An algebraic description of boundary maps used in index theory

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Dedicated to the memory of Gert Pedersen

In index theory and in noncommutative geometry one often associates C^*-algebras with geometric objects. These algebras can for instance arise from pseudodifferential operators, differential forms, convolution algebras etc.. However they are often given a priori as locally convex algebras and one looses a certain amount of information by passing to the C^*-algebra completions. In some cases, for instance for algebras containing unbounded differential operators, there is in fact no C^*-algebra that accommodates them. On the other hand, it seems that nearly all algebraic structures arising from differential geometry can be described very naturally by locally convex algebras (or by the slightly more general concept of bornological algebras). The present note can be seen as part of a program in which we analyze constructions, that are classical in K-theory for C^*-algebras and in index theory, in the framework of locally convex algebras. Since locally convex algebras have, besides their algebraic structure, only very little structure, all arguments in the study of their K-theory or their cyclic homology have to be essentially algebraic (thus in particular they also apply to bornological algebras).

This paper is triggered by an analysis of the proof of the Baum-Douglas-Taylor index theorem, [2], in the locally convex setting. Consider the extension

$$\mathcal{E}_\Psi : \quad 0 \to \mathcal{K} \to \Psi(M) \to \mathcal{C}(S^*M) \to 0$$

determined by the C^*-algebra completion $\Psi(M)$ of the algebra of pseudodifferential operators of order 0 on M and the natural extension

$$\mathcal{E}_{B^*M} : \quad 0 \to \mathcal{C}_0(T^*M) \to \mathcal{C}(B^*M) \to \mathcal{C}(S^*M) \to 0$$

determined by the evaluation map on the boundary S^*M of the ball bundle B^*M. Both extensions determine elements which we denote by $KK(\mathcal{E}_\Psi)$ and $KK(\mathcal{E}_{B^*M})$, respectively, in the bivariant K-theory of Kasparov.

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The Baum-Douglas-Taylor index theorem determines the K-homology class $KK(\mathcal{E}_\Psi)$ in $KK_1(\mathcal{C}(S^*M),\mathbb{C})$ as

$$KK(\mathcal{E}_\Psi) = KK(\mathcal{E}_{B^*M}) \cdot [\bar{\partial}_{T^*M}]$$

where $[\bar{\partial}_{T^*M}]$ is the fundamental K-homology class defined by the Dolbeault operator on T^*M. Note that multiplication by $KK(\mathcal{E}_{B^*M})$ describes the boundary map $K^0(\mathcal{C}_0(T^*M)) \to K^1(\mathcal{C}(S^*M)$ in K-homology.

This theorem which determines $KK(\mathcal{E}_\Psi)$ may be considered as a fundamental theorem in index theory, since it contains all the relevant information on the K-theoretic connections between symbols and indices of pseudodifferential operators on a given manifold. In particular, it contains the classical Atiyah-Singer theorem as well as Kasparov's bivariant version of the index theorem and determines not only the index of a given elliptic operator P, but also the K-homology class determined by P. (Note however that Kasparov proves his theorem in the equivariant case and for manifolds which are not necessarily compact. In this generality Kasparov's theorem remains the strongest result). The connection between the index theorems by Baum-Douglas-Taylor, Kasparov and Atiyah-Singer will be explained briefly in section 6.

The proof, by Baum-Douglas-Taylor, of their index theorem is a combination of a formula by Baum and Douglas [1], with a construction of Boutet de Monvel [4], [3]. The formula of Baum-Douglas determines the image under the boundary map, in the long exact sequence associated with an extension, of K-homology elements described by cycles satisfying certain conditions. ¿From this formula they derive a formula for the bivariant K-theory class determined by the Toeplitz extension on a strictly pseudoconvex domain. The construction by Boutet de Monvel identifies the extension of pseudodifferential operators on a manifold M with the Toeplitz extension on the strictly pseudoconvex domain given by the ball bundle on M with boundary given by the sphere bundle.

