],F_j) \to B$ by $h_j'(f_0^k \otimes x) := h_j(g_0^k \otimes x)$ for $k \in \mathbb{N}$, $x \in F_j$ and $j = 1,\ldots,n$.

The new system $d_j' \in B$, $h_j'\colon C_0((0,1],F_j) \to B$ satisfies $h_j'(f_0 \otimes 1)d_j' = d_j'$ and the canonical generators differ form d_j and the old images by h_j of the canonical generators of $C_0((0,1],F_j)$ by $\|f_0 - g_0\| < \delta^{1/2}$ and $\|w - 1\| < \delta^{1/2}$.

The use of $(M_2 \oplus M_3)^{\otimes k}$ instead of F_j (with minimal dimension of irreducible representations $\geq m$ in an asymptotic sense) is possible, because every irreducible representation of $(M_2 \oplus M_3)^{\otimes k}$ has dimension $\geq 2^k$ and there is a unital *-morphism from $(M_2 \oplus M_3)^{\otimes k}$ into M_ℓ for *all* $\ell > 6^k$, because $1 = 2^k x - 3^k y = 3^k(2^k - y) - 2^k(3^k - x)$ with $1 < x < 3^k$ and $1 < y < 2^k$ (but $6^k + 1$ is not the smallest value for ℓ with the property that all numbers $\ell, \ell+1, \ell+2 \ldots,$ are sums $\sum_{0 \leq j \leq k} n_j 2^j 3^{k-j}$ with $n_j \in \mathbb{N} \cup \{0\}$). □

One can read off some properties of $\mathrm{cov}(B, m)$ and $\mathrm{cov}(B)$ straight from Definition 3.1 and Remark 3.2:

Remark 3.3 *The maps* $(B, m) \mapsto \mathrm{cov}(B, m)$ *and* $B \mapsto \mathrm{cov}(B)$ *on unital* C^**–algebras* B *have the properties:*

(1) $\mathrm{cov}(B, m) \leq \mathrm{cov}(B, m+1)$*,*
(2) $\mathrm{cov}(C, m) \leq \mathrm{cov}(B, m)$ *if there exist a *-morphism* $\psi \colon B \to C$ *such that* $\psi(1) = 1$*, or that* $\psi(1_B)$ *is properly infinite and is full in* C*.*
In particular, $\mathrm{cov}(\mathcal{O}_\infty, m) = \mathrm{cov}(\mathcal{O}_2, m) = \mathrm{cov}(M_{2^\infty}, m)\mathrm{cov}(M_{2^m}, m) = 1$ *for* $m > 1$*.*
(3) If $B_1, B_2, \ldots$ *is a sequence of unital* C^**–algebras, then, for every* $m \in \mathbb{N}$*,* $\mathrm{cov}(\prod_\omega \{B_1, B_2, \ldots), m\} = \lim_\omega \mathrm{cov}(B_n, m)$ *and* $\mathrm{cov}(\prod_\omega \{B_1, B_2, \ldots\}) = \lim_\omega \mathrm{cov}(B_n)$*.* [1]
In particular, $\mathrm{cov}(B_\omega, m) = \mathrm{cov}(B, m)$*, and* $\mathrm{cov}(B_\omega) = \mathrm{cov}(B)$*.*
(4) $\mathrm{cov}(B, m) = \inf_n \mathrm{cov}(B_n, m)$*,* $\mathrm{cov}(B) = \sup_m \inf_n \mathrm{cov}(B_n, m) \leq \sup_n \mathrm{cov}(B_n)$*, if* $B_1 \subset B_2 \subset \cdots \subset B$ *are unital* C^**-subalgebras with* $\bigcup_n B_n$ *dense in* B*.*
(5) Suppose that 1_B *is finite. Then* $\mathrm{cov}(B, m) = 1$*, if and only if, there are a* C^**–algebra* A_m *of finite dimension and a unital *-morphism* $h_m \colon A_m \to B$*, where every irreducible representation of* A_m *has dimension* $\geq m$*.*
(6) $\mathrm{cov}(B) = 1$ *if* 1_B *is properly infinite.*
(7) $\mathrm{cov}(B) = \mathrm{cov}(B, m) = \infty$ *if every irreducible representation of* B *has dimension* $\leq m - 1$*.*
(8) $\mathrm{cov}(B, m) < \infty$ *for every* $m \in \mathbb{N}$ *if* B *is strictly antiliminal.*
(9) If B *has real rank zero, then* $\mathrm{cov}(B, m) = 1$ *if and only if there exist* $1 \leq p < m$*,* $F = M_{k_1} \oplus \ldots \oplus M_{k_p} \subset B$ *with* $m \leq k_j < 2m$ *for* $j = 1, \ldots, p$ *and an isometry* $d \in B$ *with* $1_F d = d$*.*
(10) Every separable C^**–subalgebra* $B_1 \subset B$ *of a unital* C^**–algebra* B *is contained in a unital* C^**-subalgebra* $1_B \in B_2 \subset B$ *with* $\mathrm{cov}(B_2, m) = \mathrm{cov}(B, m)$ *for all* $m \in \mathbb{N}$*.*

Proof. (1),(2), and (7) follow immediately from Definition Definition 3.1. (5) and (9) follow from Remark 3.2. (3): Use Remark 3.2 and (2).
(4): Use (2) for $\leq$ and (i,iii) for $\geq$. (6): Use (2). (8): Use the Glimm halving lemma [29, lem. 6.7.1] to see that 1_B is majorized by a finite sum of m-homogenous positive contractions. (10): Use Remark 3.2 and (4). □

[1] We extended $\lim_\omega$ to all sequences $(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ with $\alpha_n \in [0, \infty]$.

Proposition 3.4 *Suppose that A is an inductive limit* $\mathrm{indlim}(h_n\colon A_n \to A_{n+1})$ *of separable C^*–algebras $A_1, A_2, \ldots$. Then*

$$\mathrm{cov}(F(A), m) \le \lim \inf_{n\to\infty} \mathrm{cov}(F(A_n), m)\,.$$

In particular, $\mathrm{cov}(F(A)) \le \liminf \mathrm{cov}(F(A_n))$.

Proof. Remark 3.3(4) is not applicable, because the $F(A_n)$ are not related to $F(A)$. But Proposition 1.14 works:
$\mathrm{cov}(F(A_n/I)) \le \mathrm{cov}(F(A_n))$ for closed ideals I of A_n by Remark 3.3(2), because $F(A_n/I)$ is a quotient of $F(A_n)$ by Remark 1.15. Thus, we may suppose that $A_1, A_2, \ldots \subset A$ and $\bigcup_n A_n$ is dense in A.

By Proposition 1.14, for every $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ and for every separable unital C^*–subalgebra $E \subset \prod_\omega \{F(A_1), F(A_2), \ldots\}$, there is a unital *-morphism $E \to F(A)$. Thus $\mathrm{cov}(F(A)) \le \mathrm{cov}(E)$ by Remark 3.3(2). E can be found such that $\mathrm{cov}(E, m) = \mathrm{cov}(\prod_\omega \{F(A_1), F(A_2), \ldots\}, m)$ for every $m \in \mathbb{N}$ by Remark 3.3(10). Now apply Remark 3.3(3) and note that for $\alpha_1, \alpha_2, \ldots \in [0, \infty]$ there is a free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ with $\lim_\omega \alpha_n = \liminf_{n\to\infty} \alpha_n$. □

Proposition 3.5 *If a unital nuclear separable C^*–algebra B has decomposition rank* $\mathrm{dr}(B) < \infty$ *(cf. [26, def. 3.1]) and if B has no irreducible representation of finite dimension, then* $\mathrm{cov}(B) \le \mathrm{dr}(B) + 1$.

Proof. This follows easily from the definition of the decomposition rank [26, def. 3.1] by [26, prop. 5.1], which implies that the c.p. contractions $\varphi_{r_i}\colon M_{r_i} \to B$ of strict order zero arising in n-decomposable c.p. approximations $\varphi\colon \bigoplus_{i=1}^s M_{r_i} \to B$ and $\psi\colon B \to \bigoplus_{i=1}^s M_{r_i}$ of [26, def. 3.1] can be chosen such that (eventually) $\min\{r_1, \ldots, r_s\} \ge q$ if $\psi \circ \varphi \to \mathrm{id}_B$ (in point-norm) and B has no irreducible representation of dimension $\le q$.

Indeed, suppose that $\varphi_n\colon C_n \oplus D_n \to B$ and $\psi_n\colon B \to C_n \oplus D_n$ are completely positive contractions with suitable C^*-algebras C_n and D_n such that $\varphi_n \circ \psi_n$ tends to id_B in point-norm, $\lim_n \|\psi_n(b^*b) - \psi_n(b^*)\psi_n(b)\| = 0$ for all $b \in B$, ψ_n is unital and every irreducible representation of C_n has dimension $\le q$. Then the ultrapower $C := \prod_\omega \{C_1, C_2, \ldots\}$ has only irreducible representations of dimension $\le q$ and the restriction to B of the ultrapower $U\colon B_\omega \to C$ of the completely positive contractions $p_1 \circ \psi_n\colon B \to C_n$ is a unital *-morphism from B into C. The latter contradicts that B has no irreducible representation of dimension $\le q$. □

Recall that a quasi-trace $\tau\colon A_+ \to [0, \infty]$ is *trivial* if it takes only the values 0 and $+\infty$. The following is a reformulation of [24, prop. 5.7].

Remark 3.6 *Suppose that every lower semi-continuous 2-quasi-trace on A_+ is trivial. Then, for every $n \in \mathbb{N}$, $a \in A_+ \setminus \{0\}$ and $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$ there are $d_1, \ldots, d_n$ in $M_k \otimes A$ such that $d_i^*(1_k \otimes a)d_j = \delta_{i,j}(1_k \otimes (a - \varepsilon)_+)$ for $i, j = 1, \ldots, n$.*

Proposition 3.7 *If* $\mathrm{cov}(F(A)) < \infty$ *and if every lower semi-continuous 2-quasi-trace on A_+ is trivial, then A is weakly purely infinite.*

Proof. Let $m := \mathrm{cov}(F(A))$ and $n := 2m$. Below we show that, for $a \in A_+$ and $\varepsilon > 0$, there exists a matrix $V = [v_{j,q}]_{m,n} \in M_{m,n}(A_\omega)$ such that $V^*(a \otimes 1_m)V = (a-\varepsilon)_+ \otimes 1_n$. It follows that A is pi-m in the sense of [25, def. 4.3] (because one can use representing sequences and the isomorphism $M_{m,n}(A_\omega) \cong (M_{m,n}(A))_\omega$). Thus A is weakly purely infinite.

Let $k_0 \in \mathbb{N}$ as in Remark 3.6 for $a \in A_+$ and $\varepsilon > 0$. We find finite-dimensional C^*-algebras $F_1, \ldots, F_m$, *-morphisms $h_j\colon C_0((0,1]) \otimes F_j) \to F(A)$ and elements $g_j \in F(A)$ such that $\sum_j g_j^* b_j g_j = 1$ for $b_j := h_j(f_0 \otimes 1_{F_j})$, and that F_j has only irreducible representations of dimension $\geq k_0$ for $j = 1, \ldots, m$. (We allow $b_j = 0$ for $\mathrm{cov}(F(A), k_0) \leq j \leq m$, to simplify notation.)

For every $j = 1, \ldots, m$ we find by Remark 3.6 $d_{j,1}, \ldots, d_{j,n} \in F_j \otimes A$ such that, for $1 \leq j \leq m$ and $1 \leq p, q \leq n$

$$d_{j,p}^*(1_{F_j} \otimes a)d_{j,q} = \delta_{p,q}(1_{F_j} \otimes (a-\varepsilon)_+)\,.$$

Since $g_j \otimes 1 \in \mathcal{M}(F(A) \otimes A)$, we can define, for $j = 1, \ldots, m$ and $q = 1, \ldots, n = 2m$,

$$v_{j,q} := \rho(h_j \otimes \mathrm{id}_A(f_0 \otimes d_{j,q})(g_j \otimes 1))\,.$$

A straight calculation shows

$$v_{j,p}^* a v_{j,q} = \delta_{p,q}\rho\left(g_j^* b_j g_j \otimes (a-\varepsilon)_+\right)\,,$$

i.e. $V := [v_{j,q}]_{m,n}$ is as desired. □

Now we study situations where we can deduce strong pure infiniteness from weak pure infiniteness.

Lemma 3.8 *If A is purely infinite and $F(A)$ contains two orthogonal full hereditary C^*-subalgebras, then A is strongly purely infinite.*

Proof. Let $a, b \in A_+$ and $\varepsilon > 0$, $\delta := \varepsilon/2$. If $E_1, E_2 \subset F(A)$ are orthogonal full hereditary C^*–subalgebras, there are $e_i \in (E_i)_+$ and $g_j, h_k \in F(A)$ ($i = 1, 2$, $j = 1, \ldots, m$, $k = 1, \ldots, n$) such that $1 = \sum_j g_j^*(e_1)^2 g_j$ and $1 = \sum_k h_k^*(e_2)^2 h_k$. Thus, $a^2 = \rho(1 \otimes a^2)$ (respectively b^2) is in the ideal of A_ω generated by $\rho(e_1 \otimes a)$ (respectively $\rho(e_2 \otimes b)$), because, e.g. $1 \otimes a^2$ is in the ideal of $F(A) \otimes^{\max} A$ generated by $e_1 \otimes a$. Let $u_i \in (A^c)_+ \subset A_\omega$ with $e_i = u_i + \mathrm{Ann}(A)$. Then $u_1 a b u_2 = \rho(e_1 e_2 \otimes ab) = 0$ and a^2 (respectively b^2) is in the closed ideal of A_ω generated by $u_1 a^2 u_1 = \rho((e_1)^2 \otimes a^2)$ (respectively $u_2 b^2 u_2$).

Since A is purely infinite, A_ω is again purely infinite, cf. [24]. It follows that there are $f_1, f_2 \in A_\omega$ such that $f_1 u_1 a^2 u_1 f_1 = (a^2 - \delta)_+$ and $f_2 u_2 b^2 u_2 f_2 = (b^2 - \delta)_+$.

With $v_i := f_i u_i$ holds $\|v_1^* a^2 v_1 - a^2\| < \varepsilon$, $\|v_2^* b^2 v_2 - b^2\| < \varepsilon$ and $v_1^* a b v_2 = 0$ in A_ω. With help of representing sequences for v_1 and v_2 in $\ell_\infty(A)$ we find $d_1, d_2 \in A$ with $\|d_1^* a^2 d_1 - a^2\| < \varepsilon$, $\|d_2^* b^2 d_2 - b^2\| < \varepsilon$ and $\|d_1^* a b d_2\| < \varepsilon$. This means that A is strongly purely infinite, cf. [6], [25]. □

Lemma 3.9 *If $F(A)$ contains a full 2-homogenous element, then A has the global Glimm halving property of [5] (cf. also [6]).*

If, in addition, A is weakly purely infinite, then A is strongly purely infinite.

Proof. Let $a \in A_+$, $\varepsilon \in (0,1)$, $\delta := \varepsilon^2/2$ and $D := \overline{aAa}$. By assumption, there exists $b \in F(A)$ and $d_1, \ldots, d_n \in F(A)$ with $b^2 = 0$ and $\sum_j d_j^* b^* b d_j = 1$.

Let $e_j := \rho(d_j \otimes a^{1/2})$, $c \in A^c$ with $b = c + \mathrm{Ann}(A)$ and $f := ca = \rho(b \otimes a^{1/2})$. Then $f^2 = 0$ and $a^2 = \sum_j e_j f^* f e_j$. f and $e_1, \ldots, e_n$ are in the hereditary C^*–subalgebra of A_ω generated by a, in particular they are in D_ω. Let $h = (h_1, h_2, \ldots) \in \ell_\infty(D)$ self-adjoint with $\pi_\omega(h) = f^*f - ff^*$, $g = (g_1, g_2, \ldots) \in \ell_\infty(D)$ with $\pi_\omega(g) = f$, and let $u_k := (h_k)_-^{1/k} g_k (h_k)_+^{1/k}$ for $k = 1, 2, \ldots$. Then $u_k \in D$, $u_k^2 = 0$ and $\pi_\omega(u_1, u_2, \ldots) = f$.

There exists $k \in \mathbb{N}$ and $v_1, \ldots, v_n \in D$ such that $\|a^2 - \sum_j v_j^* u_k^* u_k v_j\| < \delta$ (use representing sequences for $e_1, \ldots, e_n \in D_\omega$).

By [25, lem. 2.2] there is a contraction $z \in A$ such that $\sum_j w_j^* u_k^* u_k w_j = (a - \varepsilon)_+$ for $w_j := v_j z h(a)$ with $h(t) := \max(0, t - \varepsilon)^{1/2} / \max(0, t^2 - \delta)^{1/2}$ on $[0, \infty]$. Hence $(a - \varepsilon)_+$ is in the ideal generated by u_k.

Thus A has the global Glimm halving property of [5].

By [6] (and [5]) A is purely infinite, if and only if, A is weakly purely infinite and has the global Glimm halving property. Then A is moreover strongly purely infinite, by Lemma 3.8. □

Theorem 3.10 *If every lower semi-continuous 2-quasi-trace on A_+ is trivial and if $F(A)$ contains a simple C^*-subalgebra B with $1 \in B$ and*

$$\mathrm{cov}(B \otimes^{\max} B \otimes^{\max} \cdots) < \infty,$$

then A is strongly purely infinite.

Proof. There is a unital *-morphism from $B \otimes^{\max} B \otimes^{\max} \cdots$, into $F(A)$ by Corollary 1.13. Since $\mathrm{cov}(B \otimes^{\max} B \otimes^{\max} \cdots) < \infty$, it follows $B \neq \mathbb{C}$ and $\mathrm{cov}(F(A)) < \infty$, cf. Remarks 3.3(ii,vii).

Thus Proposition 3.7 applies, and A is weakly purely infinite. The Glimm halving lemma (cf. [29, lem. 6.7.1]) applies to B or to $B \otimes B \otimes \cdots$ if $B \cong M_n$ with $n > 2$. Thus Lemma 3.9 applies, and A is strongly purely infinite. □

Let $\mathcal{I}(m,n) \subset C([0,1], M_{mn})$ for $m, n > 1$ denote the dimension-drop algebra given by the subalgebra of $C([0,1], M_m \otimes M_n)$ of continuous functions $f\colon [0,1] \to M_m \otimes M_n$ with $f(0) \in M_m \otimes 1_n$ and $f(1) \in 1_m \otimes M_n$. In the following we use only that the *Jian–Su algebra* $\mathcal{Z}$ (cf.[18]) is a simple unital C^*–algebra, that $\mathcal{Z}$ is an inductive limit of $\mathcal{I}(m_k, n_k)$ with $\min(m_k, n_k) \to \infty$ for $k \to \infty$, that $\mathcal{Z}$ does not contain a non-trivial projection, and that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$.

