# Non-separable AF-algebras

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**Summary.** We give two pathological phenomena for non-separable AF-algebras which do not occur for separable AF-algebras. One is that non-separable AF-algebras are not determined by their Bratteli diagrams, and the other is that there exists a non-separable AF-algebra which is prime but not primitive.

## 1 Introduction

In this paper, an AF-algebra means a $C^*$-algebra which is an inductive limit of finite dimensional $C^*$-algebras on *any* directed set. Equivalently,

**Definition 1.** *A $C^*$-algebra $A$ is called an* AF-algebra *if it has a directed family of finite dimensional $C^*$-subalgebras whose union is dense in $A$.*

When an AF-algebra $A$ is separable, we can find an increasing *sequence* of finite dimensional $C^*$-subalgebras whose union is dense in $A$. Thus for separable $C^*$-algebras, the above definition coincides with the one in many literatures (for example, [E76]). For separable $C^*$-algebras, there exists one more equivalent definition of AF-algebras:

**Proposition 2 (Theorem 2.2 of [B72]).** *A separable $C^*$-algebra $A$ is an AF-algebra if and only if it is a locally finite dimensional $C^*$-algebra, which means that for any finite subset $\mathcal{F}$ of $A$ and any $\varepsilon > 0$, we can find a finite dimensional $C^*$-subalgebra $B$ of $A$ such that* $\operatorname{dist}(x, B) < \varepsilon$ *for all $x \in \mathcal{F}$.*

To the best of the author's knowledge, it is still open that the above lemma is valid in general.

For each positive integer $n \in \mathbb{Z}_+$, $\mathbb{M}_n$ denotes the $C^*$-algebra of all $n \times n$ matrices. Any finite dimensional $C^*$-algebra $A$ is isomorphic to $\bigoplus_{i=1}^{k} \mathbb{M}_{n_i}$ for some $k \in \mathbb{Z}_+$ and ${}^t(n_1, \ldots, n_k) \in \mathbb{Z}_+^k$. Let $B \cong \bigoplus_{j=1}^{k'} \mathbb{M}_{n'_j}$ be another finite dimensional $C^*$-algebra. A $*$-homomorphism $\varphi\colon A \to B$ is determined up to unitary equivalence by the $k' \times k$ matrix $N$ whose $(j,i)$-entry is the multiplicity of the composition of the restriction of $\varphi$ to $\mathbb{M}_{n_i} \subset A_\lambda$ and the natural surjection from $B$ to $\mathbb{M}_{n'_j}$.

**Definition 3.** *Let $\Lambda$ be a directed set with an order $\preceq$. An* inductive system of finite dimensional $C^*$-algebras $(A_\lambda, \varphi_{\mu,\lambda})$ *over $\Lambda$ consists of a finite dimensional $C^*$-algebra $A_\lambda$ for each $\lambda \in \Lambda$, and a $*$-homomorphism $\varphi_{\mu,\lambda}\colon A_\lambda \to A_\mu$ for each $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$ such that $\varphi_{\nu,\mu} \circ \varphi_{\mu,\lambda} = \varphi_{\nu,\lambda}$ for $\lambda \prec \mu \prec \nu$.*

*A* Bratteli diagram *of $(A_\lambda, \varphi_{\mu,\lambda})$ is the system $(n_\lambda, N_{\mu,\lambda})$ where $n_\lambda = {}^t((n_\lambda)_1, \ldots, (n_\lambda)_{k_\lambda}) \in \mathbb{Z}_+^{k_\lambda}$ satisfies $A_\lambda \cong \bigoplus_{i=1}^{k_\lambda} \mathbb{M}_{(n_\lambda)_i}$ and $N_{\mu,\lambda}$ is $k_\mu \times k_\lambda$ matrix which indicates the multiplicities of the restrictions of $\varphi_{\mu,\lambda}$ as above.*