The original proof by Baum-Douglas of their formula for the boundary map has been streamlined substantially by Higson in [10], [11]. Higson gives in fact two proofs. One makes use of Skandalis' connection formalism the other one of Paschke duality, see also [12]. Both approaches rely on a certain amount of technical background. Higson also proves a formula in the case of odd Kasparov modules (Baum-Douglas consider only the even case). We also mention, even though this is not of direct relevance to our purposes that a simplification of the proof that the relative K-homology of Baum-Douglas coincides with the K-homology of the ideal is due to Kasparov [13].

We give algebraic proofs for the boundary map formula in the even and in the odd case. It turns out that the Baum-Douglas situation is exactly the one where the cycle representing the given K-homology element extends to a cycle for a K-homology element of the mapping cone or dual mapping cone (in the sense of [5]) associated with the given extension, respectively. This leads to a simple proof in the odd case. In the even case there is a completely direct proof which is very short. This proof depends on a new description of the boundary map which uses comparison to a free extension. We also include another, slightly longer proof, using the dual mapping cones of [5], because of its complete parallelism to the proof in the odd case using the ordinary mapping cone. The dual mapping cones have also been used in the paper by Baum-Douglas and this second proof resembles the proof by Baum-Douglas, but it has been reduced to its algebraic content.

Our discussion contains much more material than what is needed to determine the boundary map in the Baum-Douglas situation. We give different descriptions of the boundary map for bivariant K-theory, but also for more general homotopy functors. The argument for the Baum-Douglas formula itself is very short indeed and essentially contained in 4.2. In other descriptions of the boundary map we also have to study its compatibility with the Bott isomorphism. This compatibility has some interest for its own sake.

We would also like to emphasize the fact that our argument, even though formulated in the category of locally convex algebras for convenience, is completely general. Because of its algebraic nature it works in many other categories of topological algebras. In particular it can be readily applied to the category of C^*-algebras and gives there the original result of Baum-Douglas. For this, one has to use the appropriate analogs of the tensor algebra and of its ideal JA in the category of C^*-algebras as explained in [7]. We will discuss this in section 5.

1 Boundary maps and Bott maps

1.1 Locally convex algebras

By a locally convex algebra we mean an algebra over $\mathbb{C}$ equipped with a complete locally convex topology such that the multiplication $A \times A \to A$ is (jointly) continuous. This means that, for every continuous seminorm α on A, there is another continuous seminorm α' such that

$$\alpha(xy) \leq \alpha'(x)\alpha'(y)$$

for all $x, y \in A$. Equivalently, the multiplication map induces a continuous linear map $A \hat{\otimes} A \to A$ from the completed projective tensor product $A \hat{\otimes} A$. All homomorphisms between locally convex algebras will be assumed to be continuous. Every Banach algebra or projective limit of Banach algebras obviously is a locally convex algebra. But so is every algebra over $\mathbb{C}$ with a countable basis if we equip it with the "fine" locally convex topology, see e.g. [8]. The fine topology on a complex vector space V is given by the family of *all* seminorms on V. Homomorphisms between locally convex algebras will always be assumed to be continuous. We denote the category of locally convex algebras by LCA.

1.2 The boundary map for half-exact homotopy functors

By a well-known construction any half-exact homotopy functor on a category of topological algebras associates with any extension a long exact sequence which is infinite to one side. This construction is in fact quite general and works for different notions of homotopy (continuous, differentiable or $\mathcal{C}^\infty$) and on different categories of algebras as well as for different notions of extensions.

We review this construction here in some detail, since we need, for our purposes, the explicit description of the boundary map in the long exact sequence. To be specific we will work in the category of locally convex algebras with $\mathcal{C}^\infty$-homotopy. An extension will be a sequence

$$0 \to I \to A \to B \to 0$$

of locally convex algebras, where the arrows are continuous homomorphisms, which is split exact in the category of locally convex vector spaces, i.e. for which there is a continuous linear splitting $s : B \to A$. An extension will be called a split-extension if there is a continuous splitting $B \to A$ which at the same time is a homomorphism.