Lemma 3.11 $\mathrm{cov}(\mathcal{I}(m,n), \min(n,m)) \leq 2$ *and* $\mathrm{cov}(\mathcal{Z}) = 2$.

The proof follows from Proposition 3.5 and parts (2),(4) and (5) of Remark 3.3, because $\mathrm{dr}(\mathcal{I}(m,n)) = 1$. But we give an independent proof.

Proof. Let $a \in C([0,1], M_{mn})_+$ given by $a(t) := t1_{mn}$. Then $a \in \mathcal{I}(m,n)$, $a^{1/3}$ is n-homogenous and $(1-a)^{1/3}$ is m-homogenous in $\mathcal{I}(m,n)$. $1 = d_1^* a^{1/3} d_1 + d_2^*(1-a)^{1/3} d_2$ for $d_1 = a^{1/3}$ and $d_2 = (1-a)^{1/3}$. Hence, $\mathrm{cov}(\mathcal{I}(m,n), \min(n,m)) \leq 2$.

For $k \in \mathbb{N}$ there are $n, m \geq k$ such that there is a unital *-morphism from $\mathcal{I}(m,n)$ into $\mathcal{Z}$. Thus, $\mathrm{cov}(\mathcal{Z}, k) \leq 2$ by Remark 3.3(2).

$\mathrm{cov}(\mathcal{Z}, 2) > 1$ by Remark 3.3(5), because $1_{\mathcal{Z}}$ is finite and does not contain a non-trivial projection. Hence $\mathrm{cov}(\mathcal{Z}, k) = 2$ for $k = 2, 3, \ldots$. □

Corollary 3.12 *$A \otimes \mathcal{Z}$ is strongly purely infinite if every lower semi-continuous 2-quasi-trace on A_+ is trivial.*

Proof. Then every l.s.c. 2-quasi-trace $(A \otimes \mathcal{Z})_+ \to [0, \infty]$ is trivial. $F(A \otimes \mathcal{Z})$ contains a copy of $\mathcal{Z}$ unitally, because $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$. The result follows from Lemma 3.11, Remark 3.3(2) and Proposition 3.7. □

Corollary 3.13 *If A is simple and separable, and is neither stably finite nor purely infinite, then there is $k_0 < \infty$ such that, for all $m, n \geq k_0$, there is no unital *-morphism from $\mathcal{I}(m,n)$ into $F(A)$.*

Proof. The assumptions imply that every l.s.c. 2-quasi-trace on A_+ is trivial. Since A is simple and is not purely infinite, A is not weakly purely infinite. Thus $\mathrm{cov}(F(A)) = \infty$ by Proposition 3.7.

Let $k_0 := \inf\{k \in \mathbb{N}\,;\ \mathrm{cov}(F(A), k) > 2\}$. If $h\colon \mathcal{I}(m,n) \to F(A)$ is a unital *-morphism, then $\mathrm{cov}(F(A), \min(m,m)) \leq \mathrm{cov}(\mathcal{I}(m,n), \min(m,n)) \leq 2$ by Remark 3.3(2) and Lemma 3.11. Thus $\min(m,n) < k_0$. □

Corollary 3.14 *Let $\mathcal{R}$ an example of a simple separable unital nuclear C^*–algebra that is neither stably finite nor purely infinite (cf. Rørdam [30]).*
Then $\mathrm{cov}(F(\mathcal{R})) = \infty$, $F(\mathcal{R})$ is stably finite, and $F(\mathcal{R})$ does not contain a simple unital C^*–subalgebra $B \neq \mathbb{C} \cdot 1$.

Proof. $\mathrm{cov}(F(\mathcal{R})) = \infty$ by Proposition 3.7. $F(\mathcal{R})$ must be stably finite by Remark 2.13, because $\mathcal{R}$ is not (locally) purely infinite. There is no unital C^*–subalgebra $B \neq \mathbb{C} \cdot 1$ of $F(\mathcal{R})$, such that $B \otimes \mathcal{R}$ is weakly purely infinite (i.e. n–purely infinite for some n), because otherwise $a \otimes 1_n$ is properly infinite in $\rho(B \otimes \mathcal{R}) \otimes M_n \subset \mathcal{R}_\omega \otimes M_n$, for every $a \in \mathcal{R}$, which implies that $\mathcal{R}$ is n–purely infinite, a contradiction. Suppose that $B \neq \mathbb{C} \cdot 1$ is a simple C^*–subalgebra of $F(\mathcal{R})$ then there is also an antiliminal simple algebra B unitally contained in $F(\mathcal{R})$ (cf. Corollary 1.13). But then $B \otimes \mathcal{R}$ is purely infinite by [6, cor. 3.11], □

Question 3.15 *Does $F(\mathcal{R})$ contain a strictly antiliminal unital C^*–subalgebra B?*

A *positive* answer to Question 3.15 would show that:
(1) there exists a separable strictly antiliminal stably finite unital C^*–algebra that does not contain a non-trivial simple C^*–algebra unitally (because of 3.14 and because every strictly antiliminal unital C^*–algebra is the inductive limit of its separable strictly antiliminal C^*–subalgebras), and
(2) there are locally purely infinite algebras that are not weakly purely infinite (because $B \otimes \mathcal{R}$ is not weakly p.i. by the argument in the proof of 3.14, but is locally p.i. by [6, cor. 3.9(iv)]).

Question 3.16 *Suppose that A is a simple stably projection-less separable C^*–algebra and that $M_2 \oplus M_3$ is unitally contained in $F(A)$.*
Is A approximately divisible?

If $1_{F(A)} \in M_2 \oplus M_3 \subset F(A)$, then there is a unital *-morphism from the infinite tensor product

$$E := (M_2 \oplus M_3) \otimes (M_2 \oplus M_3) \otimes \cdots$$

into $F(A)$ by Corollary 1.13. E contains a simple AF–algebra D unitally (communicated by M. Rørdam, May 2004). Every simple unital AF–algebra contains a copy of $M_2 \oplus M_3$ unitally. Thus, the property in the question equivalently means that $F(A)$ contains a copy of a simple AF–algebra unitally. Every simple unital AF–algebra absorbs a copy of $\mathcal{Z}$, cf. [18]. It follows that $A \cong A \otimes \mathcal{Z}$ (cf. Section 4).

The estimate for $\mathrm{cov}(F(A), m)$ in Proposition 3.4 is not optimal, e.g. $\mathrm{cov}(F(M_{2^\infty}), m) = 1$ and $\mathrm{cov}(F(M_{2^k}), 2) = \infty$ for all $k \in \mathbb{N}$, because $F(M_{2^\infty})$ contains a copy of M_{2^∞} unitally and $F(M_{2^k}) = F(\mathbb{C}) = \mathbb{C}$.
Since $F(M_{2^k}, M_{2^{k+m}}) = M_{2^m}$, one gets better estimates if one considers in some case also also $\mathrm{cov}(F(A_{n_k}, A_{n_{k+1}}), m)$ for suitable $n_1 < n_2 < \cdots$.

4 Self-absorbing subalgebras of $F(A)$.

Suppose that $1_{F(A)} \in D \subset F(A)$ is a simple separable and nuclear unital C^*-subalgebra of $F(A)$. Then $\mathcal{D} := D^{\otimes \infty} := D \otimes D \otimes \cdots$ is unitally contained in $F(A)$ by Corollary 1.13.

Here we are interested in the question, when this implies that there is an isomorphism ψ from A onto $A \otimes \mathcal{D}$, and when ψ can be found such that ψ is approximately unitarily equivalent to the morphism $a \in A \mapsto a \otimes 1 \in A \otimes \mathcal{D}$.

Definitions 4.1 *Let A and D C^*-algebras, where D is unital. We say that A is* D-absorbing *(in a strong sense) if there exists an isomorphism ψ from A onto $A \otimes D$ that is approximately unitarily equivalent to the morphism $a \mapsto a \otimes 1$ (by unitaries in $\mathcal{M}(A \otimes D)$). We call A* stably D-absorbing *if $\mathcal{K} \otimes A$ is D-absorbing.*

A unital C^*-algebra D is self-absorbing *if D is D-absorbing.*

D has approximately inner flip *if the flip automorphism of $D \otimes D$ is approximately inner.*

If there exists $A \neq \{0\}$ such that A is D-absorbing, then D is simple and nuclear (cf. Lemma 4.9). Conversely, if D is simple, separable, unital, and nuclear, then $\mathcal{O}_2$ is D-absorbing (by classification theory).

If A and D are separable, D simple, unital and nuclear and A is $\mathcal{D}$-absorbing, then $D^{\otimes\infty}$ is unitally contained in $F(A)$. (cf. Proposition 4.11). This property is not enough to ensure that A is D-absorbing, as the following remark shows (see Appendix C for details):

Remark 4.2 *The infinite tensor product $\mathcal{O}_n \otimes \mathcal{O}_n \otimes \cdots$ is unitally contained in $F(\mathcal{O}_n)$. But the maps $\eta_{1,\infty}\colon\ a \mapsto a \otimes 1 \otimes 1 \otimes \ldots$ and $\eta_{2,\infty}\colon\ a \mapsto 1 \otimes a \otimes 1 \otimes \ldots$ from $\mathcal{O}_n$ into $\mathcal{O}_n \otimes \mathcal{O}_n \otimes \ldots$ $(i = 1, 2)$ are not approximately unitarily equivalent for $n \geq 3$. In particular, ${\mathcal{O}_n}^{\otimes\infty}$ is not self-absorbing, and the flip on $({\mathcal{O}_n}^{\otimes\infty}) \otimes ({\mathcal{O}_n}^{\otimes\infty})$ is not approximately inner.*

Let us fix some notation for this section. If D is a unital, we let $\mathcal{D} := D^{\otimes\infty} := D \otimes D \otimes \ldots$ denote the infinite tensor product of D.

We define $\eta_{k,n}\colon\ D \to D^{\otimes n}$ for $n = 2, 3, \ldots, \infty$, $k = 1, 2, \ldots$ with $k \leq n$ by $\eta_{k,n}(a) = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1$ for $a \in D$ with a on k-th position. We let $\eta_1 := \eta_{1,2}$ and $\eta_2 := \eta_{2,2}$.

The different behavior of D and $\mathcal{D}$ can be seen from the following.

Remarks 4.3 *Suppose that D is a simple separable unital nuclear C^*–algebra that contains a copy of $\mathcal{O}_2$ unitally. Then:*
(1) The morphisms η_1 and η_2 are approximately unitarily equivalent in $D \otimes D$ and $D^{\otimes\infty} \cong \mathcal{O}_2$.

An example with $D \ncong \mathcal{D}$ is $D := \mathcal{P}_\infty$ the unique p.i.s.u.n. algebra in the UCT class with $K_0(\mathcal{P}_\infty) = 0$ and $K_1(\mathcal{P}_\infty) \cong \mathbb{Z}$.
(2) The flip automorphism on $\mathcal{P}_\infty \otimes \mathcal{P}_\infty$ is not *approximately inner.*
(3) There exist simple nuclear C^*–algebras D that contains a copy of $\mathcal{O}_2$ unitally and are not *purely infinite (e.g. the examples of Rørdam [30] are stably isomorphic to those algebras).*

See Appendix C for more explanation.

It shows that infinite tensor products $\mathcal{D} = D^{\otimes\infty}$ considerably loose properties of D. $\mathcal{D}$ is stably finite or purely infinite by [6, cor. 3.11] for simple D.

Below we see that $D \cong \mathcal{D}$ and every unital *-endomorphism of $\mathcal{D}$ is approximately inner if and only if $\mathcal{D}$ is self-absorbing and separable. Therefore we use the notation $\mathcal{D}$ also for self-absorbing algebras.

This class of separable self-absorbing algebras could be of interest for a classification theory of (not necessarily purely infinite) separable nuclear C^*–algebras up to tensoring with $\mathcal{D}$:
The classification of separable stable strongly purely infinite nuclear algebras

is a classification of all separable stable nuclear C^*–algebras modulo tensor product with $\mathcal{D} = \mathcal{O}_\infty$. The strongly purely infinite algebras are contained in the (possibly larger) class of algebras A with the property that $\mathcal{O}_\infty$ is unitally contained in $F(C, A)$ for every separable *nuclear* C^*–subalgebra C of A_ω.

We list some results on self-absorbing $\mathcal{D}$ in the UCT-class, and point out some open questions on self-absorbing $\mathcal{D}$ in the UCT class that have a tracial state.

Proposition 4.4 *Let $\mathcal{D}$ a unital separable self-absorbing C^*–algebra. Then:*

(1) $\mathcal{D}$ is simple and nuclear. Either $\mathcal{D}$ is purely infinite or $\mathcal{D}$ has a unique tracial state.
(2) $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D} \otimes \cdots$.
(3) Every unital endomorphism of $\mathcal{D}$ is approximately inner by unitaries in the commutator *subgroup of $\mathcal{U}(\mathcal{D})$.*
(4) If B is separable and $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally, then B is $\mathcal{D}$-absorbing if and only if $F(B)$ contains a copy of $\mathcal{D}$ unitally.
In particular, a separable algebra A is stably *$\mathcal{D}$-absorbing if and only if $\mathcal{D} \subset F(A)$.*
(5) If the commutator subgroup of $\mathcal{U}(\mathcal{D})$ is contained in the connected component $\mathcal{U}_0(\mathcal{D})$ of 1, then every stably *$\mathcal{D}$-absorbing separable C^*-algebra is $\mathcal{D}$-absorbing.*

It is a consequence of the basic Proposition 4.11 and of Corollaries 4.12 and 4.13. See end of this section for a proof.

We use the invariant $F(A)$ to give an alternative approach to permanence properties of the class of (strongly) $\mathcal{D}$-absorbing separable C^*–algebras, as e.g. studied by A. Toms and W. Winter [32], and we give a simple *necessary and sufficient* condition under which the class is closed under extensions (and is then automatically closed under Morita equivalence).

Theorem 4.5 *Suppose that $\mathcal{D}$ is unital, separable and self-absorbing.*

(1) If B is a unital separable C^*–algebra and a copy of $\mathcal{D}$ is unitally contained in B_ω, then $B \otimes B \otimes \cdots$ is $\mathcal{D}$-absorbing.
In particular:

$$\mathcal{D} \otimes M_2 \otimes M_3 \otimes \cdots \cong M_2 \otimes M_3 \otimes \cdots$$

if $\mathcal{D}$ is quasi-diagonal.
*If for every $n \in \mathbb{N}$ there exist $p, q \geq n$ and a unital *-morphism from $\mathcal{E}(M_p, M_q)$ into $\mathcal{D}_\omega$, then $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$.*
(2) The class of stably *$\mathcal{D}$-absorbing separable C^*–algebras is closed under inductive limits, passage to hereditary C^*–subalgebras, and to quotients.*
A unital separable algebra B is $\mathcal{D}$-absorbing if B is stably $\mathcal{D}$-absorbing.
(3) The class of stably $\mathcal{D}$-absorbing separable algebras is closed under extensions, if and only if,

$$\mathcal{E}(\mathcal{D},\mathcal{D}) \cong \mathcal{E}(\mathcal{D},\mathcal{D}) \otimes \mathcal{D}\,,$$

if and only if, the commutator subgroup of $\mathcal{U}(\mathcal{D})$ is in the connected component $\mathcal{U}_0(\mathcal{D})$ of 1.

(4) If the class of stably $\mathcal{D}$-absorbing separable algebras is closed under extension, then every stably $\mathcal{D}$-absorbing algebra is $\mathcal{D}$-absorbing.

We give a proof at the end of this section.

Parts (2)-(4) together imply that the class of $\mathcal{D}$-absorbing separable algebras is closed under all above considered operations, if and only if, $uvu^{-1}v \in \mathcal{U}_0(\mathcal{D})$ for all unitaries $u, v \in \mathcal{D}$. The property $[\mathcal{U}(\mathcal{D}),\mathcal{U}(\mathcal{D})] \subset \mathcal{U}_0(\mathcal{D})$ holds for simple purely infinite algebras $\mathcal{D}$, because the natural group morphism $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \to K_1(\mathcal{D})$ is an isomorphism (J. Cuntz [10]) if $\mathcal{D}$ is simple and purely infinite. A. Toms and W. Winter [32] obtained permanence results for tensorially $\mathcal{D}$–absorbing algebras under the (perhaps stronger) assumption that $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \cong K_1(\mathcal{D})$ for self-absorbing $\mathcal{D}$.

Remark 4.6 *The Cuntz algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$, the UHF algebras M_{p^∞} (p prime), the Jiang-Su algebra $\mathcal{Z}$ and all (finite and infinite) tensor products $\mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \ldots$ are examples of self-absorbing $\mathcal{D}$ in the sense of Definition 4.1.*

Up to tensoring with $\mathcal{O}_\infty$ this list exhausts all $\mathcal{D}$ in the UCT class.

The Elliott invariants of this algebras exhaust all possible Elliott invariants of $\mathcal{D} \otimes \mathcal{Z}$ for self-absorbing $\mathcal{D}$ in the KTP class ($\supset$ UCT class).

They all have connected unitary groups, thus the class of separable $\mathcal{D}$-absorbing algebras is closed under inductive limits, extensions, passage to hereditary subalgebras and under passage to quotients.

The flip automorphism on $\mathcal{D} \otimes \mathcal{D}$ is (unitarily) homotopic to the identity for UHF-algebras $\mathcal{D}$, $\mathcal{D} = \mathcal{O}_2$ and $\mathcal{D} = \mathcal{O}_\infty$.

Clearly, M_{p^∞} has the required properties. The considered properties are invariant under infinite tensor products. $\mathcal{Z}$ has the properties by [18]. The others follow from KTP, UCT and the classification of p.i.s.u.n. algebras by means of KK-theory (see Appendix C for details, or [32] for an alternative proof). We do not know if η_1, η_2 are (unitrily) homotopic for $\mathcal{D} = \mathcal{Z}$.