A Bratteli diagram $(n_\lambda, N_{\mu,\lambda})$ satisfies $N_{\mu,\lambda} n_\lambda \leq n_\mu$ for $\lambda \prec \mu$, and $N_{\nu,\mu} N_{\mu,\lambda} = N_{\nu,\lambda}$ for $\lambda \prec \mu \prec \nu$. It is not difficult to see that when the directed set $\Lambda$ is $\mathbb{Z}_+$, any system $(n_\lambda, N_{\mu,\lambda})$ satisfying these two conditions can be realized as a Bratteli diagram of some inductive system of finite dimensional $C^*$-algebras (see 1.8 of [B72]). This does not hold for general directed set:

*Example 4.* Let $\Lambda = \{a, b, c, d, e\}$ with an order $a \succ b, c \succ d, e$. Let us define

$$n_a = (24), \quad n_b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad n_c = \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \quad n_d = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad n_e = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

and

$$N_{a,b} = \begin{pmatrix} 3 & 3 \end{pmatrix}, \quad N_{b,d} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad N_{b,e} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad N_{a,d} = \begin{pmatrix} 6 & 6 \end{pmatrix},$$
$$N_{a,c} = \begin{pmatrix} 2 & 2 \end{pmatrix}, \quad N_{c,d} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \quad N_{c,e} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad N_{a,e} = \begin{pmatrix} 6 & 6 \end{pmatrix}.$$

These matrices satisfy $N_{\mu,\lambda} n_\lambda = n_\mu$ for $\lambda, \mu \in \Lambda$ with $\mu \succ \lambda$, and

$$N_{a,b} N_{b,d} = N_{a,c} N_{c,d} = N_{a,d}, \qquad N_{a,b} N_{b,e} = N_{a,c} N_{c,e} = N_{a,e}.$$

Thus the system $(n_\lambda, N_{\mu,\lambda})$ satisfies the two conditions above. However, one can see that this diagram never be a Bratteli diagram of inductive systems of finite dimensional $C^*$-algebras.

In 1.8 of [B72], O. Bratteli showed that when the directed set $\Lambda$ is $\mathbb{Z}_+$, a Bratteli diagram of an inductive system of finite dimensional $C^*$-algebras determines the inductive limit up to isomorphism. This is no longer true for general directed set $\Lambda$ as the following easy example shows.

*Example 5.* Let $X$ be an infinite set, and $\Lambda$ be the directed set consisting of all finite subsets of $X$ with inclusion as an order. We consider the following two inductive systems of finite dimensional $C^*$-algebras.

For each $\lambda \in \Lambda$, we define a $C^*$-algebra $A_\lambda = \mathcal{K}(\ell^2(\lambda)) \cong \mathbb{M}_{|\lambda|}$ whose matrix unit is given by $\{e_{x,y}\}_{x,y \in \lambda}$. For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, we define a $*$-homomorphism $\varphi_{\mu,\lambda}\colon A_\lambda \to A_\mu$ by $\varphi_{\mu,\lambda}(e_{x,y}) = e_{x,y}$. It is clear to see that this defines an inductive system of finite dimensional $C^*$-algebras, and the inductive limit is $\mathcal{K}(\ell^2(X))$.

For each $\lambda \in \Lambda$ with $n = |\lambda|$, we set $A'_\lambda = \mathbb{M}_n$ whose matrix unit is given by $\{e_{k,l}\}_{1\leq k,l\leq n}$. For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, we define a $*$-homomorphism $\varphi'_{\mu,\lambda} \colon A'_\lambda \to A'_\mu$ by $\varphi'_{\mu,\lambda}(e_{k,l}) = e_{k,l}$. It is clear to see that this defines an inductive system of finite dimensional $C^*$-algebras, and the inductive limit is $\mathcal{K}(\ell^2(\mathbb{Z}_+))$.

The above two inductive systems give isomorphic Bratteli diagrams, but the AF-algebras $\mathcal{K}(\ell^2(X))$ and $\mathcal{K}(\ell^2(\mathbb{Z}_+))$ determined by the two inductive systems are isomorphic only when $X$ is countable.