Let $[a, b]$ be an interval in $\mathbb{R}$. We denote by $\mathbb{C}[a, b]$ the algebra of complex-valued $\mathcal{C}^\infty$-functions f on $[a, b]$, all of whose derivatives vanish in a and in b (while f itself may take arbitrary values in a and b). Also the subalgebras $\mathbb{C}(a, b], \mathbb{C}[a, b)$ and $\mathbb{C}(a, b)$ of $\mathbb{C}[a, b]$, which, by definition consist of functions f, that vanish in a, in b, or in a and b, respectively, will play an important role. The topology on these algebras is the usual Fréchet topology.

Given two complete locally convex spaces V and W, we denote by $V \hat{\otimes} W$ their completed projective tensor product (see [14], [8]). We note that $\mathbb{C}[a, b]$ is nuclear in the sense of Grothendieck [14] and that, for any complete locally convex space V, the space $\mathbb{C}[a, b]\hat{\otimes}V$ is isomorphic to the space of $\mathcal{C}^\infty$-functions on $[a, b]$ with values in V, whose derivatives vanish in both endpoints, [14], § 51.

Given a locally convex algebra A, we write $A[a, b]$, $A(a, b]$ and $A(a, b)$ for the locally convex algebras $A\hat{\otimes}\mathbb{C}[a, b]$, $A\hat{\otimes}\mathbb{C}(a, b]$ and $A\hat{\otimes}\mathbb{C}(a, b)$ (their elements are A - valued $\mathcal{C}^\infty$-functions whose derivatives vanish at the endpoints). The algebra $A(0, 1]$ is called the cone over A and denoted by CA. The algebra $A(0, 1)$ is called the suspension of A and denoted by SA. The cone extension for A is

$$0 \to SA \to CA \to A \to 0$$

(it has an obvious continuous linear splitting). This extension is fundamental for the construction of the boundary maps.

In the following we will usually consider covariant functors E on the category LCA. The contravariant case is of course completely analogous and, in fact, in later sections we will also apply the results discussed here to contravariant functors (such as K-homology).

Definition 1. *Let $E : \mathrm{LCA} \to \mathrm{Ab}$ be a functor from the category of locally convex algebras to the category of abelian groups. We say that*

- *E is diffotopy invariant, if the maps $\mathrm{ev}_t : E(A[0,1]) \to E(A)$ induced by the different evaluation maps for $t \in [0,1]$ are all the same (it is easy to see that this is the case if and only if the map induced by evaluation at $t = 0$ is an isomorphism).*
- *E is half-exact, if, for every extension $0 \to I \to A \to B \to 0$ of locally convex algebras, the induced short sequence $E(I) \to E(A) \to E(B)$ is exact.*

Definition 2. *Two homomorphisms $\alpha, \beta : A \to B$ between locally convex algebras are called diffotopic if there is a homomorphism $\varphi : A \to B[0,1]$ such that*

$$\alpha = \mathrm{ev}_0 \circ\varphi \qquad \beta = \mathrm{ev}_1 \circ\varphi.$$

A locally convex algebra A is called contractible if the endomorphisms id_A and 0 are diffotopic. If α and β are diffotopic and E is diffotopy invariant, then clearly $E(\alpha) = E(\beta)$. Moreover $E(A) = 0$ for every contractible algebra A.