The case of UCT algebras suggests:

Conjecture 4.7 *If $\mathcal{D}$ is self-absorbing and $\mathcal{D} \neq \mathcal{O}_2$ then*

$$\mathcal{D} \otimes \mathcal{O}_\infty \otimes M_2 \otimes M_3 \otimes \cdots \cong \mathcal{O}_\infty \otimes M_2 \otimes M_3 \otimes \cdots \,.$$

Recall from Proposition 1.9(4,5,9), that the natural *-morphism from the normalizer $\mathcal{N}(D_B) \subset \mathcal{M}(B)_\omega$ of $D_B \subset B_\omega$ into $\mathcal{M}(D_B)$ defines isomorphisms $\mathcal{N}(D_B)/\mathrm{Ann}(B, \mathcal{M}(B)_\omega) \cong \mathcal{M}(D_B)$ and

$$F(B) = (B' \cap \mathcal{M}(B)_\omega)/\mathrm{Ann}(B, M(B)_\omega) \cong B' \cap \mathcal{M}(D_B)$$

if B is σ–unital. It allows to improve [25, prop. 8.1] as follows:

Proposition 4.8 *Suppose that B is a separable C^*-algebra and A is a non-degenerate C^*–subalgebra of B. Let $\mathcal{U}_1 \subset \mathcal{M}(D_B)$ denote the image of the unitary group of $\mathcal{N}(D_B)$ in $\mathcal{M}(D_B)$.*

If there are unitaries $W_1, W_2, \ldots \in \mathcal{U}_1$ with $\lim_{n\to\infty} \|W_n a - aW_n\| = 0$ for every $a \in A$ and $\lim_{n\to\infty} \mathrm{dist}(W_n^* b W_n, A_\omega) = 0$ for every $b \in B$, then there is a unitary $U = \pi_\omega(u_1, u_2, \ldots) \in \mathcal{M}(B)_\omega$ with $U^*BU = A$.

*The *-isomorphism $\psi(a) := UaU^*$ from A onto B is approximately unitarily equivalent to the inclusion map $A \subset B$ by the unitaries $u_1^*, u_2^*, \ldots \in \mathcal{M}(B)$.*

If one can find the W_n even in $\mathcal{U}_0(\mathcal{M}(D_B))$ then $u_1, u_2, \ldots$ can be chosen in $\mathcal{U}_0(\mathcal{M}(B))$.

Proof. Let $G \subset \mathcal{U}(\mathcal{M}(B))$ a (countable) subgroup such that for each $n \in \mathbb{N}$ there is a sequence $(g_1, g_2, \ldots) \in G$ with $\pi_\omega(g_1, g_2, \ldots) \in \mathcal{N}(D_B) \subset \mathcal{M}(B)_\omega$ and

$$\pi_\omega(g_1, g_2, \ldots) + \mathrm{Ann}(B, \mathcal{M}(B)_\omega) = W_n \,.$$

Note that G can be found in $\mathcal{U}_0(\mathcal{M}(B))$ if $W_n \in \mathcal{U}_0(\mathcal{M}(B))$, because unitaries in

$$\mathcal{U}_0(\mathcal{M}(D_B)) \cong \mathcal{U}_0(\mathcal{N}(D_B, \mathcal{M}(B)_\omega)/\mathrm{Ann}(B, \mathcal{M}(B)_\omega))$$

lift to unitaries in $\mathcal{U}_0(\mathcal{M}(B)_\omega)$ and $\mathcal{U}_0(\mathcal{M}(B)_\omega) \subset (\mathcal{U}_0(\mathcal{M}(B)))_\omega$.

Let $(a_1, a_2, \cdots)$ and $(b_1, b_2, \ldots)$ dense sequences in the unit-ball of A respectively of B. Consider the sequence of functions $f_1, f_2, \ldots$ on G given by $f_{2k-1}(g) := \|ga_k - a_k g\|$ and $f_{2k}(g) := \mathrm{dist}(g^{-1}b_k g, A)$ for $k \in \mathbb{N}$. Then $G \subset \mathcal{M}(B)$ and $(f_1, f_2, \ldots)$ satisfy the assumptions of Remark A.2: indeed, use the representing sequences for W_n and apply the assumptions on W_n. It follows, that there is a sequence $(v_1, v_2, \ldots)$ in $G \subset \mathcal{U}(\mathcal{M}(B))$ such that $\lim_n f_k(v_n) = 0$ for all $k \in \mathbb{N}$. This means that $\mathrm{id}\colon A \hookrightarrow B$ and $(v_1, v_2, \ldots)$ satisfy the assumptions of [25, prop. 8.1]. The proof of [25, prop. 8.1] shows that there is a sequence of unitaries $u_1, u_2, \ldots \in G$, such that $U := \pi_\omega(u_1, u_2, \ldots)$ is as required. □

Lemma 4.9 *Suppose that D and E are C^*–algebras $a \in D_+$, that $h\colon C_0((0,1], D) \to E$ is a *-morphism with $h(f_0 \otimes a) \neq 0$. If there is a net $\{U_\tau\}$ unitaries in $\mathcal{M}(E \otimes D)$ such that $\{U_\tau^*(h(f_0 \otimes d) \otimes a)U_\tau\}$ converges to $h(f_0 \otimes a) \otimes d$ for all $d \in D$, then:*

(1) D is simple and nuclear.

(2) If there are $\delta > 0$ and a lower semi-continuous 2-quasi-trace $\mu\colon E_+ \to [0, \infty]$ with $0 < \mu(h((f_0 - \delta)_+ \otimes (a - \delta)_+)) < \infty$, then all l.s.c. 2-traces $\nu\colon D_+ \to [0, \infty]$ are additive and are proportional to the trace

$$a \in D_+ \mapsto \mu(h((f_0 - \delta)_+ \otimes a)) \,.$$

Proof. (1): Use inner automorphisms of $E \otimes D$ composed with slice maps from $E \otimes D$ into D.

(2): Since D is simple and nuclear, every l.s.c. 2-quasi-trace ν on D_+ is an additive trace, and there is an extended l.s.c. 2-quasi-trace $\lambda\colon (E \otimes D)_+ \to$

$[0,\infty]$ with $\lambda(e\otimes d)=\mu(e)\nu(d)$ for $d\in D_+$, $e\in E_+$ with $\nu(d)<\infty$ and $\mu(e)<\infty$, cf. [6, rem. 2.29, proof of cor. 3.11(iv)]. ν is semi-finite and faithful if ν is non-trivial, in particular $0<\nu((a-\delta)_+)<\infty$ for $\delta>0$.

Then $\mu(h((f\otimes d))\nu(b)\le\mu(h((f\otimes b))\nu(d)$ for $d\in D_+$, $f:=(f_0-\delta)_+$ and $b:=(a-\delta)_+$, because $h(f\otimes d)\otimes b$ is the limit of $U_\tau(h(f\otimes b)\otimes d)U_\tau^*$ and λ is l.s.c. A similar argument shows "$\ge$". Thus $\nu(d)=\gamma\mu(h(f\otimes d))$ for all $d\in D_+$, where $\gamma:=\nu(b)/\mu(h(f\otimes b))$. □

Remark 4.10 *Suppose that A and D are separable where D is unital. Consider the following conditions for A and D:*

- *(β) The two *-morphisms $\mathrm{id}_A\otimes\eta_1$ and $\mathrm{id}_A\otimes\eta_2$ from $A\otimes D$ into $A\otimes(D\otimes D)$ are approximately unitarily equivalent by unitaries in the connected component $\mathcal{U}_0(\mathcal{M}(A\otimes D\otimes D))$ of the unitaries in $\mathcal{M}(A\otimes D\otimes D)$.*
- *(β') The *-morphisms $\mathrm{id}_A\otimes\eta_{1,\infty}$ and $\mathrm{id}_A\otimes\eta_{2,\infty}$ from $A\otimes D$ into $A\otimes(D\otimes D\otimes\cdots)$ are approximately unitarily equivalent by unitaries in $\mathcal{U}_0(\mathcal{M}(A\otimes D\otimes D\otimes\cdots))$.*

Then:

1. *(β) implies (β'), (β') implies that D is simple and nuclear, and that for every unital endomorphism φ of $D\otimes D\otimes\cdots$ the endomorphism $\mathrm{id}_A\otimes\varphi$ of $A\otimes D\otimes D\otimes\ldots$ is approximately inner by unitaries in $\mathcal{U}_0(A\otimes D\otimes D\otimes\cdots)$. In particular ($\beta$) holds with $D\otimes D\otimes\ldots$ in place of D.*
2. *If A_+ has a non-trivial lower semi-continuous (extended) 2-quasi-trace, then (β') implies that D has a unique tracial state.*
3. *The condition (β') is satisfied if the morphisms $\eta_{1,\infty}$ and $\eta_{2,\infty}$ from D into $D\otimes D\otimes\cdots$ are approximately unitarily equivalent and if $\mathcal{M}(A)$ contains a copy of $\mathcal{O}_2$ unitally (e.g. if A is stable).*
4. *The condition (β') is satisfied for every A if the morphisms $\eta_{1,\infty}$ and $\eta_{2,\infty}$ from $D\to D\otimes D\otimes\cdots$ are approximately unitarily equivalent by unitaries in the connected component $\mathcal{U}_0(D\otimes D\otimes\cdots)$ of 1 in $\mathcal{U}_0(D\otimes D\otimes\cdots)$.*
5. *If $A\cong A\otimes D\otimes D\otimes\cdots$, then ($\beta$') implies ($\beta$).*

The morphisms η_k and $\eta_{k,\infty}$ are above defined. Recall that $\mathcal{U}(A\otimes D\otimes D\otimes\ldots)$ is connected in norm-topology if A is stable and σ-unital (by a result of J. Cuntz and N. Higson).

Proof. (2) follows from from Lemma 4.9(2).

(4) is obvious.

(3): Since $\eta_{1,\infty}$ and $\eta_{2,\infty}$ are approximately unitarily equivalent, we get from Lemma 4.9(1) that $\mathcal{D}$ is simple and nuclear.

By classification theory, $\mathcal{O}_2\cong\mathcal{O}_2\otimes\mathcal{D}$, because $\mathcal{D}$ is simple, separable and nuclear. The group of unitaries $\mathcal{U}(\mathcal{O}_2)$ is connected (cf. [10]).

(1): It is obvious that (β) implies (β'). D is simple and nuclear by Lemma 4.9(1).

Let B_1 and B_2 unital algebras, and psi_1, ψ_2 unital morphisms from B_1 into B_2. We use the notation $\psi_1\sim\psi_2$ if $\mathrm{id}_A\otimes\psi_1$ and $\mathrm{id}_A\otimes\psi_2$ are approximately unitarily equivalent by unitaries in $\mathcal{U}_0(\mathcal{M}(A\otimes B_2))$.

There are obvious composition rules:
$\psi_1 \sim \psi_2$ and $\psi_2 \sim \psi_3$ imply $\psi_1 \sim \psi_3$. If $\lambda\colon B_2 \to B_3$ and $\mu\colon B_0 \to B_1$ are unital, and if $\psi_1 \sim \psi_2$, then $\lambda \circ \psi_1 \sim \lambda \circ \psi_2$ and $\psi_1 \circ \mu \sim \psi_2 \circ \mu$.

For $n \in \mathbb{N}$ and permutations σ of $\{1, \ldots, n\}$, we define *-morphisms

$$\psi_\sigma\colon D^{\otimes n} \to D^{\otimes n} \otimes 1 \otimes 1 \otimes \ldots \subset \mathcal{D}$$

by

$$\psi_\sigma(d_1 \otimes d_2 \otimes \ldots \otimes d_n) = d_{\sigma(1)} \otimes d_{\sigma(2)} \otimes \ldots \otimes d_{\sigma(n)} \otimes 1 \otimes 1 \otimes \cdots .$$

Further let $\epsilon_n := \psi_\sigma$ for $\sigma = \mathrm{id}$ of $\{1, \ldots, n\}$. For $m < n$ we define $\nu_{m,n}\colon D^{\otimes m} \to D^{\otimes n}$ by $\epsilon_m = \epsilon_n \circ \mu_{m,n}$, i.e.

$$\nu_{m,n}(d_1 \otimes \cdots \otimes d_m) = d_1 \otimes \cdots \otimes d_m \otimes 1 \otimes \cdots \otimes 1 .$$

The condition (β') implies that $\psi_\sigma \sim \epsilon_n$ for every transposition σ. Since every permutation is a product of transpositions, one can see by the rules for the relation $\sim$ that $\psi_\sigma \sim \epsilon_n$.

Let τ_1 and τ_2 denote the endomorphisms of $\mathcal{D}$ given by

$$\tau_1(d_1 \otimes d_2 \otimes \cdots \otimes d_n \otimes \cdots) = d_1 \otimes 1 \otimes d_2 \otimes 1 \otimes \cdots \otimes 1 \otimes d_n \otimes 1 \otimes \cdots$$

respectively

$$\tau_2(d_1 \otimes d_2 \otimes \cdots \otimes d_n \otimes \cdots) = 1 \otimes d_1 \otimes 1 \otimes d_2 \otimes \cdots \otimes 1 \otimes d_n \otimes 1 \otimes \cdots$$

Since there is a permutation σ of $\{1, \ldots, 2n\}$ with $\tau_\sigma \circ \epsilon_n = \psi_\sigma \circ \nu_{n,2n}$, we get that ($\beta$') implies that $\tau_k \circ \epsilon_n \sim \epsilon_n$ for $k = 1, 2$, $n \in \mathbb{N}$. It follows that

$$\tau_1 \sim \mathrm{id}_{\mathcal{D}} \sim \tau_2 .$$

We denote by γ the isomorphism from $\mathcal{D}$ onto $\mathcal{D} \otimes \mathcal{D}$ onto $\mathcal{D}$ with

$$\gamma\left((d_1 \otimes d_2 \otimes \cdots) \otimes (e_1 \otimes e_2 \otimes \ldots)\right) = (d_1 \otimes e_1 \otimes d_2 \otimes e_2 \otimes \cdots)$$

for $d_1, e_1, d_2, e_2, \ldots \in D$.

Then $\gamma \circ \eta_k = \tau_k \sim \mathrm{id}$ for $k = 1, 2$. It follows $\eta_1 = \gamma^{-1} \circ \tau_1 \sim \gamma^{-1} \circ \tau_2 = \eta_2$

Let $\psi\colon \mathcal{D} \to \mathcal{D}$ unital. Then

$$\psi \sim \gamma\eta_1\psi = \gamma(\psi \otimes \mathrm{id})\eta_1 \sim \gamma(\psi \otimes \mathrm{id})\eta_2 = \gamma\eta_2 = \tau_2 \sim \mathrm{id} .$$

(5): Conditions (β) and (β') are preserved if one passes over to isomorphic algebras, e.g. if $E \cong D$, then A and E satisfy (β), with E in place of D, if and only if, A and D satisfy (β).

Let $B := D \otimes D$, $\mathcal{D} := D \otimes D \otimes \cdots$, and let $\tau\colon B \to B$ denote the flip map $\tau\colon b_1 \otimes b_2 \mapsto b_2 \otimes b_1$.

Suppose that A and D satisfy condition (β'). Then A and $\mathcal{D} \otimes B \cong \mathcal{D}$ satisfy by part (1), that for every isomorphism φ of $\mathcal{D} \otimes B$ the isomorphism $\mathrm{id}_A \otimes \varphi$

of $A \otimes \mathcal{D} \otimes B$ is approximately unitarily equivalent to $\mathrm{id} = \mathrm{id}_A \otimes \mathrm{id}_{\mathcal{D}} \otimes \mathrm{id}_B$ by unitaries in $\mathcal{U}_0(A \otimes \mathcal{D} \otimes B)$ This applies to $\varphi := \mathrm{id}_{\mathcal{D}} \otimes \tau$.

If there is an isomorphism λ from A onto $A \otimes \mathcal{D}$ then there is a unital morphism Ψ from $\mathcal{M}(A \otimes (\mathcal{D} \otimes B))$ onto $\mathcal{M}(A \otimes B)$ with $\Psi(A \otimes (\mathcal{D} \otimes 1_B)) = A \otimes 1_B$ and $\Psi(a \otimes d \otimes b) = \lambda(a \otimes d) \otimes b$ for $a \in A$, $d \in \mathcal{D}$ and $b \in B$. It follows, that $\mathrm{id}_A \otimes \tau = \Psi \circ (\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{D}} \otimes \tau) \circ \Psi^{-1}$ is approximately unitarily equivalent to $\mathrm{id}_A \otimes \mathrm{id}_B$ by unitaries in $\mathcal{U}_0(\mathcal{M}(A \otimes B))$.

In particular, A and D satisfy condition (β), because $\tau \circ \eta_1 = \eta_2$. □

The following proposition is the basic observation of this section. It generalizes [25, thm. 8.2] and observations of Effros and Rosenberg [15]. The proof uses Proposition 4.8. Here we consider a property that is a bit stronger than D-absorption.

Proposition 4.11 *Suppose A and D are separable, and that D is unital. Then the following are equivalent:*

(1) There is an isomorphism φ from A onto $A \otimes D$ that is approximately unitarily equivalent to $a \in A \mapsto a \otimes 1$ by unitaries in $\mathcal{U}_0(A \otimes D)$.
(2) A and D satisfy condition (β) of Remark 4.10 and $F(A)$ contains a copy of D unitally.
(3) A and D satisfy condition (β') of Remark 4.10 and $F(A)$ contains a copy of D unitally.
(4) There is an isomorphism ψ from A onto $A \otimes D \otimes D \otimes \cdots$ that is approximately unitarily equivalent to $a \mapsto a \otimes 1$ by unitaries in $\mathcal{U}_0(\mathcal{M}(A \otimes D \otimes D \otimes \cdots))$.
(5) A and D satisfy (β') and $A \cong A \otimes D \otimes D \otimes \cdots$.

In part (5) we don't suppose that the isomorphism from A onto $A \otimes \mathcal{D}$ is approximately unitarily equivalent to $a \mapsto a \otimes 1 \otimes 1 \otimes \cdots$.

Proof.
(1)⇒(2): Let $\varphi\colon A \to A \otimes D$ as in part (1). Then $a \mapsto \varphi(a) \otimes 1$ is approximately unitarily equivalent to $a \mapsto a \otimes 1 \otimes 1$ by unitaries in $\mathcal{U}_0(A \otimes D \otimes D)$. The same must happen for $a \mapsto (\mathrm{id}_A \otimes \sigma)(\varphi(a) \otimes 1)$, because $\mathrm{id}_A \otimes \sigma$ extends to an automorphism of $\mathcal{M}(A \otimes D \otimes D)$. If we let $a := \varphi^{-1}(f)$ for $f \in A \otimes D$, then this shows that $f \to (\mathrm{id}_A \otimes \sigma)(f \otimes 1)$ and $f \to f \otimes 1$ are approximately unitarily equivalent, In particular, A and D satisfy condition (β) of Remark 4.10, and D is simple and nuclear by Remark 4.10(1).

The non-degenerate endomorphism $a \mapsto \varphi^{-1}(a \otimes 1)$ of A is approximately unitarily equivalent to id_A. If $u_1, u_2, \ldots \in \mathcal{M}(A)$ is a sequence of unitaries with $\lim u_n^* \varphi^{-1}(a \otimes 1) u_n = a$ for $a \in A$, then

$$\varphi_n\colon\ d \in D \to u_n^* \mathcal{M}(\varphi^{-1})(1 \otimes d) u_n \in \mathcal{M}(A)$$

is a unital *-monomorphism with $\lim \|[\varphi_n(d), a]\| = 0$, i.e.

$$\pi_\omega(\varphi_1(d), \varphi_2(d), \ldots) \in (A, \mathcal{M}(A))^c\,.$$

Since $F(A) \cong (A, \mathcal{M}(A))^c/\mathrm{Ann}(A, \mathcal{M}(A)_\omega)$ and D is simple, it follows that $F(A)$ contains a copy of D unitally.