In a similar way, we can find two inductive systems of finite dimensional $C^*$-algebras whose Bratteli diagrams are isomorphic, but the inductive limits are $\bigotimes_{x\in X} \mathbb{M}_2$ and $\bigotimes_{k=1}^{\infty} \mathbb{M}_2$ which are not isomorphic when $X$ is uncountable.

By Example 5, we can see that G. A. Elliott's celebrated theorem of classifying (separable) AF-algebras using $K_0$-groups (Theorem 6.4 of [E76]) does not follow for non-separable AF-algebras, because $K_0$-groups are determined by Bratteli diagrams. Example 5 is not so interesting because the inductive system $(A'_\lambda, \varphi'_{\mu,\lambda})$ has many redundancies and does not come from directed families of finite dimensional $C^*$-subalgebras. More interestingly, we can get the following whose proof can be found in the next section:

**Theorem 6.** *There exist two non-isomorphic AF-algebras $A$ and $B$ such that they have directed families of finite dimensional $C^*$-subalgebras which define isomorphic Bratteli diagrams.*

The author could not find such an example in which every finite dimensional $C^*$-subalgebras are isomorphic to full matrix algebras $\mathbb{M}_n$ (cf. Problem 8.1 of [D67]).

As another pathological fact on non-separable AF-algebras, we prove the next theorem in Section 3.

**Theorem 7.** *There exists a non-separable AF-algebra which is prime but not primitive.*

It had been a long standing problem whether there exists a $C^*$-algebra which is prime but not primitive, until N. Weaver found such a $C^*$-algebra in [W03]. Note that such a $C^*$-algebra cannot be separable.

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## 2 Proof of Theorem 6

In this section, we will prove Theorem 6. Let $X$ be an infinite set, and $Z$ be the set of all subsets $z$ of $X$ with $|z| = 2$.

For each $z \in Z$, we define a $C^*$-algebra $M_z$ by $M_z = \mathbb{M}_2$. Elements of the direct product $\prod_{z\in Z} M_z$ will be considered as norm bounded functions $f$ on $Z$ such that $f(z) \in M_z$ for $z \in Z$. For each $z \in Z$, we consider $M_z \subset \prod_{z\in Z} M_z$ as a direct summand. We denote by $\bigoplus_{z\in Z} M_z$ the direct sum of $M_z$'s which is an ideal of $\prod_{z\in Z} M_z$.

**Definition 8.** *For each $z \in Z$, we fix a matrix unit $\{e^z_{i,j}\}^2_{i,j=1}$ of $M_z = \mathbb{M}_2$. For each $x \in X$, we define a projection $p_x \in \prod_{z\in Z} M_z$ by*

$$p_x(z) = \begin{cases} e^z_{1,1} & \text{if } x \in z, \\ 0 & \text{if } x \notin z. \end{cases}$$

*We denote by $A$ the $C^*$-subalgebra of $\prod_{z\in Z} M_z$ generated by $\bigoplus_{z\in Z} M_z$ and $\{p_x\}_{x\in X}$.*

**Definition 9.** *For each $z = \{x_1, x_2\} \in Z$, we fix a matrix unit $\{e^z_{x_i,x_j}\}^2_{i,j=1}$ of $M_z = \mathbb{M}_2$. For each $x \in X$, we define a projection $q_x \in \prod_{z\in Z} M_z$ by*

$$q_x(z) = \begin{cases} e^z_{x,x} & \text{if } x \in z, \\ 0 & \text{if } x \notin z. \end{cases}$$

*We denote by $B$ the $C^*$-subalgebra of $\prod_{z\in Z} M_z$ generated by $\bigoplus_{z\in Z} M_z$ and $\{q_x\}_{x\in X}$.*

The following easy lemma illustrates an difference of $A$ and $B$.

**Lemma 10.** *For $x, y \in X$ with $x \neq y$, we have $p_x p_y = e^{\{x,y\}}_{1,1} \neq 0$, and $q_x q_y = 0$.*

*Proof.* Straightforward.