Let $\alpha : A \to B$ be a continuous homomorphism between locally convex algebras. The mapping cone $C_\alpha \subset A \oplus B(0,1]$ is defined to be

$$C_\alpha = \{(x,f) \in A \oplus B(0,1] \,|\, \alpha(x) = f(1)\}$$

Similarly, the mapping cylinder Z_α is

$$Z_\alpha = \{(x,f) \in A \oplus B[0,1] \,|\, \alpha(x) = f(1)\}$$

Lemma 3. *(a) The maps $Z_\alpha \to A$, $(x,f) \mapsto x$ and $A \to Z_\alpha$, $x \mapsto (x, \alpha(x)1)$ are homotopy inverse to each other, i.e. their compositions both ways are diffotopic to the identity on Z_α and on A, respectively.*

(b) If there is a continuous linear map $s : B \to A$ such that $\alpha \circ s = \mathrm{id}_B$, then the natural exact sequence

$$0 \to C_\alpha \to Z_\alpha \to B \to 0$$

is an extension (i.e. admits a continuous linear splitting).

(c) If E is half-exact and $\pi : C_\alpha \to A$ is defined by $\pi((x,f)) = x$, then the sequence

$$E(C_\alpha) \xrightarrow{E(\pi)} E(A) \xrightarrow{E(\alpha)} E(B)$$

is exact.

Lemma 4. *Let $0 \to I \to A \xrightarrow{q} B \to 0$ be an extension of locally convex algebras. Denote by $e : I \to C_q$ the map defined by $e(x) = (x, 0) \in C_q \subset A \oplus CB$.*

(a) The following diagram commutes

$$\begin{array}{ccccccc} & & I & \to & A & \to & B \\ & & \downarrow e & & \| & & \downarrow \\ 0 \to & SB & \xrightarrow{\kappa} & C_q & \xrightarrow{\pi} & A & \to 0 \end{array}$$

and the natural map $\kappa : SB \to C_q$ defined by $\kappa(f) = (0, f)$ makes the second row exact.

(b) One has $E(C_e) = 0$ and the map $E(e) : E(I) \to E(C_q)$ is an isomorphism.

Proof. (a) Obvious. (b) This follows from the exact sequences $0 = E(CI) \to E(C_e) \to E(SCB) = 0$ and $E(C_e) \to E(I) \to E(C_q)$ (cf. 3 (c)).

Proposition 5. *Let*

$$0 \to I \xrightarrow{j} A \xrightarrow{q} B \to 0$$

be an extension of locally convex algebras. Then there is a long exact sequence

$$\begin{array}{l} \xrightarrow{\partial} E(SI) \xrightarrow{E(Sj)} E(SA) \xrightarrow{E(S\alpha)} E(SB) \\ \qquad \xrightarrow{\partial} E(I) \xrightarrow{E(j)} E(A) \xrightarrow{E(q)} E(B) \end{array}$$

which is infinite to the left. The boundary map ∂ is given by $\partial = E(e)^{-1}E(\kappa)$.

Proof. Let $\pi : C_q \to A$ be as above. Consider the following diagram

$$\begin{tikzcd} & & E(I) \arrow[r] \arrow[d, "\cong"] & E(A) \arrow[r] \arrow[d, equal] & E(B) \\ & E(SB) \arrow[r] \arrow[ur, "\partial"] \arrow[d, "\cong"] & E(C_q) \arrow[r] \arrow[d, equal] & E(A) & \\ E(SA) \arrow[r] \arrow[ur, "E(Sq)"] & E(C_\pi) \arrow[r] & E(C_q) & & \end{tikzcd}$$

The rows are exact and the diagram is commutative except possibly for the first triangle. By a well known argument one shows that the composition of the maps $SA \xrightarrow{Sq} SB \to C_\pi$ is diffotopic to the natural map $SA \to C_\pi$ composed with the self-map of SA that switches the orientation of the interval $[0, 1]$.

In the category of locally convex algebras we can also define an algebraic suspension and algebraic mapping cones in the following way.