(2)⇒(3): is obvious.

(5)⇒(3): Property (β') implies that D is *simple* and nuclear, cf. Remark 4.10(1). If λ is an isomorphism from $A \otimes D \otimes D \otimes \ldots$ onto A, then λ extends to a unital *-isomorphism from $\mathcal{M}(A \otimes D \otimes D \otimes \ldots)$ onto $\mathcal{M}(A)$. For $d \in D$ let

$$\varphi_n(d) := \lambda(1_{\mathcal{M}(A)} \otimes 1 \otimes \cdots \otimes 1 \otimes d \otimes 1 \otimes \cdots) \in \mathcal{M}(A \otimes D \otimes D \otimes \cdots)$$

with d on n-th position. This defines unital *-morphisms from D into $\mathcal{M}(A)$ with $\lim \|[\varphi_n(d), a]\| = 0$. Now deduce (3) as in the proof of the implication (1)⇒(2).

(3)⇒(4): By Remark 4.10(1), D must be simple and nuclear, and condition (β) is satisfied for A and $\mathcal{D} := D \otimes D \otimes \cdots$ (in place of D).

By Corollary 1.13, there is also a copy of $\mathcal{D} := D \otimes D \otimes \cdots$ unitally contained in $F(A)$, because A and D are separable, and D is unital, simple and nuclear.

Let $A \subset B := A \otimes \mathcal{D}$ (and identify A with $A \otimes 1_{\mathcal{D}}$). We show that A and B satisfy the assumptions of Proposition 4.8:

Let $h\colon \mathcal{D} \to F(A)$ a unital *-morphism. There is an isomorphism λ from $A \otimes \mathcal{D} \otimes \mathcal{D}$ into B_ω with $\lambda(a \otimes 1 \otimes 1) = a \in A_\omega \subset B_\omega$, $\lambda(a \otimes d \otimes 1) = \rho_A(h(d) \otimes a) \in D_A \subset A_\omega$, and $\lambda(a \otimes 1 \otimes d) = a \otimes d \in B$. λ is give by application of

$$(\rho_A \circ \sigma) \otimes \mathrm{id}_{\mathcal{D}}\colon\ A \otimes^{max} F(A) \otimes \mathcal{D} \to A_\omega \otimes \mathcal{D} \subset B_\omega$$

on $\mathrm{id}_A \otimes h \otimes \mathrm{id}_{\mathcal{D}}$. (Here σ means the flip isomorphism $a \otimes b \mapsto b \otimes a$).

I.e. $A \otimes \mathcal{D} \otimes \mathcal{D}$ may be considered as a non-degenerate C^*–subalgebra of $\overline{A(B_\omega)A} = D_B$.

The image of λ is a non-degenerate subalgebra of D_B. Thus

$$\mathcal{M}(\lambda)\colon\ \mathcal{M}(A \otimes \mathcal{D} \otimes \mathcal{D}) \to \mathcal{M}(D_B)$$

exists and is unital. Since A and $\mathcal{D}$ satisfy (β), we find a sequence of unitaries $W_n = \mathcal{M}(\lambda)(V_n) \in \mathcal{U}_0(\mathcal{M}(D_B))$ with $\lim_n W_n^* \lambda(a \otimes 1 \otimes d) W_n = \lambda(a \otimes d \otimes 1)$ for all $a \in A$ and $d \in D$. Thus $(W_1, W_2, \ldots)$ satisfies the assumptions of Proposition 4.8. It follows that there is an isomorphism ψ from A onto $B = A \otimes \mathcal{D}$ that is approximately inner by unitaries in $\mathcal{U}_0(A \otimes \mathcal{D})$, i.e. ψ is as stipulated in (3).

(4)⇒(1): If we apply the above verified implications (1)⇒(2) to A and $\mathcal{D} := D \otimes D \otimes \cdots$ in place of D, then we get that condition (β) is satisfied for A and $\mathcal{D}$. It follows that $\mathcal{D}$ is simple and nuclear.

By assumption, there is an isomorphism $\psi\colon\ A \to A \otimes \mathcal{D}$ from A onto $A \otimes \mathcal{D}$ such that $a \in A \mapsto a \otimes 1 \in A \otimes \mathcal{D}$ is approximately unitarily equivalent to ψ by unitaries in $\mathcal{U}_0(\mathcal{M}(A \otimes \mathcal{D}))$.

It follows that $a \in A \mapsto \psi^{-1}(a \otimes 1_{\mathcal{D}}) \in A$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{A}}$ by unitaries in $\mathcal{U}_0(\mathcal{M}(A))$. Let $\lambda\colon \mathcal{D} \to \mathcal{D} \otimes D$ the isomorphism given by $\lambda(d_1 \otimes d_2 \otimes \cdots) := (d_2 \otimes d_3 \otimes \cdots) \otimes d_1$. Then $\varphi := (\psi^{-1} \otimes \mathrm{id}_D) \circ (\mathrm{id}_A \otimes \lambda) \circ \psi$ is an isomorphism from A onto $A \otimes D$ and is approximately unitarily equivalent to $a \in A \mapsto \psi^{-1}(a \otimes 1_{\mathcal{D}}) \otimes 1_D \in A \otimes D$ by unitaries in $\mathcal{U}_0(\mathcal{M}(A \otimes D))$. Thus, the isomorphism $\varphi\colon A \to A \otimes D$ is approximately unitarily equivalent to $a \mapsto a \otimes 1$ by unitaries in $\mathcal{U}_0(A \otimes D)$.

(4)⇒(5): Since (4) implies (1), it implies also (2) and (3). Thus (4) implies condition (β') for A and D. $A \otimes A \otimes D \otimes D \otimes \ldots$ is part of (4). □

Corollary 4.12 *Suppose that D is unital and separable, and let $\mathcal{D} := D^{\otimes\infty} := D \otimes D \otimes \cdots$. Following properties (1)–(4) of D are equivalent:*

(1) Any two endomorphisms φ and ψ of $\mathcal{D}$ are approximately unitarily equivalent by commutators $u_n = v_n^* w_n^* v_n w_n$ *of unitaries v_n, w_n in $\mathcal{D}$.*
(2) The flip automorphism $\sigma\colon d \otimes e \mapsto e \otimes d$ of $\mathcal{D} \otimes \mathcal{D}$ is approximately inner,
(3) $\mathcal{D}$ is self-absorbing.
(4) The morphisms $\eta_{1,\infty}\colon d \mapsto d \otimes 1 \otimes 1 \otimes \cdots$ and $\eta_{2,\infty}\colon d \mapsto 1 \otimes d \otimes 1 \otimes \cdots$ from D into $\mathcal{D}$ are approximately unitarily equivalent in $\mathcal{D}$.

Proof. (1)⇒(2): Since $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$ by some isomorphism $\psi\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, we get that $\psi^{-1}\sigma\psi$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{D}}$. Thus, σ is an approximately inner automorphism of $\mathcal{D} \otimes \mathcal{D}$.

(2)⇒(3): $\mathcal{D}$ is simple and nuclear by Lemma 4.9(1). Let $A := \mathcal{K} \otimes \mathcal{D}$, then $A \cong A \otimes \mathcal{D} \otimes \mathcal{D} \otimes \cdots$ (by any isomorphism from $\mathcal{D}$ to $\mathcal{D} \otimes \mathcal{D} \otimes \cdots$).

Since $\eta_1 = \sigma \circ \eta_2$, we get that $\mathrm{id}_A \otimes \eta_1$ and $\mathrm{id}_A \otimes \eta_2$ are approximately unitarily equivalent by unitaries in $\mathcal{O}_2 \otimes \mathcal{D} \otimes \mathcal{D} \subset \mathcal{M}(A \otimes \mathcal{D} \otimes \mathcal{D})$. The unitary group of $\mathcal{O}_2 \otimes \mathcal{D} \otimes \mathcal{D} \cong \mathcal{O}_2$ is connected. Thus, A and $\mathcal{D}$ satisfy condition (β) (with $\mathcal{D}$ in place of D).

It follows that Proposition 4.11 can be applied on A and $\mathcal{D}$. It leads to an isomorphism ψ from A onto $A \otimes \mathcal{D}$ that is approximately unitarily equivalent to $a \mapsto a \otimes 1$. Since $\mathcal{D}$ is unital, ψ defines an isomorphism from $\mathcal{D} \cong e_{1,1} \otimes \mathcal{D}$ onto $\mathcal{D} \otimes \mathcal{D}$, that is approximately unitarily equivalent to $d \mapsto d \otimes 1$, i.e. $\mathcal{D}$ is self-absorbing.

(3)⇒(4): If $\mathcal{D} := D^{\otimes\infty}$ is self-absorbing, then $A := \mathcal{K} \otimes D$ and D satisfy part (4) of Proposition 4.11. Thus, A and D fulfill condition (β) by the implication (4)⇒(2) of 4.11. But this means that $\mathrm{id}_{\mathcal{D}} \otimes \eta_k\colon \mathcal{D} \otimes D \to \mathcal{D} \otimes (D \otimes D)$, with $k = 1, 2$ are approximately unitarily equivalent. The latter is an equivalent formulation of (4).

(4)⇒(1): $A := \mathcal{K}$ and D satisfy condition (β') of Remark 4.10. Thus, by part (1) of 4.10, $\mathrm{id}_{\mathcal{K}} \otimes \psi$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{K}} \otimes \mathrm{id}_{\mathcal{D}}$ for every unital endomorphism of $\mathcal{D} := D \otimes D \otimes \cdots$. This implies that any two unital endomorphisms of $\mathcal{D}$ are approximately unitarily equivalent.

It implies that $u \otimes u^* \otimes 1 \otimes \cdots \in \mathcal{U}(\mathcal{D})$ for $u \in \mathcal{U}(D^{\otimes n})$ is in the norm closure of the set of commutators $\{wv^*w^*v\,;\ v, w \in \mathcal{U}(\mathcal{D})\}$ in $\mathcal{U}(\mathcal{D})$. Indeed: the flip σ_n on $D^{\otimes n} \otimes D^{\otimes n}$ extends to an isomorphism λ of $\mathcal{D}$ with $\lambda(a \otimes b \otimes 1 \otimes \cdots) =$

$a\otimes b\otimes 1\otimes\cdots$ for $a,b\in D^{\otimes n}$. Since λ is approximately inner, we get a sequence of unitaries $v_n\in\mathcal{U}(\mathcal{D})$ with $u\otimes u^*\otimes 1\otimes\cdots = w\lambda(w^*) = \lim_n wv_n^*w^*v_n$ for $w:=u\otimes 1\otimes 1\otimes\cdots$.

If $X\subset\mathcal{D}$ is a finite subset of the contractions in $\mathcal{D}$ and if $v\in\mathcal{U}(\mathcal{D})$, then for every $\varepsilon>0$ there exists $n\in\mathbb{N}$ and $u\in\mathcal{U}(D^{\otimes n})$, such that $\|v^*dv-w^*dw\|<\varepsilon$ for $d\in X$ where $w:=u\otimes u^*\otimes 1\otimes\cdots$. Recall here that one can find n such that the elements of $X\cup\{v\}$ have distance $<\varepsilon/9$ from $D^{\otimes n}\otimes 1\otimes\cdots\subset\mathcal{D}$.

It follows that unital endomorphisms φ and ψ of $\mathcal{D}$ are approximately unitarily equivalent by unitaries w_n in the set of commutators in $\mathcal{U}(\mathcal{D})$. □

Proposition 4.11 (with $A:=\mathcal{K}\otimes D$) and Corollary 4.12 immediately imply the following corollary:

Corollary 4.13 *If D is a unital and separable, then D is self-absorbing (in the sense of Definitions 4.1) if and only if $D\cong D\otimes D\otimes\cdots$ and all endomorphisms of D are approximately unitarily equivalent by unitaries in the commutator subgroup of $\mathcal{U}(D)$.*

Proof. If $D\cong D\otimes D\otimes\cdots$ and all endomorphisms of $\mathcal{D}:=D\otimes D\otimes\cdots$ are unitarily equivalent, then $\mathcal{D}\cong D$ is self-absorbing by Corollary 4.12(3). If D is self-absorbing, then the implication (1)⇒(4) of Proposition 4.11 applies to $A:=\mathcal{K}\otimes D$ and D. Thus, there is an isomorphism ψ from $\mathcal{K}\otimes D$ onto $(\mathcal{K}\otimes D)\otimes\mathcal{D}$, such that ψ is approximately unitarily equivalent to $a\in\mathcal{K}\otimes D\mapsto a\otimes 1$. Since D is unital, this implies that $D\cong D\otimes D\otimes\ldots$. □

Corollary 4.14 *If A is separable and if there is a unital *-morphism from $M_2\oplus M_3$ into $F(A)$ then $A\cong\mathcal{Z}\otimes A$.*

It could be that $1_{F(A)}\in M_2\oplus M_3\subset F(A)$ does *not* imply approximate divisibility of A in general, cf. Question 3.16.

Proof. Let $E:=(M_2\oplus M_3)\otimes(M_2\oplus M_3)\otimes\cdots$. There is a sequence of unital *-morphisms h_n from $\mathcal{E}(M_{p_n},M_{q_n})$ into E such that $\gcd(p_n,q_n)=1$ and $p_n,q_n\geq n$. This defines a unital morphism from $\mathcal{Z}$ into E_ω. Since $\mathcal{Z}$ is self-absorbing, this implies $E\otimes\mathcal{Z}\cong E$ by Theorem 4.5(1). There is a unital *-morphism from E into $F(A)$ by Corollary 1.13. Thus $\mathcal{Z}\subset F(A)$ unitally. Since $\mathcal{U}(\mathcal{Z})=\mathcal{U}_0(\mathcal{Z})$ and $\mathcal{Z}$ is tensorially self-absorbing, $A\cong A\otimes\mathcal{Z}$ by Proposition 4.4(4,5). □

Proof of Proposition 4.4:

(1,2,3): $\mathcal{D}\cong\mathcal{D}\otimes\mathcal{D}\otimes\cdots$ and every unital endomorphism of $\mathcal{D}$ is approximately inner by unitaries in the commutator group, cf. by Corollary 4.13. In particular, the flip automorphism of $\mathcal{D}\otimes\mathcal{D}\cong\mathcal{D}$ is approximately inner. Thus $\mathcal{D}$ is simple and nuclear and has at most one tracial state by Lemma 4.9. Since $\mathcal{D}$ is tensorially non-prime, it follows from [6, cor. 3.11(i)], that either $\mathcal{D}$ is purely infinite or $\mathcal{D}$ is stably finite. If a unital nuclear C^*-algebra $\mathcal{D}$ is stably finite then $\mathcal{D}$ admits tracial state (by results of B. Blackadar, J. Cuntz and U. Haagerup).

(3): See Corollary 4.13 or Corollary 4.12(1).

(4): The pair of algebras $(B, \mathcal{D})$ satisfies condition (β') by part (3) of Remark 4.10, because the flip automorphism on $\mathcal{D} \otimes \mathcal{D}$ is approximately inner by part (2) and Corollary 4.12(2). By the equivalences (1)$\Leftrightarrow$(3) of Proposition 4.11, B is $\mathcal{D}$-absorbing if and only if there is a copy of $\mathcal{D}$ unitally contained in $F(B)$

$\mathcal{M}(\mathcal{K} \otimes A)$ contains a unital copy of $\mathcal{O}_2$ for every A, and $F(A) \cong F(\mathcal{K} \otimes A)$ for separable A.

(5): By part (3), the maps η_2 and η_2 from $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D} \otimes \cdots$ into $\mathcal{D} \otimes \mathcal{D} \cong \mathcal{D}$ are approximately unitarily equivalent by unitaries in the commutator subgroup of $\mathcal{U}(\mathcal{D})$.

By assumption, the commutator subgroup is contained in $\mathcal{U}_0(\mathcal{D})$. Thus, by Remark 4.10(4), the pair of algebras $(A, \mathcal{D})$ satisfies condition (β') for every separable algebra A. Now Proposition 4.11 applies: A is $\mathcal{D}$-absorbing if and only if $F(A)$ contains a copy of $\mathcal{D}$ unitally. $\square$

Proof of Theorem 4.5:

(1): Let $B^{\otimes\infty} := B \otimes B \otimes \cdots$ There is a unital *-morphism

$$\psi\colon\ B_\omega \to (B \otimes B \otimes \cdots)^c \cong F(B \otimes B \otimes \cdots)\,.$$

It is the ultrapower $\psi := (\psi_1, \psi_2, \ldots)_\omega$ of the morphisms $\psi_n\colon B \to B^{\otimes\infty}$ given by $\psi_n(b) := 1_n \otimes b \otimes 1_\infty$, where $1_{n+1} := 1_n \otimes 1$ and $1_\infty := 1 \otimes 1 \otimes \cdots$.

If $\varphi\colon \mathcal{D} \to B_\omega$ is unital, then $\psi \circ \varphi$ is a unital *-morphism from $\mathcal{D}$ into $F(B \otimes B \otimes \cdots)$. Since $\mathcal{D}$ is simple, a copy of $\mathcal{D}$ is unitally contained in $F(B \otimes B \otimes \cdots)$. Thus $B \otimes B \otimes \cdots$ is stably $\mathcal{D}$-absorbing by Proposition 4.4(4).

If $\mathcal{D}$ is quasi-diagonal, then $\mathcal{D}$ is unitally contained in B_ω for $B := M_2 \otimes M_3 \otimes \cdots$.

Let $\psi_n\colon\ \mathcal{E}(M_{p_n}, M_{q_n}) \to \mathcal{D}_\omega$ unital *-morphisms, where $p_n, q_n \geq n$. Then

$$\psi_\omega\colon\ \prod_\omega \mathcal{E}(M_{p_n}, M_{q_n}) \to (\mathcal{D}_\omega)_\omega$$

is a unital morphism. One can see, that there is a unital *-morphism from $\mathcal{Z}$ into $\prod_\omega \mathcal{E}(M_{p_n}, M_{q_n})$. (If $\lim_\omega \gcd(p_n, q_n) = \infty$ this is trivial, because then it contains an ultrapower of matrix algebras.)

Thus, there is a unital morphism from $\mathcal{Z}$ into $(\mathcal{D}_\omega)_\omega$. On the other hand, $(\mathcal{D}_\omega)_\omega$ is the quotient of $\ell_\infty(\ell_\infty(\mathcal{D})) \cong \ell_\infty(\mathcal{D})$ induced by some other character ω_1 on its center $\ell_\infty(\ell_\infty) \cong \ell_\infty$, i.e. $(\mathcal{D}_\omega)_\omega \cong \mathcal{D}_{\omega_1}$. We obtain that $\mathcal{Z} \subset \mathcal{D}_{\omega_1}$ for some free ultrafilter on $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$. Since $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D} \otimes \cdots$ and $\mathcal{Z}$ is self-absorbing, $\mathcal{D}$ is $\mathcal{Z}$-absorbing.