**Definition 11.** *Let $\lambda$ be a finite subset of $X$. We denote by $A_\lambda$ the $C^*$-subalgebra of $A$ spanned by $\bigoplus_{z\subset\lambda} M_z$ and $\{p_x\}_{x\in\lambda}$, and by $B_\lambda$ the $C^*$-subalgebra of $B$ spanned by $\bigoplus_{z\subset\lambda} M_z$ and $\{q_x\}_{x\in\lambda}$,*

**Lemma 12.** *There exist isomorphisms*

$$A_\lambda \cong B_\lambda \cong \bigoplus_{z\subset\lambda} \mathbb{M}_2 \oplus \bigoplus_{x\in\lambda} \mathbb{C}$$

*for each finite set $\lambda \subset X$ such that two inclusions $A_\lambda \subset A_\mu$ and $B_\lambda \subset B_\mu$ have the same multiplicity.*

*Proof.* For $x \in \lambda$, let us denote $p'_x \in A_\lambda$ by $p'_x = p_x - \sum_{y \in \lambda \setminus \{x\}} e_{1,1}^{\{x,y\}}$. Then we have an orthogonal decomposition

$$A_\lambda = \sum_{z \subset \lambda} M_z + \sum_{x \in \lambda} \mathbb{C} p'_x.$$

This proves $A_\lambda \cong \bigoplus_{z \subset \lambda} \mathbb{M}_2 \oplus \bigoplus_{x \in \lambda} \mathbb{C}$. Similarly we have $B_\lambda \cong \bigoplus_{z \subset \lambda} \mathbb{M}_2 \oplus \bigoplus_{x \in \lambda} \mathbb{C}$. Now it is routine to check the last statement.

**Proposition 13.** *Two $C^*$-algebras $A$ and $B$ are AF-algebras, and the directed families $\{A_\lambda\}$ and $\{B_\lambda\}$ of finite dimensional $C^*$-subalgebras give isomorphic Bratteli diagrams.*

*Proof.* Follows from the facts

$$\overline{\bigcup_{\lambda \subset X} A_\lambda} = A, \quad \overline{\bigcup_{\lambda \subset X} B_\lambda} = B$$

and Lemma 12.

*Remark 14.* ¿From Proposition 13, we can show that $K_0(A)$ and $K_0(B)$ are isomorphic as scaled ordered groups. In fact, they are isomorphic to the subgroup $G$ of $\prod_{z \in Z} \mathbb{Z}$ generated by $\bigoplus_{z \in Z} \mathbb{Z}$ and $\{g_x\}_{x \in X}$, where $g_x \in \prod_{z \in Z} \mathbb{Z}$ is defined by

$$g_x(z) = \begin{cases} 1 & \text{if } x \in z, \\ 0 & \text{if } x \notin z. \end{cases}$$

The order of $G$ is the natural one, and its scale is

$$\{g \in G \mid 0 \leq g(z) \leq 2 \text{ for all } z \in Z\}.$$

¿From this fact and Elliott's theorem (Theorem 6.4 of [E76]), we can show the next lemma, although we give a direct proof here.

**Proposition 15.** *When $X$ is countable, $A$ and $B$ are isomorphic.*

*Proof.* Let us list $X = \{x_1, x_2, \ldots\}$. We define a $*$-homomorphism $\varphi\colon A \to B$ as follows. For $z = \{x_k, x_l\}$, we define $\varphi(e^z_{i,j}) = e^z_{x_{n_i}, x_{n_j}}$ where $n_1 = k, n_2 = l$ when $k < l$ and $n_1 = l, n_2 = k$ when $k > l$. For $x_k \in X$, we set

$$\varphi(p_{x_k}) = q_{x_k} + \sum_{i=1}^{k-1} \left( e^{\{x_i, x_k\}}_{x_i, x_i} - e^{\{x_i, x_k\}}_{x_k, x_k} \right).$$

Now it is routine to check that $\varphi$ is an isomorphism from $A$ to $B$.

Proposition 15 is no longer true for uncountable $X$. To see this, we need the following lemma.