Definition 6. *Let $A[t] = A \hat{\otimes} \mathbb{C}[t]$ denote the algebra of polynomials with coefficients in A. The topology is defined by choosing the fine topology on $\mathbb{C}[t]$. We denote by $C^{\mathrm{alg}}A$ and $S^{\mathrm{alg}}A$ the ideals $tA[t]$ and $t(1-t)A[t]$ of polynomials vanishing in 0 or in 0 and 1, respectively.*

Clearly, $C^{\mathrm{alg}}A$ is contractible and we have an extension $0 \to S^{\mathrm{alg}}A \to C^{\mathrm{alg}}A \to A \to 0$. The associated long exact sequence shows that $E(S^{\mathrm{alg}}A) = E(SA)$ for every half-exact diffotopy functor E.

The algebraic mapping cone C_α^{alg} for a homomorphism $\alpha : A \to B$ is defined as the subalgebra of $A \oplus C^{\mathrm{alg}}B$ consisting of all pairs (x, f) such that $\alpha(x) = f(1)$. Again it is easily checked that the natural map $E(C_\alpha^{\mathrm{alg}}) \to E(C_\alpha)$ is an isomorphism for every half-exact diffotopy functor E (compare the long exact sequences associated with the extensions $0 \to S^{\mathrm{alg}}B \to C_\alpha^{\mathrm{alg}} \to A \to 0$ and $0 \to SB \to C_\alpha \to A \to 0$).

1.3 The universal boundary map

A description of the boundary map which is, at the same time, elementary and universal can be obtained by comparing a given extension to a free extension.

Let V be a complete locally convex space. Consider the algebraic tensor algebra

$$T_{alg}V = V \,\oplus\, V{\otimes}V \,\oplus\, V^{\otimes^3} \oplus \,\ldots$$

with the usual product given by concatenation of tensors. There is a canonical linear map $\sigma : V \to T_{alg}V$ mapping V into the first direct summand. We equip $T_{alg}V$ with the locally convex topology given by the family of all seminorms of the form $\alpha \circ \varphi$, where φ is any homomorphism from $T_{alg}V$ into a locally convex algebra B such that $\varphi \circ \sigma$ is continuous on V, and α is a continuous seminorm on B. We further denote by TV the completion of $T_{alg}V$ with respect to this locally convex structure. TV has the following universal property:

for every continuous linear map $s : V \to B$ where B is a locally convex algebra, there is a unique homomorphism $\tau_s : TV \to B$ such that $s = \varphi \circ \sigma$.

(Proof. τ_s maps $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ to $s(x_1)s(x_2)\ldots s(x_n) \in B$.)

For any locally convex algebra A we have the natural extension

$$0 \to JA \to TA \xrightarrow{\pi} A \to 0.$$

Here π maps a tensor $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ to $x_1 x_2 \ldots x_n \in A$ and JA is defined as $\operatorname{Ker}\pi$. This extension is (uni)versal in the sense that, given any extension $\mathcal{E} : 0 \to I \to D \to B \to 0$ of a locally convex algebra B with continuous linear splitting s, and any continuous homomorphism $\alpha : A \to B$, there is a morphism of extensions

$$\begin{array}{ccccccc}
0 \to & JA & \to & TA & \to & A & \to 0 \\
 & \downarrow \gamma & & \downarrow \tau & & \downarrow \alpha & \\
0 \to & I & \to & D & \to & B & \to 0
\end{array}$$

The map $\tau : TA \to D$ maps $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ to $s'(x_1)s'(x_2)\ldots s'(x_n) \in D$, where $s' := s \circ \alpha$.

Definition 7. *If $\alpha = \mathrm{id} : B \to B$, then the map $\gamma : JB \to I$ (defined to be the restriction of τ) is called the* classifying *map for the extension $\mathcal{E}$.*

The classifying map depends on s only up to diffotopy. In fact, if $\bar{s}$ is a second continuous linear splitting, then the classifying maps associated to $ts+(1-t)\bar{s}$ define a diffotopy between γ and the classifying map associated with $\bar{s}$. Thus, up to diffotopy, an extension has a unique classifying map. More generally, let $s : A \to D$ be a continuous linear map between locally convex algebras and I a closed ideal in D such that $s(xy) - s(x)s(y)$ is in I for all $x, y \in A$. Then the restriction γ_s, of τ_s to JA, maps JA into I. We have the following useful observation.