(2): By Proposition 4.4(4), A is stably $\mathcal{D}$-absorbing if and only if a copy of $\mathcal{D}$ is unitally contained in $F(A)$. If J is a closed ideal of A, then there are unital *-morphisms from $F(A)$ into $F(J)$ and from $F(A)$ onto $F(A/J)$, cf. Remark 1.15(3). Thus, J and A/J are stably $\mathcal{D}$-absorbing if A is $\mathcal{D}$-absorbing.

If $E \subset A$ is a hereditary C^*-subalgebra and if J denotes the closed ideal of A generated by E, then $\mathcal{K} \otimes E \cong \mathcal{K} \otimes J$. Hence, E is stably $\mathcal{D}$-absorbing if A is stably $\mathcal{D}$-absorbing.

Suppose that $A = \operatorname{indlim}(h_n \colon B_n \to B_{n+1})$, where $B_1, B_2, \ldots$ are separable. Let $h_{n,\infty} \colon B_n \to A$ denote the corresponding natural morphisms. Then $A_n := h_{n,\infty}(B_n)$ is an increasing sequence of C^*-subalgebras of A, such that $\bigcup_n A_n$ is dense in A. If B_n is stably $\mathcal{D}$-absorbing, then its quotient A_n is stably $\mathcal{D}$-absorbing.

It follows that $\mathcal{D}$ is unitally contained in $F(A_n)$ for $n \in \mathbb{N}$. Since $\mathcal{D}$ and A are separable, we get that $\mathcal{D}$ is unitally contained in $F(A)$ by Proposition 1.14.

Suppose that B is unital and stably $\mathcal{D}$-absorbing, i.e. there is an isomorphism ψ from $\mathcal{K}\otimes B$ onto $\mathcal{K}\otimes B\otimes\mathcal{D}$ that is approximately unitarily equivalent to $a \mapsto a \otimes 1$ for $a \in \mathcal{K} \otimes B$.

Then there exist a unitary $u \in \mathcal{M}(\mathcal{K} \otimes B \otimes \mathcal{D})$ such that

$$u^*\psi(e_{1,1} \otimes 1_B)u = e_{1,1} \otimes 1_B \otimes 1_{\mathcal{D}}\,.$$

Then there is a unique isomorphism φ from B onto $B \otimes \mathcal{D}$ with

$$u^*\psi(e_{1,1} \otimes b)u = e_{1,1} \otimes \varphi(b)$$

for $b \in B$, and φ is approximately unitarily equivalent to $b \mapsto b \otimes 1$.

(3): Suppose that the commutator group $[\mathcal{U}(\mathcal{D}), \mathcal{U}(\mathcal{D})]$ of $\mathcal{U}(\mathcal{D})$ is contained in $\mathcal{U}_0(\mathcal{D})$. Let $z_1, z_2, \ldots$ a sequence that is dense in $\mathcal{U}(\mathcal{D})$. For $n \in \mathbb{N}$ there are $u_n, v_n \in \mathcal{U}(\mathcal{D} \otimes \mathcal{D})$ with

$$\|((v_nu_n)^*u_nv_n)^*\eta_1(z_k)((v_nu_n)^*u_nv_n) - \eta_2(z_k)\| < 1/n\,,$$

and there is a continuous map $w \colon [0, 1/2] \to \mathcal{U}(\mathcal{D} \otimes \mathcal{D})$ with $w_0 = 1$ and $w_{1/2} = (v_nu_n)^*u_nv_n$. We define unital completely positive maps $T_n \colon \mathcal{D} \to \mathcal{E}(\mathcal{D}, \mathcal{D})$ by $T_n(d)_t := (w_t)^*\eta_1(d)(w_t)$ for $t \in [0, 1/2]$ and $T_n(d)_t := (2t - 1)\eta_2(d) + 2(1 - t)T_n(d)_{1/2}$ for $t \in (1/2, 1]$. Then T_n is $2/n$-multiplicative on $\{z_1, \ldots, z_n\}$. Thus, the restriction of the ultrapower T_ω to $\mathcal{D} \subset \mathcal{D}_\omega$ defines a unital *-morphism

$$\Psi \colon \mathcal{D} \to \mathcal{E}(\mathcal{D}, \mathcal{D})_\omega\,.$$

Let A a separable C^*-algebra and J a closed ideal of A such that J and A/J are stably $\mathcal{D}$-absorbing. Then there exist unital subalgebras $D_0 \subset F(J)$ and $D_1 \subset F(A/J)$ that are isomorphic to $\mathcal{D}$. Thus $\mathcal{E}(\mathcal{D}, \mathcal{D}) \cong \mathcal{E}(D_0, D_1)$, and, by Proposition 1.17, there exits a unital *-morphism $h \colon \mathcal{E}(\mathcal{D}, \mathcal{D}) \to F(A)$. The superposition $h_\omega \circ \Psi$ is a unital *-morphism from $\mathcal{D}$ into $F(A)_\omega$. Since $\mathcal{D}$ is simple and separable, there is a copy of $\mathcal{D}$ unitally contained even in $F(A)$ itself, cf. Proposition 1.14 (with $A_n = A$). Hence, A is stably $\mathcal{D}$-absorbing.

Conversely, suppose that the class of separable stably $\mathcal{D}$-absorbing algebras is closed under extensions. Then $\mathcal{E}(\mathcal{D}, \mathcal{D}) \cong \mathcal{D} \otimes \mathcal{E}(\mathcal{D}, \mathcal{D})$, because $\mathcal{E}(\mathcal{D}, \mathcal{D})$ is a unital extension of the $\mathcal{D}$–absorbing algebra $\mathcal{D} \oplus \mathcal{D}$ by $C_0(0, 1) \otimes \mathcal{D}$:

$$0 \to C_0((0,1), \mathcal{D} \otimes \mathcal{D}) \to \mathcal{E}(\mathcal{D}, \mathcal{D}) \to \mathcal{D} \otimes 1 \oplus 1 \otimes \mathcal{D} \to 0\,.$$

In particular, there is a unital *-morphism $\psi\colon\, \mathcal{D} \to \mathcal{E}(\mathcal{D}, \mathcal{D})$. It is given by a point-norm continuous path of unital *-morphisms $\psi_t\colon\, \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ with $\psi_0(\mathcal{D}) \subset \mathcal{D} \otimes 1$ and $\psi_1(\mathcal{D}) \subset 1 \otimes \mathcal{D}$. For $u, v \in \mathcal{U}(\mathcal{D})$ we have that $\psi_0(u^*v^*uv)$ is in $\mathcal{U}_0(\mathcal{D} \otimes \mathcal{D})$ by the path $w_t := \psi_0(u)^*\psi_t(v)^*\psi_0(u)\psi_t(v)$ with $w_0 = \psi_0(u^*v^*uv)$ and $w_1 = 1 \otimes 1$. If ι denotes an isomorphism from $\mathcal{D} \otimes \mathcal{D}$ onto $\mathcal{D}$, then $\iota \circ \psi$ is approximately inner. Since $\iota(\psi_0(u^*v^*uv)) \in \mathcal{U}_0(\mathcal{D})$, and since $\mathcal{U}_0(\mathcal{D})$ is a closed and open normal subgroup of $\mathcal{U}(\mathcal{D})$, it follows $u^*v^*uv \in \mathcal{U}_0(\mathcal{D})$.

(4): If the class of stably $\mathcal{D}$-absorbing separable C^*–algebras is closed under extensions, then $[\mathcal{U}(\mathcal{D}), \mathcal{U}(\mathcal{D})] \subset \mathcal{U}_0(\mathcal{D})$. The latter implies that every stably $\mathcal{D}$-absorbing algebra is $\mathcal{D}$-absorbing. □

We conclude this section with some remarks and questions:
(1) If $\eta_1, \eta_2\colon\, \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ are homotopic then for every separable C^*–algebra A there is a natural isomorphism

$$KK(\mathcal{D}, A \otimes \mathcal{D}) \cong K_0(A \otimes \mathcal{D})\,.$$

(Here we do not assume that the UCT is valid for $\mathcal{D}$.)
(2) In particular, $KK(\mathcal{D}, \mathcal{D}) \cong K_0(\mathcal{D})$ with ring-structure given by tensor product of projections, and $KK^1(\mathcal{D}, \mathcal{D}) \cong K_1(\mathcal{D})$.
(3) Let $\mathcal{D}$ be a self-absorbing algebra.
Are η_1 and η_2 homotopic?
Is $\mathrm{cov}(\mathcal{D}) < \infty$? Is $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$?
Is $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \to K_1(\mathcal{D})$ an isomorphism if $\mathcal{D}$ is self-absorbing?
Is always $K_1(\mathcal{D}) = 0$ for self-absorbing unital $\mathcal{D}$?
(4) Does there exist a nuclear C^*-algebra A such that A is stably projection-less, that the flip automorphism of $A \otimes A$ is approximately inner (by unitaries in $\mathcal{M}(A \otimes A)$) and with $K_*(A) = K_*(\mathbb{C})$?

A Elementary properties of ultrapowers.

One has to take a more general and flexible approach to ultrapowers to get a tool for our proofs: It is useful for our applications to consider bounded subsets X_n of the closed unit-ball of a Banach spaces B_n (or of $\mathcal{L}(B_n, B_n)$). This is the most general form of bounded metric spaces (with an given uniform bound for the diameters). But the needed selection results are part of elementary set theory (and are rather elementary).

Let $\omega \subset \wp(\mathbb{N})$ a (fixed) free ultrafilter on $\mathbb{N}$. Then $X_1 \times X_2 \times \cdots$ with semi-metric

$$d_\omega((s_1, s_2, \ldots), (t_1, t_2, \ldots)) := \lim_{\omega} \|s_n - t_n\|$$

defines a metric space that is isometric to the (closed) subset

$$X_\omega := \{\pi_\omega(s_1, s_2, \ldots)\,;\; s_1 \in X_1, s_2 \in X_2, \ldots\}$$

of the Banach space

$$\prod_\omega \{B_1, B_2, \ldots\} := \ell_\infty(B_1, B_2, \ldots)/c_\omega(B_1, B_2, \ldots)\,.$$

We suppose now that on each X_n there is given a sequence of functions $f_n^{(1)}, f_n^{(2)}, \ldots$ with $f_n^{(k)}\colon X_n \to [0,\infty)$ for $k = 1, 2, \ldots$. Further we suppose that for fixed $k \in \mathbb{N}$ the sequence has a common estimate $\gamma_k < \infty$ for the Lipschitz constants of $f_n^{(k)}$ for $n = 1, 2, \ldots$. (This condition can be relaxed in applications by ω-lim-existence conditions.)

We can define functions $f_\omega^{(k)}\colon X_\omega \to [0,\infty]$ for $k \in \mathbb{N}$ by

$$f_\omega^{(k)}(\pi_\omega(s_1, s_2, \ldots)) := \omega\text{-}\lim_n f_n^{(k)}(s_n)\,,$$

because $\omega\text{-}\lim_n f_n^{(k)}(t_n) - f_n^{(k)}(s_n) = 0$, if $\omega\text{-}\lim_n \|t_n - s_n\| = 0$.

The basic lemma is:

Lemma A.1 *Let $X_1, X_2, \ldots$ any sequence of sets and suppose that for each $n \in \mathbb{N}$ there is given a sequence $f_n^{(1)}, f_n^{(2)}, \ldots$ of functions $f_n^{(k)}\colon X_n \to [0,\infty)$ for $k = 1, 2, \ldots$. For $k \in \mathbb{N}$, let*

$$f_\omega^{(k)}(s_1, s_2, \ldots) := \omega\text{-}\lim_n f_n^{(k)}(s_n)\,.$$

Suppose that for every $m \in \mathbb{N}$ and $\varepsilon > 0$, there is $s = (s_1, s_2, \ldots) \in X_1 \times X_2 \times \cdots$ with $f_\omega^{(k)}(s) < \varepsilon$ for $k = 1, \ldots, m$.

Then there is $t = (t_1, t_2, \ldots) \in X_1 \times X_2 \times \ldots$ with

$$\omega\text{-}\lim_n f_n^{(k)}(t_n) = f_\omega^{(k)}(t) = 0$$

for all $k \in \mathbb{N}$.

Moreover, then there is a sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that there are $s_\ell \in X_{n_\ell}$ with $f_{n_\ell}^{(k)}(s_\ell) < 2^{-\ell}$ for $k \le \ell$, $\ell = 1, 2, \ldots$.

The second part is almost trivial by the fact that any subsequence of a zero-sequence is a zero-sequence. It does not imply the first part because the infinite set $\{n_1, n_2, \ldots\} \subset \mathbb{N}$ is not necessarily contained in the given free ultrafilter ω on $\mathbb{N}$.

Proof. We define subsets $X_{n,m} \subset X_n$ by $X_{n,0} := X_n$

$$X_{n,m} := \{s \in X_n\,;\; \max(f_n^{(1)}(s), \ldots, f_n^{(m)}(s)) < 1/m\}\,.$$

Then $X_{n,m+1} \subset X_{n,m}$. We let $m(n) := \sup\{m \le n\,;\; X_{n,m} \neq \emptyset\}$. For every $k \in \mathbb{N}$, the set Y_k of $n \in \mathbb{N}$ with $k < m(n)$ is in the free ultrafilter ω, because there are $s_n \in X_n$ with $\omega\text{–}\lim_n f_n^{(j)}(s_n) < (2k)^{-1}$ for $1 \leq j \leq k+1$. $Y_k \in \omega$ (for all $k \in \mathbb{N}$) implies $\omega\text{–}\lim_n 1/m(n) = 0$.

By definition of $m(n)$ we find $t_n \in X_{n,m(n)} \subset X_n$. Then $\omega\text{–}\lim_n f_n^{(j)}(t_n) = 0$ for every $j \in \mathbb{N}$, because $f_n^{(j)}(t_n) \leq 1/m(n)$ for $n > j$.

Second part: We find $n_1 < n_2 < \ldots$ with $m(n_\ell) > 2^\ell$, because $\omega\text{–}\lim_n 1/m(n) = 0$. Now let $s_\ell := t_{n_\ell} \in X_{n_\ell}$. □

A special case of Lemma A.1 is:

Remark A.2 *Let ω a (fixed) free ultrafilter on $\mathbb{N}$, and let X a bounded subset of a Banach space B. Suppose that $f_1, f_2, \ldots$ is a sequence of functions $f_k \colon X \to [0,2]$.*

If for every $m \in \mathbb{N}$ and $\varepsilon > 0$ there is a sequence $s_1, s_2, \ldots \in X$ such that $\omega\text{-}\lim_n f_k(s_n) < \varepsilon$ for $k = 1, \ldots, m$, then there is a sequence $(t_1, t_2, \ldots)$ in X such that $\lim_n f_k(t_n) = 0$ for all $k \in \mathbb{N}$.

Let A a C^*–algebra, $0 < \gamma < \infty$ and suppose that $X_n \subset \mathcal{L}(A)$ are subsets with $\|T\| \leq \gamma$ for all $T \in X_n$ $(n = 1, 2, \ldots)$. Then $\prod_\omega \{X_1, X_2, \ldots\}$ denotes the set of ultrapowers $T_\omega \colon A_\omega \to A_\omega$ for $T_\omega = (T_1, T_2, \ldots)_\omega$ defined by $T_\omega(\pi_\omega(a_1, a_2, \ldots)) := \pi_\omega(T_1(a_1), T_2(a_2), \ldots)$ where $(a_1, a_2, \ldots) \in \ell_\infty(A)$ and $T_n \in X_n$ for all $n \in \mathbb{N}$.

Lemma A.3 *Suppose that $C \subset A_\omega$ is a separable subset, $0 < \gamma < \infty$ and $X_n \subset \mathcal{L}(A)$ are subsets with $\|T\| \leq \gamma$ for all $T \in X_n$ and $n = 1, 2, \ldots$.*

Then the set of restricted maps $T_\omega | C \colon C \to A_\omega$ with $T_\omega \in \prod_\omega \{X_1, X_2, \ldots\}$ is point-norm closed.

Proof. Let $S \colon C \to A_\omega$ a map with the property that for every finite sequence $c^{(1)}, \ldots, c^{(m)} \in C$ and $\varepsilon > 0$ there is $T_\omega \in \prod_\omega \{X_1, X_2, \ldots\}$ with

$$\|S(c^{(j)}) - T_\omega(c^{(j)})\| < \varepsilon$$

for $j = 1, \ldots, m$. We get that S has Lipschitz constant $< 2\gamma$.

Let $c^{(1)}, c^{(2)}, \ldots$ a dense sequence in C, and $(a_1^{(j)}, a_2^{(j)}, \ldots) \in \ell_\infty(A)$, $(b_1^{(j)}, b_2^{(j)}, \ldots) \in \ell_\infty(A)$ representing sequences for $c^{(j)}$ respectively $S(c^{(j)})$, $j = 1, 2, \ldots$. Then the functions $f_n^{(j)}(T) := \|b_n^{(j)} - T(a_n^{(j)})\|$ on X_n satisfy the assumptions of Lemma A.1. Thus, there are $S_n \in X_n$ with $S_\omega(c^{(j)}) = S(c^{(j)})$ for all $j \in \mathbb{N}$. Since S_ω and S are Lipschitz, it follows that $S = S_\omega | C$. □

Proposition A.4 *Suppose that B is a C^*–algebra and J a closed ideal of B, that $P_1, P_2, \ldots$ is a sequence of polynomials in in non-commuting variables x, x^* with coefficients in B_ω, that $\mathcal{V}_n \subset \mathcal{L}(B)$ are subsets of linear operators of norm $\leq \gamma < \infty$, and that $C \subset B_\omega$ is a separable subset.*

If for each $n \in \mathbb{N}$, $\varepsilon > 0$ and every finite subset $Y \subset C$, there is a contraction $a \in J_\omega$ with $\|P_k(a, a^*)\| < \varepsilon$ for $k = 1, \ldots, n$, and $\|S_\omega(y) - a^* y a\| < \varepsilon \cdot \|y\|$ for suitable $S_n \in \mathcal{V}_n$ and all $y \in Y$.

Then there exist $T_n \in \mathcal{V}_n$ $(n = 1, 2, \ldots)$ and a contraction $x_0 \in J_\omega$ with $P_k(x_0, x_0^*) = 0$ for all $k \in \mathbb{N}$ and $T_\omega(c) = x_0^* c x_0$ for all $c \in C$.