**Lemma 16.** *There exists a surjection $\pi_A\colon A \to \bigoplus_{x\in X}\mathbb{C}$ defined by $\pi_A(M_z) = 0$ for $z \in Z$ and $\pi_A(p_x) = \delta_x$ for $x \in X$. Its kernal is $\bigoplus_{z\in Z} M_z$ which coincides with the ideal generated by the all commutators $xy - yx$ of $A$. The same is true for $B$.*

*Proof.* Let $\pi_A$ be the quotient map from $A$ to $A/\bigoplus_{z\in Z} M_z$. Then $A/\bigoplus_{z\in Z} M_z$ is generated by $\{\pi_A(p_x)\}_{x\in X}$ which is an orthogonal family of non-zero projections. This proves the first statement. Since $\bigoplus_{x\in X}\mathbb{C}$ is commutative, the ideal $\bigoplus_{z\in Z} M_z$ contains all commutators. Conversely, the ideal generated by the commutators of $A$ contains $\bigoplus_{z\in Z} M_z$ because $\mathbb{M}_2$ is generated by its commutators. This shows that $\bigoplus_{z\in Z} M_z$ is the ideal generated by the all commutators of $A$. The proof goes similarly for $B$.

**Proposition 17.** *When $X$ is uncountable, $A$ and $B$ are not isomorphic.*

*Proof.* To the contrary, suppose that there exists an isomorphism $\varphi\colon A \to B$. By Lemma 16, $\bigoplus_{z\in Z} M_z$ is the ideal generated by the all commutators in both $A$ and $B$. Hence $\varphi$ preserves this ideal $\bigoplus_{z\in Z} M_z$. Thus we get the following commutative diagram with exact rows;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{z\in Z} M_z & \longrightarrow & A & \xrightarrow{\pi_A} & \bigoplus_{x\in X}\mathbb{C} & \longrightarrow & 0 \\
 & & \big\downarrow\varphi & & \big\downarrow\varphi & & \big\downarrow & & \\
0 & \longrightarrow & \bigoplus_{z\in Z} M_z & \longrightarrow & B & \xrightarrow{\pi_B} & \bigoplus_{x\in X}\mathbb{C} & \longrightarrow & 0.
\end{array}$$

Since the family of projections $\{q_x\}_{x\in X}$ in $B$ is mutually orthogonal, the surjection $\pi_B\colon B \to \bigoplus_{x\in X}\mathbb{C}$ has a splitting map $\sigma_B\colon \bigoplus_{x\in X}\mathbb{C} \to B$ defined by $\sigma_B(\delta_x) = q_x$. Hence by the diagram above, the surjection $\pi_A\colon A \to \bigoplus_{x\in X}\mathbb{C}$ also has a splitting map $\sigma_A\colon \bigoplus_{x\in X}\mathbb{C} \to A$. Let us set $p'_x = \sigma_A(\delta_x)$ for $x \in X$. Choose a countable infinite subset $Y$ of $X$. For each $y \in Y$, the set

$$\mathcal{F}_y = \big\{x \in X \;\big|\; x \neq y, \|(p_y - p'_y)(\{x, y\})\| \geq 1/2\big\}$$

is finite, because $p_y - p'_y \in \ker \pi_A = \bigoplus_{z\in Z} M_z$. Since $X$ is uncountable, we can find $x_0 \in X$ with $x_0 \notin Y \cup \bigcup_{y\in Y}\mathcal{F}_y$. Since

$$\mathcal{F}_{x_0} = \big\{x \in X \;\big|\; x \neq x_0, \|(p_{x_0} - p'_{x_0})(\{x, x_0\})\| \geq 1/2\big\}$$

is finite, we can find $y_0 \in Y\backslash\mathcal{F}_{x_0}$. We set $z = \{x_0, y_0\}$. ¿From $y_0 \notin \mathcal{F}_{x_0}$, we have $\|(p_{x_0} - p'_{x_0})(z)\| < 1/2$, and from $x_0 \notin \mathcal{F}_{y_0}$, we have $\|(p_{y_0} - p'_{y_0})(z)\| < 1/2$. However, $p_{x_0}(z) = p_{y_0}(z) = e^z_{1,1}$ and $p'_{x_0}(z)$ is orthogonal to $p'_{y_0}(z)$. This is a contradiction. Thus $A$ and $B$ are not isomorphic.