Suppose that, in addition, $A \subset B_\omega$ is σ-unital (respectively is separable) and $a \in \mathrm{Ann}(A) \cap J_\omega$ (respectively $a \in (A,B)^c \cap J_\omega = A' \cap J_\omega$) then there is $x_0 \in \mathrm{Ann}(A) \cap J_\omega$ (respectively $x_0 \in (A,B)^c \cap J_\omega$) with $P_k(x_0, x_0^*) = 0$ for all $k \in \mathbb{N}$.

If one takes as $\mathcal{V}_n$ the set of maps $b \mapsto d^*bd$ with a contraction $d \in B$ (respectively $J = B$), then the assumption on $\mathcal{V}_n$ and a (respectively on J and a) are trivially satisfied if $\max_{k \le n} \|P_k(a^*, a)\| < \varepsilon$.

Proof. The linear operators $S_\omega \colon B_\omega \to B_\omega$ for $S_\omega := (S_1, S_2, \ldots)_\omega$ with $S_n \in \mathcal{V}_n$ have norm $< 2\gamma$. Let $c^{(1)}, c^{(2)}, \ldots$ a dense sequence in C. We find representing sequences $c_1^{(k)}, c_2^{(k)}, \ldots \in B$ for $c^{(k)}$ with $\|c_n^{(k)}\| \le \|c^{(k)}\|$, $k, n \in \mathbb{N}$.

$P_k(x^*, x)$ is the sum of products of $d^{(k,j)} \in B_\omega$, x and x^*. $j = 1, \ldots, \ell_k$. There are representing sequences $d_1^{(k,j)}, d_2^{(k,j)}, \ldots \in B$ of $d^{(k,j)}$ with norms $\le \|d^{(k,j)}\|$. The corresponding non-commutative polynomials $P_n^{(k)}(x^*, x)$ with coefficients in B have the property that $\sup_n \|P_n^{(k)}(b_n^*, b_n)\| < \infty$ for every $(b_1, b_2, \ldots) \in \ell_\infty(B)$ and satisfy

$$\pi_\omega(P_1^{(k)}(b_1^*, b_1), P_2^{(k)}(b_2^*, b_2), \ldots) = P_k(b_\omega^*, b_\omega)\,.$$

Let $X_n = \mathcal{V}_n \times \{b \in J\,;\ \|b\| \le 1\}$ for $n \in \mathbb{N}$. We define

$$f_n^{(k)}(T, b) := \|T(c_n^{(k)}) - b^* c_n^{(k)} b\| + \|P_n^{(k)}(b^*, b)\|$$

for $(T, b) \in X_n$ and $k = 1, 2, \ldots$.

Then $(X_n, f_n^{(1)}, f_n^{(2)}, \ldots)$ $(n = 1, 2, \ldots)$ satisfies the assumptions of Lemma A.1.

Thus there exists $t = ((T_1, b_1), (T_2, b_2), \ldots) \in X_1 \times X_2 \times \cdots$ with $\omega\text{-}\lim_n f_n^{(k)}(T_n, b_n) = 0$. Then $T_\omega = (T_1, T_2, \ldots)_\omega$ and $x_0 := \pi_\omega(b_1, b_2, \ldots)$ are as desired.

To get x_0 in $\mathrm{Ann}(A,B) \cap J_\omega$ or in $(A,B)^c$ we have to add to the polynomials $P_1, P_2, \ldots$ the polynomials $Q_1(x, x^*) = x a_0$ and $Q_2(x, x^*) = a_0 x$ respectively $Q_n(x, x^*) = x a_n - a_n x$, where $a_0 \in A$ is a strictly positive contraction and $a_1, a_2, \ldots$ is dense in the unit ball of A. $\square$

Lemma A.5 *If $T_1, T_2, \ldots \in \mathcal{L}(B, B)$ is a bounded sequence of positive maps and $A \subset B_\omega$ is a σ-unital C^*–subalgebra. Then there are contractions $b_1, b_2, \ldots \in B_+$ such that $\|S_n\| \le \|T_\omega|A\|$ and $S_\omega|A = T_\omega|A$ for $S_n := T_n(b_n(.)b_n)$.*

Proof. Let $d \in A_+$ a strictly positive contraction for A and let $e = (e_1, e_2, \ldots) \in \ell_\infty(B)$ a positive contraction with $\pi_\omega(e) = d$. Then $\|T_\omega(d^{1/k})\| \le \|T_\omega|A\| =: \gamma$ for all $k \in \mathbb{N}$.

Let $X_n := \{t e_n^{1/j}\,;\ j \in \mathbb{N},\, 0 < t \le 1\}$ and consider the functions $f_n^{(k)}(b) := \max(\|e_n^{1/k} - b e_n^{1/k}\|, \|T_n(b^2)\| - \gamma)$ on $X_n \subset B$.

Then $(X_n, f_n^{(1)}, f_n^{(2)}, \ldots)$ $(n = 1, 2, \ldots)$ satisfy the assumptions Lemma A.1, because $\|e^{1/j} e^{1/k} - e^{1/k}\| \le k/j$ and $\|T_\omega(e^{2/j})\| \le \gamma$ for $j \in \mathbb{N}$.

By Lemma A.1, there is a positive contraction $g = (g_1, g_2, \ldots) \in \ell_\infty(A)$ with $g_n \in X_n$ such that $\pi_\omega(g)d = d$ and $\|T_n(g_n^2)\| \leq 1$. Thus, $S_\omega|D = T_\omega|D$ for $D := \overline{dA_\omega d} \supset A$ and $S_n := T_n(g_n(\cdot)g_n)$. □

B Proofs of results in Section 1.

Proof of Proposition 1.3:

Let $a_1, a_2, \ldots \in A_+$ a sequence that is dense in the set of positive contractions in A.

Consider the non-commutative polynomials $P_1(x, x^*) := x^* - x$, $P_2(x, x^*) := a - x^*xa$, $P_3(x, x^*) := (b + c)x^*x$, $P_{3+n}(x, x^*) := a_nx - xa_n$ for $n = 1, 2, \ldots$. An approximate zero for the polynomials $P_k(x, x^*)$ is given by $x_n = a^{1/n}$: $P_k(x_n, x_n^*) = 0$ for $k \neq 2$ and $\|P_2(x_n, x_n^*)\| \leq 2/n$. Thus, by Proposition A.4, there is a self-adjoint contraction $e' \in A' \cap B_\omega$ with $a = e'e'a$ and $(b + c)e' = 0$. Thus $e := e'e' \in (A, B)^c$ is a positive contraction with $ea = a$ and $eb = ec = 0$.

If $z \in A_+$ is a strictly positive element of A, then almost the same argument shows that there is a positive contraction $p \in B_\omega$ with $p(z + b) = z + b$, i.e. $py = yp = y$ for all $y \in C^*(A, b)$.

Let I a closed ideal of B with $b \in I_\omega$, and let $S_1, S_2, \ldots \in \mathcal{V}$ with $S_\omega(c) = bcb$. Consider the non-commutative polynomials $Q_1 := P_1$, $Q_2(x, x^*) := b - x^*xb$, $Q_3(x, x^*) = (e + c)x^*x$ $Q_{3+n} := P_{3+n}$ for $n = 1, 2, \ldots$.

We show below that the sequence of polynomials $(Q_1, Q_2, \ldots)$ have contractive approximate solutions $x_n \in I_\omega$ such that for every $n \in \mathbb{N}$ there is a sequence $S_1^{(n)}, S_2^{(n)}, \ldots$ of contractions in $\mathcal{V}$ with $x_n^*yx_n = S_\omega^{(n)}(y)$ for all $y \in A$.

By Proposition A.4, there exist contractions $T_n \in \mathcal{V}$ $(n = 1, 2, \ldots)$ and a contraction $f' \in I_\omega$ with $P_k(f', (f')^*) = 0$ for all $k \in \mathbb{N}$ and $T_\omega(c) = (f')^*cf'$ for all $c \in A$. Thus $(f')^* = f' \in (A, B)^c$, and $f := f'f' \in (A, B)^c$ is a positive contraction in $A' \cap I_\omega$ with $fe = fc = 0 = b - fb$ and $T_\omega(c) = cf$ for all $c \in A$. In particular, $fa = fea = 0$.

Let $E := C^*(A, b)$, $K := \overline{\mathrm{span}(EbE)}$. Then K is a closed ideal of E, $E = A + K$, $K \subset I_\omega$ and K is the closed span of $\bigcup_n (bA + bAb + Ab)^n$. It follows that every element $d \in K$ is the limit of finite sums $d_n = \sum_n u_nbv_n$ with $u_n \in A \cup \{p\}$ and $v_n \in E \subset B_\omega$. Furthermore, $bEe = \{0\} = bEc$, because $b(A + \mathbb{C}b)^ne = \{0\}$ and $b(A + \mathbb{C}b)^nc = \{0\}$ for $n \in \mathbb{N}$. Thus $(e + c)K = K(e + c) = \{0\}$. Since E is separable, K contains a strictly positive contraction $h \in K_+$.

We find in $C^*(h)_+ \subset K_+$ a sequence of positive contractions $x_1, x_2, \ldots$ with $x_nx_{n+1} = x_n$, $\|h - x_nh\| < 1/n$ and $\lim_{n\to\infty} \|x_nc - cx_n\| = 0$ for all $c \in E$, cf. [29, thm. 3.12.14]. Note that $x_n(e + h) = 0$ for all $n \in \mathbb{N}$, that $\lim \|b - x_n^*x_nb\| = 0$ and $x_n \in I_\omega$.

We show that for every $d \in K$ there is a sequence $R_1, R_2, \ldots \in \mathcal{V}$ with $\sup_n \|R_n\| \leq \|d\|^2$ and $R_\omega(y) = d^*yd$: By assumption there is a bounded sequence $S_1, S_2, \ldots \in \mathcal{V}$ with $S_\omega(y) = b^*yb$ for all $y \in A$. Let (first) d be a finite sum $d = \sum_n u_n b v_n$ with $u_n \in A \cup \{p\}$ and $v_n \in E \subset B_\omega$, and let $(u_1^{(n)}, u_2^{(n)}, \ldots)$ and $(v_1^{(n)}, v_2^{(n)}, \ldots)$ in $\ell_\infty(B)$ be representing sequences for u_n respectively v_n, with $\|v_k^{(n)}\| \leq \|v_n\|$ and $\|u_k^{(n)}\| \leq \|u_n\|$. Then the map R_k, defined by

$$R_k(y) := \sum_{m,n} (v_k^{(m)})^* S_k((u_k^{(m)})^* y u_k^{(n)}) v_k^{(n)} ,$$

is in $\mathcal{V}$, $\|R_k\| \leq \|S_k\| (\sum_n \|v_n\|)^2 (\sum_n \|u_n\|)^2$ and

$$R_\omega(y) = \sum_{m,n} (v_m)^* S_\omega((u_m)^* y u_n) v_n .$$

Since $py = yp = y$ for $y \in A$, we get $R_\omega(y) = d^*yd$ for $y \in A$.

By Lemma A.5 we find another sequence $R'_1, R'_2, \ldots \in \mathcal{V}$ with $R'_\omega(y) = d^*yd$ for $y \in A$ and $\|R'_n\| \leq \|d\|^2$. This happens for every $d \in K$ by Lemma A.3, because every $d \in K$ can be approximated in norm by finite sums $\sum_n u_n b v_n$ of the above considered type. Thus, Proposition A.4 applies to $Q_1, Q_2, \ldots$, $\mathcal{V}_n := \mathcal{V}$, $\gamma = 1$ and I (in place of J there).

Now we can repeat the above arguments with $c, J, e+f, CB(B,B)$ and $c - x^*xc, (f+e)x^*x$ in place of $b, I, e+c, \mathcal{V}$ and Q_2, Q_3 . We get a self-adjoint contraction $g' \in A' \cap J_\omega$, such that with $g := g'g' \in (A,B)^c$, $g \in J_\omega$, $gc = c$, $ge = gf = 0$. Then e, f, g are as stipulated. □

Proof of Proposition 1.6: Suppose that A is a separable C^*–subalgebra of C. The set of all positive elements in $A' \cap I$ of norm < 1 build an an approximate unit for I by Definition 1.5.

Let $b \in C_+$ with $\pi_I(b) \in \pi_I(A)' \cap C/I$. Then $ab - ba \in I$ for all $a \in A$. $[b, A]$ is contained in a separable C^*–subalgebra D of I. Let $d \in D_+$ strictly positive. Since I is a σ-ideal of C there exists a positive contraction $e \in A' \cap I$ with $ed = d$. Then $c := (1-e)b(1-e)$ satisfies $c \in A' \cap C$ and $\pi_I(c) = \pi_I(b)$. Thus

$$0 \to A' \cap I \to A' \cap C \to \pi_I(A)' \cap (C/I) \to 0$$

is short exact.

Let $D \subset \pi_I(A)' \cap (C/I)$ a separable C^*–subalgebra and $B \subset A' \cap C$ a separable C^*–algebra with $\pi_I(B) = D$. If d denotes a strictly positive element of $B \cap I$, then there is a positive contraction $e \in C^*(A \cup B)' \cap I$ with $ed = d$.

There is a *-morphism $\lambda\colon\ C_0(0,1] \otimes B \to A' \cap C$ with $\lambda(f_0^n \otimes b) = (1-e)^n b$ for $b \in B$ and $n \in \mathbb{N}$. It follows $\lambda(C_0(0,1] \otimes (B \cap I)) = \{0\}$ and $\pi_I(\lambda(f)) = \pi_I(f(1))$ for $f \in C_0((0,1], B) \cong C_0(0,1] \otimes B$. Thus there is a *-morphism $\psi\colon\ C_0((0,1] \otimes D) \to A' \cap C$ with $\psi(f_0 \otimes h) = \lambda(f_0 \otimes b)$ for $b \in B$ with $\pi_I(b) = h$. ψ satisfies $\pi_I \circ \psi(f_0 \otimes h) = h$ for $h \in D$.

Since $\mathrm{Ann}(\pi_I(A), C/I) \subset \pi_I(A)' \cap (C/I)$, for every positive $f \in \mathrm{Ann}(\pi_I(A), C/I)$ there is a positive element $b \in A' \cap C$ with $\pi_I(b) = f$. Let

$a_0 \in A_+$ a strictly positive element of A. Then $ba_0 \in I$. There is a positive contraction $e \in C^*(b)' \cap I$ with $eba_0 = ba_0$. It follows that $c := b(1-e) \in C_+$ satisfies $ca_0 = 0$ and $\pi_I(c) = f$. □

Lemma B.1 *Suppose that A is a σ-unital non-degenerate C^*-subalgebra of a C^*–algebra D, that $E \subset A$ is a full and hereditary σ-unital C^*–subalgebra of A, and let $D_E := \overline{EDE}$. Then the natural map from $A' \cap \mathcal{M}(D)$ into $E' \cap \mathcal{M}(D_E)$ is a *-isomorphism (onto $E' \cap \mathcal{M}(D_E)$).*

Proof. The natural *-morphism is given by $\iota(T)c = Tc$ for $T \in A' \cap \mathcal{M}(D)$ and $c \in D_E$. $TD_E \subset D_E$, because T commutes with $E \subset A$. If $\iota(T) = 0$, then $TAEA = ATEA = \{0\}$ because T commutes with A. It follows $TA = \{0\}$ and $T = 0$, because span(AEA) is dense in A and span(AD) is dense in D. Thus ι is a *-monomorphism from $A' \cap \mathcal{M}(D)$ into $E' \cap \mathcal{M}(D_E)$, and it suffices to construct a *-morphism $\kappa\colon\ E' \cap \mathcal{M}(D_E) \to A' \cap \mathcal{M}(D)$ with $\iota \circ \kappa = \mathrm{id}$.

One can see, that $(A \otimes \mathcal{K})' \cap \mathcal{M}(D \otimes \mathcal{K}) = (A' \cap \mathcal{M}(D)) \otimes 1$ and $A' \cap \mathcal{M}(D) = \mathcal{M}(A)' \cap \mathcal{M}(D)$ for all non-degenerate pairs $A \subset D$.

There is an element $g \in A \otimes \mathcal{K}$ such that g^*g is a strictly positive element of $A \otimes \mathcal{K}$ and gg^* is a strictly positive element of $E \otimes \mathcal{K}$, cf. [7]. The polar decomposition $g = v(g^*g)^{1/2} = (gg^*)^{1/2}v$ of g in $(A \otimes \mathcal{K})^{**}$ defines an isomorphism ψ from $E \otimes \mathcal{K}$ onto $A \otimes \mathcal{K}$ by $\psi(e) := v^*ev$. Clearly, ψ extends to an isomorphism from $\mathcal{M}(D_E \otimes \mathcal{K})$ onto $\mathcal{M}(D \otimes \mathcal{K})$ such that $\psi(T) = v^*Tv$ in $(D \otimes \mathcal{K})^{**}$. It maps to $\mathcal{M}(D \otimes \mathcal{K})$ because $\psi(T)x = \lim_n (g^*g + 1/n)^{-1/2} g^*Tg(g^*g + 1/n)^{-1/2}x$ for $x \in D \otimes \mathcal{K}$.

If $T \in \mathcal{M}(D_E \otimes \mathcal{K})$ commutes with $E \otimes \mathcal{K}$, then $\psi(T)$ commutes with $A \otimes \mathcal{K}$ and $\psi(T)y = Ty$ for all $y \in D_E \otimes \mathcal{K}$, because $Tg(g^*g + 1/n)^{-1/2}(gg^*)^{1/k}y = g(g^*g + 1/n)^{-1/2}(gg^*)^{1/k}Ty$ for all $y \in D_E \otimes \mathcal{K}$.

Thus, there is a *-morphism κ from $E' \cap \mathcal{M}(D_E)$ into $A' \cap \mathcal{M}(D)$ with $\kappa(S) \otimes 1 = \psi(S \otimes 1)$. We have $\iota(\kappa(S))(c) \otimes p = \psi(S \otimes 1)(c \otimes p) = (S \otimes 1)(c \otimes p)$ for $c \in D_E$ and $p \in \mathcal{K}$. Hence $\iota \circ \kappa = \mathrm{id}$. □

Proof of Proposition 1.9: (1) is obvious.

(2)+(3): Let $Y = \{y_1, y_2, \ldots\} \subset B_\omega$, $a_0 \in A_+$ a strictly positive element of A, and $c := (1 + \|d\|)^{-1}d$ with $d := a_0 + \sum_n 2^{-n}(1 + \|y_n\|^2)^{-1}(y_ny_n^* + y_n^*y_n)$. By Corollary 1.7 there exists a positive contraction $e \in B_\omega$ with $ec = c$. Thus $ea_0 = a_0 = a_0e$ and $ey = y = ye$ for all $y \in Y$.