Combining Proposition 13 and Proposition 17, we get Theorem 6.

## 3 A prime AF-algebra which is not primitive

In this section, we construct an AF-algebra which is prime but not primitive. Although we follow the idea of Weaver in [W03], our construction of the $C^*$-algebra and proof of the main theorem is much easier than the ones there. A similar construction can be found in [K04], but the proof there uses general facts of topological graph algebras.

Let $X$ be an uncountable set, and $\Lambda$ be the directed set of all finite subsets of $X$. For $n \in \mathbb{N}$, we set $\Lambda_n = \{\lambda \subset X \mid |\lambda| = n\}$. We get $\Lambda = \coprod_{n=0}^{\infty} \Lambda_n$.

**Definition 18.** *For $n \in \mathbb{Z}_+$ and $\lambda \in \Lambda_n$, we define*

$$l(\lambda) = \{t\colon \{1, \ldots, n\} \to \lambda \mid t \text{ is a bijection}\}.$$

*For $\emptyset \in \Lambda$, we define $l(\emptyset) = \{\emptyset\}$.*

Note that $|l(\lambda)| = n!$ for $\lambda \in \Lambda_n$ and $n \in \mathbb{N}$.

**Definition 19.** *For $n \in \mathbb{N}$ and $\lambda \in \Lambda_n$, we define $M_\lambda \cong \mathbb{M}_{n!}$ whose matrix unit is given by $\{e^{(\lambda)}_{s,t}\}_{s,t \in l(\lambda)}$.*

**Definition 20.** *Take $\lambda \in \Lambda_n$ and $\mu \in \Lambda_m$ with $\lambda \cap \mu = \emptyset$. For $t \in l(\lambda)$ and $s \in l(\mu)$, we define $ts \in l(\lambda \cup \mu)$ by*

$$(ts)(i) = \begin{cases} t(i) & \text{for } i = 1, \ldots, n \\ s(i-n) & \text{for } i = n+1, \ldots, n+m. \end{cases}$$

Note that when $\mu = \emptyset$, we have $t\emptyset = t$.

**Definition 21.** *For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, we define a $*$-homomorphism $\iota_{\mu,\lambda}\colon M_\lambda \to M_\mu$ by*

$$\iota_{\mu,\lambda}\big(e^{(\lambda)}_{s,t}\big) = \sum_{u \in l(\mu \setminus \lambda)} e^{(\mu)}_{su,tu} \quad \text{for } s, t \in l(\lambda).$$

Note that $\iota_{\lambda,\lambda}$ is the identity map of $M_\lambda$, and that $\iota_{\lambda_3,\lambda_2} \circ \iota_{\lambda_2,\lambda_1} \neq \iota_{\lambda_3,\lambda_1}$ for $\lambda_1 \subsetneq \lambda_2 \subsetneq \lambda_3$. For $\lambda_1, \lambda_2 \in \Lambda_n$ and $\mu \in \Lambda_m$ with $\lambda_1 \neq \lambda_2$ and $\lambda_1 \cup \lambda_2 \subset \mu$, the images $\iota_{\mu,\lambda_1}(M_{\lambda_1})$ and $\iota_{\mu,\lambda_2}(M_{\lambda_2})$ are mutually orthogonal.

**Definition 22.** *For $\lambda \in \Lambda$, we define a $*$-homomorphism $\iota_\lambda\colon M_\lambda \to \prod_{\mu \in \Lambda} M_\mu$ by*

$$\iota_\lambda(x)(\mu) = \begin{cases} \iota_{\mu,\lambda}(x) & \text{if } \lambda \subset \mu, \\ 0 & \text{otherwise,} \end{cases}$$

*for $x \in M_\lambda$. We set $N_\lambda = \iota_\lambda(M_\lambda) \subset \prod_{\mu \in \Lambda} M_\mu$ and $f^{(\lambda)}_{s,t} = \iota_\lambda(e^{(\lambda)}_{s,t}) \in N_\lambda$ for $s, t \in l(\lambda)$.*

For $\lambda \in \Lambda_n$, We have $N_\lambda \cong \mathbb{M}_{n!}$ and $\{f^{(\lambda)}_{s,t}\}_{s,t\in l(\lambda)}$ is a matrix unit of $N_\lambda$.

**Lemma 23.** *For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, $s, t \in l(\lambda)$ and $s', t' \in l(\mu)$, we have $f^{(\lambda)}_{s,t} f^{(\mu)}_{s',t'} = f^{(\mu)}_{su,t'}$ when $s' = tu$ with some $u \in l(\mu \setminus \lambda)$, and $f^{(\lambda)}_{s,t} f^{(\mu)}_{s',t'} = 0$ otherwise.*

*Proof.* Straightforward.

**Lemma 24.** *For $\lambda, \mu \in \Lambda$, we have $0 \neq N_\lambda N_\mu \subset N_\mu$ if $\lambda \subset \mu$, $0 \neq N_\lambda N_\mu \subset N_\lambda$ if $\lambda \supset \mu$, and $N_\lambda N_\mu = 0$ otherwise.*

*Proof.* If $\lambda \subset \mu$, we have $0 \neq N_\lambda N_\mu \subset N_\mu$ by Lemma 23. Similarly we have $0 \neq N_\lambda N_\mu \subset N_\lambda$ if $\lambda \supset \mu$. Otherwise, we can easily see $N_\lambda N_\mu = 0$ from the definition.

**Corollary 25.** *For each $n$, the family $\{N_\lambda\}_{\lambda\in\Lambda_n}$ of $C^*$-algebras is mutually orthogonal.*

**Corollary 26.** *Take $\lambda, \lambda' \in \Lambda$ with $\lambda \subset \lambda'$. Let $p_{\lambda'}$ be the unit of $N_{\lambda'}$. Then $N_\lambda \ni a \mapsto ap_{\lambda'} \in N_{\lambda'}$ is an injective $*$-homomorphism.*

**Definition 27.** *We define $A = \overline{\sum_{\lambda\in\Lambda} N_\lambda} \subset \prod_{\mu\in\Lambda} M_\mu$.*

**Proposition 28.** *The set $A$ is an AF-algebra.*

*Proof.* For each $\mu \in \Lambda$, $A_\mu = \sum_{\lambda\subset\mu} N_\lambda$ is a finite dimensional $C^*$-algebra by Lemma 24. For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, we have $A_\lambda \subset A_\mu$. Hence $A = \overline{\bigcup_{\mu\in\Lambda} A_\mu}$ is an AF-algebra.

**Lemma 29.** *Every non-zero ideal $I$ of $A$ contains $N_\lambda$ for some $\lambda \in \Lambda$.*

*Proof.* As in the proof of Proposition 28, we set $A_\mu = \sum_{\lambda\subset\mu} N_\lambda$ for $\mu \in \Lambda$. Since $A = \overline{\bigcup_{\mu\in\Lambda} A_\mu}$, we have $I = \overline{\bigcup_{\mu\in\Lambda}(I \cap A_\mu)}$ for an ideal $I$ of $A$. Hence if $I$ is nonzero, we have $I \cap A_{\mu_0} \neq 0$ for some $\mu_0 \in \Lambda$. Thus we can find a non-zero element $a \in I$ in the form $a = \sum_{\lambda\subset\mu_0} a_\lambda$ for $a_\lambda \in N_\lambda$. Since $a \neq 0$, we can find $\lambda_0 \in \Lambda$ with $\lambda_0 \subset \mu_0$ such that $a_{\lambda_0} \neq 0$ and $a_\lambda = 0$ for all $\lambda \subsetneq \lambda_0$. Take $x_0 \in X$ with $x_0 \notin \mu_0$. Set $\lambda'_0 = \lambda_0 \cup \{x_0\}$. Let $p_{\lambda'_0}$ be the unit of $N_{\lambda'_0}$. For $\lambda \subset \mu_0$, $a_\lambda p_{\lambda'_0} \neq 0$ only when $\lambda \subset \lambda_0$. Hence we have $ap_{\lambda'_0} = a_{\lambda_0} p_{\lambda'_0}$. By Corollary 26, $a_{\lambda_0} p_{\lambda'_0}$ is a non-zero element of $N_{\lambda'_0}$. Hence we can find a non-zero element in $I \cap N_{\lambda'_0}$. Since $N_{\lambda'_0}$ is simple, we have $N_{\lambda'_0} \subset I$. We are done.