If $e \in B_\omega$ is any positive contraction with $ea_0 = a_0$ then $ea = a = ae$ for all $a \in D_{A,B} \supset A$. In particular, $e \in (A, B)^c$. If $b \in (A, B)^c \subset \{a_0\}' \cap B_\omega$, then $(eb - b)$ and $(be - b)$ are in $\mathrm{Ann}(a_0, B_\omega) = \mathrm{Ann}(A, B_\omega)$. Thus $e + \mathrm{Ann}(A, B_\omega) = 1$ in $F(A, B)$.

(4): The natural *-morphism is given by

$$b \in \mathcal{N}(D_{A,B}) \mapsto L_b \in \mathcal{M}(D_{A,B}) \subset \mathcal{L}(D_{A,B}),$$

where $L_b(a) := ba$ for $a \in D_{A,B}$, and involution on $\mathcal{M}(D_{A,B})$ is defined by $t^*(a) := t(a^*)^*$ for $a \in D_{A,B}$ and $t \in \mathcal{M}(D_{A,B})$. Clearly, this is a *-morphism with kernel $\mathrm{Ann}(A, B_\omega)$. $D_{A,B}$ embeds naturally into $\mathcal{M}(D_{A,B})$ by $b \mapsto L_b$ for $b \in D_{A,B}$.

Let $t \in \mathcal{M}(D_{A,B})_+$ and let $a_0 \in A$ a strictly positive contraction. Then $c_n := a_0^{1/n} t a_0^{1/n} \in D_{A,B}$ converges to t in the strict topology. In particular, $L_{c_n} \colon C^*(a_0) \to B_\omega$ converges in point-norm topology to $t|C^*(a_0)$.

Let $\mathcal{S}$ denote the set of maps $L_b \colon B \to B$ with $b \in B_+$ and $\|b\| \le 1$. Then $L_{c_n} \in \mathcal{S}^\omega$ for every $n \in \mathbb{N}$. By Lemma A.3 (or by [22, proof of lem. 2.13]) there exists a sequence $L_{b_n} \in \mathcal{S}$ with $t|C^*(a_0) = (L_{b_1}, L_{b_2}, \ldots)_\omega | C^*(a_0)$. Thus, $b := \pi_\omega(b_1, b_2, \ldots) \in B_\omega$ satisfies $b \ge 0$ and $b a_0^{1/n} = t(a_0^{1/n})$ for $n \in \mathbb{N}$. Since, a_0 is a strictly positive element of $D_{A,B}$, it follows that $b \in \mathcal{N}(D_{A,B})$ and $L_b = t$.

(5): Since $\mathrm{Ann}(A, B_\omega) = \mathrm{Ann}(D_{A,B}, B_\omega)$, the kernel is $\mathrm{Ann}(A, B_\omega) \subset (A, B)^c$. Clearly, the image of $(A, B)^c$ in $\mathcal{M}(D_{A,B})$ commutes with A. If $c \in \mathcal{M}(D_{A,B})$ commutes with A and is the image of $b \in \mathcal{N}(D_{A,B})$, then $[b, A] \subset \mathrm{Ann}(A, B_\omega)$. Thus $[b, a_1 a_2] = 0$ for $a_1, a_2 \in A$. Since $A = A \cdot A$, it follows $b \in (A, B)^c$. Hence the natural epimorphism from $\mathcal{N}(D_{A,B})$ onto $\mathcal{M}(D_{A,B})$ defines a *-isomorphism η from $F(A, B) = (A, B)^c / \mathrm{Ann}(A, B_\omega)$ onto $A' \cap \mathcal{M}(D_{A,B})$ with $\rho_{A,B}(g \otimes a) = \eta(g) a$ for $g \in F(A, B)$ and $a \in A$.

(6): If $e = e^2 \ge 0$ is the unit of $(A, B)^c$, and $b \in B_\omega$ is a positive contraction with $be = 0$, then $e + \mathrm{Ann}(A, B_\omega)$ is the unit of $F(A, B)$ and $ba = b\rho_A(1 \otimes a) = bea = 0$ for $a \in A$, i.e. $b \in \mathrm{Ann}(A, B_\omega)$. Since $\mathrm{Ann}(A, B_\omega)$ a closed ideal of $(A, B)^c$, it follows $b = 0$. Thus e is the unit of B_ω.

If f is the unit element of B_ω and $(f_1, f_2, \ldots) \in \ell_\infty(B)$ is a representing sequence of positive contractions for f, then $g := \sum_n 2^{-n} f_n$ satisfies $\|g\| \le 1$ and $f_n \le 2^{1/n} g^{1/n^2}$. Hence $f = h$ for $h := \pi_\omega(g^{1/4}, g^{1/9}, \ldots)$. It follows that zero can not be in the spectrum of g, i.e. that B is unital.

The other implications are obvious.

(7): Clearly, if B is unital and $1_B \in A$, then $\mathrm{Ann}(A, B_\omega) = \{0\}$.

If $\mathrm{Ann}(A, B_\omega) = \{0\}$ then $(A, B)^c \cong F(A, B)$. Thus $(A, B)^c$ and B are unital by parts (1) and (6). Let $a_0 \in A$ is a strictly positive contraction for A, then $1_B \in D_{A,B} = \overline{a_0 B_\omega a_0}$ by Remark 2.7. It follows that a_0 is invertible in B_ω, i.e. $1_B \in A$.

(8): Let $E := \overline{dAd}$. Then E is a full σ-unital hereditary C^*–subalgebra of A and $\overline{dD_{A,B}d} = D_{E,B} = \overline{ED_{A,B}E}$.

A natural *-morphism ι from $A' \cap \mathcal{M}(D_{A,B})$ into $E' \cap \mathcal{M}(D_{E,B})$ is given by $\iota(T)c := Tc$ for $T \in A' \cap \mathcal{M}(D_{A,B})$ and $c \in D_{E,B}$. It is a *-isomorphism from $A' \cap \mathcal{M}(D_{A,B})$ onto $E' \cap \mathcal{M}(D_{E,B})$ by Lemma B.1, because A is a σ-unital non-degenerate subalgebra of $D_{A,B}$, $E \subset A$ is a full hereditary σ-unital C^*-subalgebra of A, and $D_{E,B} = \overline{ED_{A,B}E}$.

Let $\eta_1 \colon F(A, B) \to A' \cap \mathcal{M}(D_{A,B})$ and $\eta_2 \colon F(E, B) \to E' \cap \mathcal{M}(D_{E,B})$ the isomorphisms from part (5), then $\psi := \eta_2^{-1} \circ \iota \circ \eta_1$ is a *-isomorphism from $F(A, B)$ onto $F(E, B)$ with $\rho_{E,B}(\psi(g) \otimes a) = \rho_{A,B}(g \otimes a)$ for $a \in E \subset A$ and $g \in F(A, B)$.

(9): Suppose that $C \subset B$ is a hereditary C^*–subalgebra with $A \subset C_\omega \subset B_\omega$. Then $D_{A,C} = D_{A,B} \subset C_\omega$. Since A is σ–unital, the natural *-morphisms $\mathcal{N}(D_{A,C}) \to \mathcal{M}(D_{A,C})$ and $\mathcal{N}(D_{A,B}) \to \mathcal{M}(D_{A,C})$ are epimorphisms by part (4), and map $(A, C)^c$ respectively $(A, B)^c$ onto $A' \cap \mathcal{M}(D_{A,C})$. Thus $(A, B)^c = (A, C)^c + \mathrm{Ann}(A, B_\omega)$. Because $\mathrm{Ann}(A, C_\omega) = \mathrm{Ann}(A, B_\omega) \cap C_\omega$, it follows $F(A, B) \cong F(A, C)$. □

Proof of Proposition 1.12:
Let H_∞ denote the free semi-group on countably many generators $X := \{x_1, x_2, \ldots\}$ with involution given by $(y_1 \cdot y_2 \cdots y_n)^* := y_n \cdots y_2 \cdot y_1$ for $y_i \in X$, and let $C^*(H_\infty)$ be the full C^*–hull $C^*(\ell_1(H_\infty))$ of the Banach *-algebra $\ell_1(H_\infty)$. $C^*(H_\infty)$ is projective in the category of all C^*–algebras.

Since $(C^*(A, B), B)^c \subset (A, B)^c$ and $\mathrm{Ann}(C^*(A, B), B_\omega) \subset \mathrm{Ann}(A, B_\omega)$, it suffices to consider the case where $B \subset A$ to get (1) also for general separable $A \subset B_\omega$. So we proof the strong result (2) in case $B \subset A$.

Let $a_0 \in A_+$ a strictly positive contraction for A with $\|a_0\| = 1$. $b_1 := a_0, b_2, \ldots \in A_+$, $d_1 := 1, d_2, \ldots \in D_+$ sequences that are dense in the set of positive contractions of norm one in A respectively in D, and let $f_0 \in B_+$ denote a strictly positive contraction for B. For each $n \in \mathbb{N}$ there are

(1) a sequence $c^{(n)}_1, c^{(n)}_2, \ldots \in B_+$ with $\pi_\omega(c^{(n)}_1, c^{(n)}_2, \ldots) = b_n$ and $\|c^{(n)}_k\| = 1$,
(2) a sequence $e^{(n)}_1, e^{(n)}_2, \ldots \in B_+$ with $e_n := \pi_\omega(e^{(n)}_1, e^{(n)}_2, \ldots) \in B^c$, $\|e^{(n)}_k\| = 1$, and $e_n + \mathrm{Ann}(B) = d_n$, and
(3) a sequence $\mu^{(n)}_1, \mu^{(n)}_2, \ldots \in B^*$ of pure states on B with $\mu^{(n)}_\omega(f_0 e_n) = \|f_0 e_n\| = \|\rho_B(d_n \otimes f_0)\|$.

For $k \in \mathbb{N}$, we define *-morphisms $\theta_k \colon C^*(H_\infty) \to B$ by $\theta_k(x_n) := e^{(n)}_k$ for the generators $\{x_1, x_2, \ldots\}$ of H_∞. Further let $G := C^*(e_1, e_2, \ldots)$, and $Y_n := \{c^{(j)}_k ;\ k, j \le n\}$.

$$\theta_\omega = (\theta_1, \theta_2, \ldots)_\omega \colon\ h \in C^*(H_\infty) \mapsto \pi_\omega(\theta_1(h), \theta_2(h), \ldots) \in G \subset B^c \subset B_\omega$$

is an epimorphism from $C^*(H_\infty)$ onto G.

Let h_1 a strictly positive contraction for $(\theta_\omega)^{-1}(G \cap \mathrm{Ann}(B))$ and h_2 a strictly positive contraction for $C^*(H_\infty)$ with $\theta_\omega(h_2) + \mathrm{Ann}(B) = 1$ in $F(B)$.

Below we select sub-sequences $(\theta_{k_m})_{m\in\mathbb{N}}$ and $(\mu^{(n)}_{k_m})_{m\in\mathbb{N}}$ of $(\theta_k)_{k\in\mathbb{N}}$ respectively $(\mu^{(n)}_k)_{k\in\mathbb{N}}$ (for $n = 1, 2, \ldots$) such that the morphism $\varphi := (\theta_{k_1}, \theta_{k_2}, \ldots)_\omega$ from $C^*(H_\infty) \subset C^*(H_\infty)_\omega$ into $(A, B)^c = A' \cap B_\omega$ satisfies $\varphi(h_1)a_0 = 0$, $\varphi(h_2)a_0 = a_0$ and $\lambda(\varphi(x_n)f_0) = \|\rho_B(d_n \otimes f_0)\|$ for $\lambda := (\mu_{k_1}, \mu_{k_2}, \ldots)_\omega$, i.e.

$$\lim_{m\to\omega} \mu^{(n)}_{k_m}(\theta_{k_m}(x_n)f_0) = \|\rho_B(d_n \otimes f_0)\| .$$

Indeed, we define for each $m \in \mathbb{N}$ the subsets $Q_m, R_m, S_m, T_m \subset \mathbb{N}$ as the set of $k \in \mathbb{N}$ with $\|\theta_k(x_j)y - y\theta_k(x_j)\| < 1/m$ for all $y \in Y_m$ and $j \le m$, $|\mu^{(j)}_k(\theta_k(x_j)f_0) - \|e_j f_0\|| < 1/m$ for $j \le m$, $\|\theta_k(h_1)b^{(1)}_j\| < 1/m$ for $j \le m$, respectively $\|b_j^{(1)} - \theta_k(h_2)b_j^{(1)}\| < 1/m$ for $j \leq m$. Then $Q_m, R_m, S_m, T_m \in \omega$ and, hence, $W_m := Q_m \cap R_m \cap S_m \cap T_m \in \omega$. In particular, W_m is infinite. Since $W_1 \supset W_2 \supset W_3 \supset \cdots$ and W_m is *not* finite, we find $k_m \in W_m$ such that $k_1 < k_2 < \cdots$. The sub-sequence $k_1, k_2, \ldots$ is as desired.

The above defined map $h \in C^*(H_\infty) \mapsto \varphi(h) + \mathrm{Ann}(A, B) \in F(A, B)$ maps h_2 to the unit of $F(B)$ and h_1 to zero. Thus it defines a unital *-morphism $\gamma_1 \colon D \to F(A, B)$ with

$$\gamma_1(\theta_\omega(h) + \mathrm{Ann}(B)) = \varphi(h) + \mathrm{Ann}(A, B)$$

for $h \in C^*(H_\infty)$. Since $\mathrm{Ann}(B)$ is an ideal of $B^c \supset (A, B)^c$ and contains $\mathrm{Ann}(A, B_\omega)$ we can compose γ_1 with the morphism $F(A, B) \to F(B)$ and get $\gamma_2 \colon D \to F(B)$ with

$$\gamma_2(\theta_\omega(h) + \mathrm{Ann}(B)) = \varphi(h) + \mathrm{Ann}(B)\,.$$

Then $\|\rho_B(\gamma_2(d_n) \otimes f_0)\| = \|\varphi(x_n)f_0\| \geq \|\rho_B(d_n \otimes f_0)\|$ for $n = 1, 2, \ldots$. Thus, $\|\rho_B(\gamma_2(d) \otimes f_0\| \geq \|\rho_B(d \otimes f_0)\| > 0$ for all $d \in D_+ \setminus \{0\}$, i.e. $\gamma_2 \colon D \to F(B)$ is faithful.

By Corollary 1.8 there exists a *-morphism $\psi \colon C_0((0, 1], D) \to (A, B)^c$ with $\psi(f) + \mathrm{Ann}(A, B_\infty) = \gamma_2(f(1))$ for $f \in C_0((0, 1], D)$. ψ is as desired. □

Proof of Corollary 1.13: If A is separable and $C, B_1, B_2, \ldots \subset F(A)$ are separable unital C^*-subalgebras, then we get by induction unital separable C^*–subalgebras $C_1 := C \subset C_2 \subset \ldots \subset F(A)$ and unital *-morphisms $\psi_n \colon C_n \otimes^{\max} B_n \to F(A)$ with $\psi_n | 1 \otimes B_n$ faithful and $\psi_n | C_n = \mathrm{id}$. Here we let $C_{n+1} := \psi_n(C_n \otimes^{\max} B_n)$. This follows from Corollary 1.8 and part (2) of Proposition 1.12 (with B,A and D replaced by A, $C^*(\lambda(C^*((0, 1], C_n)), A)$ and D_n respectively).

Note that $C \subset C_2 \subset C_3 \subset \ldots$ and that there is a natural unital *-homomorphism ψ from $C \otimes^{\max} B_1 \otimes^{\max} B_2 \otimes^{\max} \ldots$ onto the closure of $\bigcup_n C_n \subset F(A)$ with the properties as stipulated. □

Proof of Proposition 1.14: Let $C^*(H_\infty)$ as in the proof of Proposition 1.12, and let $a_k \in A_k$ a strictly positive contraction of A_k with $\|(1 - a_k)a_{k-1}\| < 2^{-k-1}$, and $a_0 := \sum_{k \in \mathbb{N}} 2^{-k} a_k \in A_+$. There are *-morphisms $\varphi_k \colon C^*(H_\infty) \to A_k^c \subset (A_k, A)^c$ such that the morphisms $\psi_k(h) := \varphi(h) + \mathrm{Ann}(A_k) \in F(A_k)$ have the property that

$$\psi_\omega \colon C^*(H_\infty) \to \prod_\omega \{F(A_1), F(A_2), \ldots\}$$

maps $C^*(H_\infty)$ onto A. Let $h_1 \in C^*(H_\infty)_+$ a strictly positive element of the kernel of ψ_ω, and let $h_2 \in C^*(H_\infty)_+$ a strictly positive contraction for $C^*(H_\infty)$ with $\psi_\omega(h_2) = 1$.

Since $C^*(H_\infty)$ is projective, there are *-morphisms $\varphi_n^{(k)} \colon C^*(H_\infty) \to A$ with $(\varphi_1^{(k)}, \varphi_2^{(k)}, \ldots)_\omega = \varphi_k$. It turns out that for suitable $\lambda_m = \varphi_{\ell_m}^{(k_m)}$ holds:

$$\lambda := (\lambda_1, \lambda_2, \ldots)_\omega \colon\ C^*(H_\omega) \to A_\omega$$

has the properties $\lambda(C^*(H_\omega)) \subset A^c$, $\lambda(h_1)a_0 = 0$ and $\lambda(h_2)a_0 = a_0$. Indeed: apply Remark A.2 with $X = \{\varphi_n^k\,;\ n, k \in \mathbb{N}\} \subset \mathcal{L}(C^*(H_\omega), A)$ and functions $f_k \colon X \to [0, 2]$ given by

$$f^k(\varphi) := \max\{\|\varphi(h_1)a_0\|,\ \|\varphi(h_2)a_0 - a_0\|,\ \|[\varphi(x_j), b_i]\|\,;\ i, j \le k\}\,,$$

where $b_1, b_2, \ldots$ is a dense sequence in the unit ball of A. □

Proof of Corollary 1.16: Clearly, J_ω is an essential ideal of B_ω if J is an essential ideal of B. Since $(A, J)^c = J_\omega \cap (A, B)^c$ is a σ-ideal of $(A, B)^c$ (cf. Corollary 1.7), we get from Proposition 1.6 that $(A, J)^c$ is a non-degenerate C^*–subalgebra of J_ω. If the image $d + \mathrm{Ann}(A, B_\omega)$ in $F(A, B)$ of $d \in (A, B)^c_+$ is orthogonal to $F(A, J)$, then $(A, J)^c d \subset \mathrm{Ann}(A, B_\omega)$.

Let $a_0 \in A_+$ a strictly positive element of A. We have $J_\omega d a_0 = \{0\}$, because $(A, J)^c$ is non-degenerate. Thus $da_0 = 0$ and $d \in \mathrm{Ann}(A, B_\omega)$. Hence $F(A, J)$ is an essential ideal of $F(A, B)$. □

Proof of Proposition 1.17: Note that $\mathcal{E}(D_0, D_1)$ is naturally isomorphic to the quotient of $\mathrm{cone}(D_0) \otimes^{max} \mathrm{cone}(D_1)$ by the ideal generated by

$$((f_0 \otimes 1_{D_0}) \otimes 1) + (1 \otimes (f_0 \otimes 1_{D_1})) - 1\,.$$

Here, $\mathrm{cone}(D_0) \subset C([0, 1], D_0)$ means the unitization of $C_0((0, 1], D_0)$. We denote the natural epimorphism from $\mathrm{cone}(D_0) \otimes^{max} \mathrm{cone}(D_1)$ onto $\mathcal{E}(D_0, D_1)$ by η.

Let $\pi := (\pi_J)_\omega \colon B_\omega \to (B/J)_\omega$ denote the the ultrapower of the epimorphism π_J from B onto B/J. (The kernel of π is J_ω and $\pi(B_\omega) = (B/J)_\omega$.)

Let $A_1 := C^*(A + J)$. Note that $\pi(A_1) = \pi(A) \subset (B/J)_\omega$, $(A_1, B)^c \subset (A, B)^c$, and that $\pi \colon (A_1, B)^c \to (B/J)_\omega$ maps $F(A_1, B) = (A_1, B)^c$ *onto* $(\pi(A), B/J)^c = F(\pi(A), B/J)$ (cf. Remark 1.15). Thus, we can suppose, that $J \subset A \subset B_\omega$.

It suffices to find a unital *-morphism H from $\mathrm{cone}(D_0) \otimes^{max} \mathrm{cone}(D_1)$ into $(A, B)^c = A' \cap B_\omega$ with $H((f_0 \otimes 1) \otimes 1) + H(1 \otimes (f_0 \otimes 1)) = 1$. Below we construct *-homomorphisms $h_1 \colon C_0((0, 1], D_1) \to (A, B)^c$ and $h_0 \colon C_0((0, 1], D_0) \to (A, B)^c$ with commuting images, such that $h_0(f_0 \otimes 1) + h_1(f_0 \otimes 1) = 1$ and $\pi(h_1(f)) = f(1)$ for all $f \in C_0((0, 1], D_1)$. There is a unique unital *-morphism

$$H \colon\ \mathrm{cone}(D_0) \otimes^{max} \mathrm{cone}(D_1) \ \to\ (A, B)^c$$

with $H(g \otimes 1) = h_0(g)$ for all $g \in C_0((0, 1], D_0)$ and $H(1 \otimes f) = h_1(f)$ for $f \in C_0((0, 1], D_0)$. Then H has the desired property and $\pi(H(1 \otimes f)) = f(1) \in D_1$ for $f \in \mathrm{cone}(D_1)$. The unital *-morphism $h \colon \mathcal{E}(D_0, D_1) \to (A, B)^c = F(A, B)$ with $h \circ \eta = H$ satisfies $\pi(h(f)) = f(1)$ for $f \in \mathrm{cone}(D_1)$.

$J_\omega \cap (A, B)^c$ is a σ-ideal of $(A, B)^c$ (cf. Corollary 1.7) and

$$0 \to A' \cap J_\omega \to (A, B)^c \to (\pi(A), B/J)^c$$

is short-exact and strongly locally liftable (cf. Remark 1.15). By Proposition 1.6, there exists a *-morphism $\varphi\colon\ C_0((0,1], D_1) \to (A,B)^c$ with $\pi(\varphi(f)) = f(1) \in D_1$ for $f \in C_0((0,1], D_1)$. In particular, $1 - \varphi(f_0 \otimes 1) \in J_\omega$. Let $D_2 := \varphi(C_0((0,1], D_1))$. Then $\varphi(C_0((0,1), D_1)) = J_\omega \cap D_2 \subset J_\omega \cap (A,B)^c$. The unital C^*–subalgebra $G := C^*(A, D_2)$ of B_ω is separable. $J_\omega \cap G$ contains $1 - \varphi(f_0 \otimes 1)$, J, and $\varphi((0,1), D_1) = J_\omega \cap D_2$. Let g_0 a strictly positive element of $J_\omega \cap G$. Since J_ω is a σ-ideal of B_ω (by Corollary 1.7), there is a positive contraction $e \in G' \cap J_\omega$ with $eg_0 = g_0$. Then $eb = be$ for all $b \in G \supset A$ and $ej = j$ for all $j \in J_\omega \cap G \supset J$. In particular, $e \in (A,B)^c$ and $(1-e)(1-\varphi(f_0 \otimes 1)) = 0$. Since e commutes element-wise with D_2, we can modify φ as follows:
There is a unique *-morphism $h_1\colon\ C_0((0,1], D_1) \to B_\omega$ with

$$h_1(f_0^n \otimes d) = (1-e)^n \varphi(f_0^n \otimes d)$$

for $d \in D_1$ and $n \in \mathbb{N}$. The *-morphism h_1 maps $C_0((0,1], D_1)$ into $(A,B)^c$ and $\pi(h_1(f)) = f(1) \in D_1$ for $f \in C_0((0,1], D_1)$. Note that $h_1(f_0 \otimes 1) = (1-e)$. Now let $G_1 := C^*(eG, e) \subset J_\omega$. Then e is a strictly positive element of G_1 and is in the center of G_1, because $e \in (G,J)^c$.

By Proposition 1.12 there exists a *-morphism ψ from $C_0((0,1], D_0)$ into $(G_1, J)^c = G_1' \cap J_\omega$ with $\psi(f_0 \otimes 1)b = b$ for all $b \in G_1$, i.e.

$$\theta\colon\ d \in D_0 \mapsto \psi(f_0 \otimes d) + \mathrm{Ann}(G_1, J_\omega) \in F(G_1, J)$$

is a unital *-morphism from D_0 into $F(G_1, J)$. Since $e \in G_1$ commutes with the image of ψ we can modify ψ as follows:
There is a unique *-morphism $h_0\colon\ C_0((0,1], D_0) \to B_\omega$ with

$$h_0(f_0^n \otimes d) = e^n \psi(f_0^n \otimes d) = \rho_{G_1}(\theta(d) \otimes e^n)$$

for $d \in D_0$ and $n \in \mathbb{N}$.

Let $b \in G$ and $d \in D_0$, then

$$be^n h_2(f_0^n \otimes d) = h_2(f_0^n \otimes d) be^n = e^n \psi(f_0^n \otimes d) b$$

for all $b \in G$, $n \in \mathbb{N}$. Thus, h_0 maps $C_0((0,1] \otimes D_0)$ into $G' \cap B_\omega$, i.e. the image of h_0 is in $(A,B)^c$ and commutes element-wise with the image of h_1. Furthermore, $h_0(f_0 \otimes 1) = \psi(f_0 \otimes 1)e = e$ because $e \in G_1$. Hence, h_1, h_0 define h (via H) with the stipulated properties. □

C Some calculations with KTP

For convenience of the reader we add here some calculations that help to verify some of the remarks in Section 4.

Proof of Remark 4.6: Suppose that $\mathcal{D}$ is self-absorbing. By Proposition 4.4, $\mathcal{D}$ is simple, is nuclear, has a unique tracial state or is purely infinite, and $\mathcal{D} \subset F(\mathcal{D})$.

If $[1] = 0$ in $K_0(\mathcal{D})$ then $\mathcal{D}$ can not have a tracial state. Thus $\mathcal{D}$ is purely infinite and $\mathcal{O}_2$ is unitally contained in $\mathcal{D} \subset F(\mathcal{D})$. Hence, $\mathcal{D} \cong \mathcal{O}_2$ by [23] (or [20, p. 135]).

If $\mathcal{D}$ is tensorially self-absorbing, then $\mathcal{D} \otimes \mathcal{O}_\infty$ is tensorially self-absorbing simple p.i.s.u.n. algebra with $K_*(\mathcal{D} \otimes \mathcal{O}_\infty) = K_*(\mathcal{D})$.

Suppose that $K_0(\mathcal{D}) \neq 0$, that $\mathcal{D}$ is a p.i.s.u.n. algebra and that $\mathcal{D}$ satisfies the KTP, i.e. that with $A = B = \mathcal{D}$ there are (unnaturally) splitting short-exact sequences

$$0 \to \mathrm{Tens}(A, B, \alpha) \to K_\alpha(A \otimes B) \to \mathrm{Tor}(A, B, \alpha) \to 0$$

for $A = B = \mathcal{D}$ and $\alpha \in \{0, 1\}$. Here

$$\mathrm{Tens}(A, B, \alpha) := (K_\alpha(A) \otimes K_0(B)) \oplus (K_{1-\alpha}(A) \otimes K_1(B))$$

and,

$$\mathrm{Tor}(A, B, \alpha) := \mathrm{Tor}(K_0(A), K_{1-\alpha}(B)) \oplus \mathrm{Tor}(K_1(A), K_\alpha(B)).$$

The monomorphism $K_\alpha(\mathcal{D}) \otimes K_0(\mathcal{D}) \to K_\alpha(\mathcal{D} \otimes \mathcal{D})$ is induced by $[x]_\alpha \otimes [p]_0 \mapsto [x \otimes p]_\alpha$ for projections $p \in \mathcal{D}$ and projections or unitaries in $\mathcal{D}$.

The isomorphisms $K_\alpha(\mathcal{D}) \otimes [1_\mathcal{D}]_{K_0} \cong K_\alpha(\mathcal{D} \otimes \mathcal{D})$ imply that $K_1(\mathcal{D}) \otimes K_1(\mathcal{D}) = 0$, and that $\mathrm{Tor}(K_\alpha(\mathcal{D}), K_\alpha(\mathcal{D})) = 0$ for $\alpha = 0, 1$. Thus $K_0(\mathcal{D})$ and $K_1(\mathcal{D})$ are torsion-free (because all Abelian groups with a non-zero torsion element have some $\mathbb{Z}_p$ or some p-Prüfer-group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ as direct summand, cf. [16, cor. 27.3]). Therefore, $K_1(\mathcal{D}) \otimes K_1(\mathcal{D}) = 0$ implies $K_1(\mathcal{D}) = 0$. The flip on $\mathcal{D} \otimes \mathcal{D}$ induces the flip on $K_0(\mathcal{D}) \otimes K_0(\mathcal{D}) \cong K_0(\mathcal{D} \otimes \mathcal{D})$, i.e. $[1] \otimes_\mathbb{Z} x = x \otimes_\mathbb{Z} [1]$ in $K_0(\mathcal{D}) \otimes K_0(\mathcal{D})$ for $x \in K_0(\mathcal{D})$. This means that there are non-zero $m, n \in \mathbb{Z}$ with $m[1] = nx$. Thus $K_0(\mathcal{D})$ is a unital subring of the rational numbers $\mathbb{Q}$ (if $K_0(\mathcal{D}) \neq 0$).
If we now suppose in addition that $\mathcal{D}$ satisfies the UCT, then the classification of simple p.i.s.u.n. algebras yields $\mathcal{D} = \mathcal{O}_\infty$ if $K_0(\mathcal{D}) \cong \mathbb{Z}$, and

$$\mathcal{D} = \mathcal{O}_\infty \otimes (\bigotimes_{p \in X} M_{p^\infty}),$$

where X is the set of prime numbers with $1/p \in K_0(\mathcal{D}) \subset \mathbb{Q}$ if $\mathcal{D} \ncong \mathcal{O}_2, \mathcal{O}_\infty$.

If $\mathcal{D}$ is not purely infinite, then $\mathcal{D}$ has a unique tracial state τ, and τ defines an order preserving isomorphism from $K_0(\mathcal{D})$ onto the subring $\tau(K_0(\mathcal{D}))$ of the rational numbers. It is an order isomorphism if and only if $(K_0(\mathcal{D}), K_0(\mathcal{D})_+)$ is weakly unperforated.

Thus, the given list of algebras exhausts all possible Elliott invariants that could appear for the algebras $\mathcal{D} \otimes \mathcal{Z}$ in the UCT-class. $\square$

Proof of Remarks 4.3: (1): The Cuntz algebra $\mathcal{O}_2$ is isomorphic to $\mathcal{D} := D \otimes D \otimes \ldots$, because $\mathcal{O}_2$ is unitally contained in $F(\mathcal{D})$ (cf. proof of Remark 2.17). Since $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes D$, η_1 and η_2 map D into (different) unital copies of of $\mathcal{O}_2$ in $D \otimes D$. Thus $[\eta_1] = 0 = [\eta_2]$ in $KK(D, D \otimes D)$. It implies that η_1 and η_2 are approximately unitarily equivalent (they even are unitarily homotopic by a basic result of classification).

(2): a) $\mathcal{P}_\infty \otimes \mathcal{P}_\infty$ is stably isomorphic to to $\mathcal{O}_\infty$ (by the classification theorem for simple p.i.s.u.n. algebras in the UCT-class and by the KTP).

b) The unit of $\mathcal{O}_\infty$ is Murray–von-Neumann equivalent to the the Bott projection $p(U \otimes 1, 1 \otimes U) \in M_2(\mathcal{P}_\infty \otimes \mathcal{P}_\infty)$ (defined below) from a unitary $U \in \mathcal{P}_\infty$ such that $[U] = 1$ in $\mathbb{Z} \cong K_1(\mathcal{P}_\infty)$. This follows from the KTP and the definition of the isomorphism $K_1(\mathcal{P}_\infty) \otimes K_1(\mathcal{P}_\infty) \cong K_0(\mathcal{P}_\infty \otimes \mathcal{P}_\infty)$ in the KTP.

c) The K_0-class of a Bott projection $p(V, W)$ for commuting unitaries V, W reverses its sign if V and W will be interchanged:
Let V, W commuting unitaries in a unital algebra B, and let $h_{V,W}$ denote the *-morphism from $C(S^1) \otimes C(S^1)$ into B with $h_{V,W}(u_0 \otimes 1) = V$ and $h_{V,W}(1 \otimes u_0) = W$. The Bott projection $p(V, W) \in M_2(B)$ is the image $h_{V,W} \otimes \mathrm{id}_2(p_{\mathrm{Bott}}) \in M_2(C^*(V, W)) \subset M_2(B)$ of the canonical Bott projection $p_{\mathrm{Bott}} \in M_2(C(S^1) \otimes C(S^1))$.

p_{Bott} is contained in the unital subalgebra $(C_0(\mathbb{R}) \otimes C_0(\mathbb{R})) + \mathbb{C} \cdot 1 \cong C(S^2)$ of $(C_0(\mathbb{R}) + \mathbb{C}1) \otimes (C_0(\mathbb{R}) + \mathbb{C}1) \cong C(S^1) \otimes C(S^1)$ and $[p_{\mathrm{Bott}}] - [1 \otimes e_{1,1}]$ generates $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$. Let $D := \{z \in \mathbb{C}\,;\ |z| \leq 1\}$ the closed unit disk in $\mathbb{C}$, $S^1 = \partial D$ its boundary and $\psi\colon\ z = x + iy \in \mathbb{C} \cong \mathbb{R}^2 \mapsto (1 + |z|^2)^{-1/2} z \in D$ the natural homeomorphism from $\mathbb{R}^2$ onto $D \setminus S^1$. The 6-term exact K_*-sequence of the corresponding exact sequence

$$0 \to C_0(\mathbb{R}) \otimes C_0(\mathbb{R}) \to C(D) \to C(S^1) \to 0$$

defines a boundary isomorphism ∂ from $K_1(C(S^1))$ onto $K_0(C_0(\mathbb{R}) \otimes C_0(\mathbb{R}))$. This isomorphism is functorial with respect to *-morphisms $\widehat{\chi}$ of $C(S^1)$ respectively of $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ that are induced by continuous maps χ from (D, S^1) into (D, S^1).

The flip $(x, y) \in \mathbb{R}^2 \mapsto (y, x) \in \mathbb{R}^2$ is induced by $\psi^{-1}(\chi(\psi(x + iy)))$, where χ is the homeomorphism of D given by $\chi(w) := i\overline{w}$ for $w \in D$. The homeomorphism $\chi|S^1$ reverses the orientation of S^1, hence

$$K_1(\widehat{\chi|S^1})\colon\ K_1(C(S_1)) \to K_1(C(S_1))$$

is the isomorphism $n \mapsto -n$ of $K_1(C(S_1)) \cong \mathbb{Z}$. Therefore, the flip automorphism of $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ defines the automorphism of $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$ that changes signs. The restriction of $h_{V,W}$ to $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ defines a group morphism $\mu_{V,W}$ from $\mathbb{Z} \cong K_0(C_0(R^2))$ to $K_0(C^*(V, W))$ (and then to $K_0(B)$ for commuting unitaries $V, W \in B$) with $\mu_{V,W}(1) = [p_{V,W}] - [1 \otimes e_{1,1}]$.

d) By a) and c), the flip map on $\mathcal{P}_\infty \otimes \mathcal{P}_\infty \cong (\mathcal{O}_\infty)^{st}$ defines an automorphism of $\mathcal{O}_\infty \otimes \mathcal{K}$ of order 2 that reverses the sign of elements $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$.

In particular, the flip of $\mathcal{P}_\infty \otimes \mathcal{P}_\infty$ is *not* approximately inner.

(3): The examples of Rørdam are not stably finite. □

Proof of Remark 4.2 Let $\delta_n(d) := \sum_{1\leq j\leq n} s_j d s_j^*$ for $d \in \mathcal{O}_n$ and the canonical generators $s_1, \ldots, s_n$ of $\mathcal{O}_n$. Since $\delta_n \colon \mathcal{O}_n \to \mathcal{O}_n$, is unital and is homotopic to id, δ_n is approximately unitarily equivalent to id (by classification theory). Thus, $\mathcal{O}_n$ is unitally contained in $F(\mathcal{O}_n)$. By Corollary 1.13 this implies that $\mathcal{D} := \mathcal{O}_n \otimes \mathcal{O}_n \otimes \cdots$ is unitally contained in $F(\mathcal{O}_n)$. Since $\mathcal{O}_n \not\cong \mathcal{O}_2$ we get that $\mathcal{O}_2$ is not unitally contained in $\mathcal{D}$, i.e. $0 \neq [1] \in K_0(\mathcal{D})$ (cf. proof of 2.17). Moreover, $\mathcal{D}$ is a p.i.s.u.n. algebra in the UCT-class and $(n-1)K_*(\mathcal{D}) = \{0\}$, because $\mathcal{O}_n$ is a p.i.s.u.n. algebra in the UCT class and $\mathcal{D} \cong \mathcal{O}_n \otimes \mathcal{D}$.

Suppose that $\eta_{1,\infty}$ and $\eta_{2,\infty}$ are approximately unitarily equivalent in $\mathcal{D}$. Then $\mathcal{D}$ is self-absorbing by Corollary 4.12. Since $(n-1)K_*(\mathcal{D}) = \{0\}$, Remark 4.6 implies that $K_*(\mathcal{D}) \cong 0$, which contradicts $0 \neq [1] \in K_0(\mathcal{D})$. □

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