**Lemma 30.** *If an ideal $I$ of $A$ satisfies $N_{\lambda_0} \subset I$ for some $\lambda_0 \in \Lambda$, then $N_\lambda \subset I$ for all $\lambda \supset \lambda_0$.*

*Proof.* Clear from Lemma 24 and the simplicity of $N_\lambda$.

**Proposition 31.** *The $C^*$-algebra is prime but not primitive.*

*Proof.* Take two non-zero ideals $I_1, I_2$ of $A$. By Lemma 29, we can find $\lambda_1, \lambda_2 \in \Lambda$ such that $N_{\lambda_1} \subset I_1$ and $N_{\lambda_2} \subset I_2$. Set $\lambda = \lambda_1 \cup \lambda_2 \in \Lambda$. By Lemma 30, we have $N_\lambda \subset I_1 \cap I_2$. Thus $I_1 \cap I_2 \neq 0$. This shows that $A$ is prime.

To prove that $A$ is not primitive, it suffices to see that for any state $\varphi$ of $A$ we can find a non-zero ideal $I$ such that $\varphi(I) = 0$ (see [W03]). Take a state $\varphi$ of $A$. By Corollary 25, the family $\{N_\lambda\}_{\lambda \in \Lambda_n}$ of $C^*$-algebras is mutually orthogonal for each $n \in \mathbb{N}$. Hence the set

$$\Omega_n = \{\lambda \in \Lambda_n \mid \text{the restriction of } \varphi \text{ to } N_\lambda \text{ is non-zero}\}$$

is countable for each $n \in \mathbb{N}$. Since $X$ is uncountable, we can find $x_0 \in X$ such that $x_0 \notin \lambda$ for all $\lambda \in \bigcup_{n \in \mathbb{N}} \Omega_n$. Let $I = \overline{\sum_{\lambda \ni x_0} N_\lambda}$. Then $I$ is an ideal of $A$ by Lemma 24. Since $\lambda \ni x_0$ implies $\varphi(N_\lambda) = 0$, we have $\varphi(I) = 0$. Therefore $A$ is not primitive.

This finishes the proof of Theorem 7.

*Remark 32.* Let $(A_\lambda, \varphi_{\mu,\lambda})$ be an inductive system of finite dimensional $C^*$-algebras over a directed set $\Lambda$, and $A$ be its inductive limit. It is not hard to see that the AF-algebra $A$ is prime if and only if the Bratteli diagram of the inductive system satisfies the analogous condition of (iii) in Corollary 3.9 of [B72]. Hence, the Bratteli diagram of an inductive system of finite dimensional $C^*$-algebras determines the primeness of the inductive limit, although it does not determine the inductive limit itself. However the primitivity of the inductive limit is not determined by the Bratteli diagram. In fact, in a similar way to the construction of Example 5, we can find an inductive system of finite dimensional $C^*$-algebras whose Bratteli diagram is isomorphic to the one coming from the directed family $\{A_\lambda\}$ constructed in the proof of Proposition 28, but the inductive limit is separable. This AF-algebra is primitive because it is separable and prime (see, for example, Proposition 4.3.6 of [P79]).

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