Lemma 8. *If $s' : A \to D$ is a second continuous linear map which is congruent to s in the sense that $s(x) - s'(x) \in I$ for all $x \in A$, then $\gamma_s, \gamma_{s'} : JA \to I$ are diffotopic.*

Proof. The diffotopy is induced by the linear map $\hat{s} : A \to D[0,1]$, where $\hat{s}_t = ts + (1-t)s'$.

Denote the classifying map $JB \to SB$ for the cone extension $0 \to SB \to CB \xrightarrow{p} B \to 0$ by ψ_B. Let E be a half-exact diffotopy functor. Comparing the long exact sequneces for the extension $0 \to JB \to TB \to B \to 0$ and for the cone extension $0 \to SB \to CB \to B \to 0$ gives

$$\begin{tikzcd}
E(SB) \arrow[r, "\partial_B"] \arrow[d, "="] & E(JB) \arrow[r] \arrow[d, "E(\psi_B)"] & E(TB) \arrow[r] \arrow[d] & E(B) \arrow[d] \\
E(SB) \arrow[r, "="] & E(SB) \arrow[r] & E(CB) \arrow[r] & E(B)
\end{tikzcd}$$

Moreover $E(CB) = E(TB) = 0$, since CB and TB are contractible. Therefore the boundary map, which we denote here by ∂_B, is an isomorphism and $\partial_B = E(\psi_B)^{-1}$.

Proposition 9. *Let $0 \to I \to A \to B \to 0$ be an extension of locally convex algebras with classifying map $\gamma : JB \to I$. Then the boundary map $\partial : E(SB) \to E(I)$ in the long exact sequence associated with this extension is given by the formula $\partial = E(\gamma) \circ \partial_B = E(\gamma) \circ E(\psi_B)^{-1}$.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
 & I \arrow[r] \arrow[d, "e"] & A \arrow[r] \arrow[d] & B \arrow[d, equal] & \\
0 \arrow[r] & C_q \arrow[r, "\pi"] & Z_q \arrow[r] & B \arrow[r] & 0 \\
 & JB \arrow[r] \arrow[u, "\kappa\circ\psi_B"] & TB \arrow[r] \arrow[u] & B \arrow[u, equal] &
\end{tikzcd}$$

The maps $e \circ \gamma$ and $\kappa \circ \psi_B$ are both classifying maps for the extension in the middle row. By uniqueness of the classifying map they are diffotopic. Therefore $\partial = E(e)^{-1}E(\kappa) = E(e)^{-1}E(e)E(\gamma)E(\psi_B)^{-1}$.

1.4 The Toeplitz extension and Bott periodicity

The algebraic Toeplitz algebra $\mathcal{T}^{\mathrm{alg}}$ is the unital complex algebra with two generators v and v^* satisfying the identity $v^*v = 1$. It is a locally convex algebra with the fine topology. There is a natural homomorphism $\mathcal{T}^{\mathrm{alg}} \to \mathbb{C}[z, z^{-1}]$ to the algebra of Laurent polynomials. The kernel is isomorphic to the algebra

$$M_\infty(\mathbb{C}) = \varinjlim_k M_k(\mathbb{C})$$

of matrices of arbitrary size (to see this note that the kernel is the ideal generated by the idempotent $e = 1 - vv^*$. The isomorphism maps an element $v^n e(v^*)^n$ of the kernel to the matrix unit E_{nm} in $M_\infty(\mathbb{C})$). $M_\infty(\mathbb{C})$ is a locally convex algebra with the fine topology (which is also the inductive limit topology in the representation as an inductive limit).

Given a locally convex algebra A, we consider also the algebra $M_\infty A$ defined by

$$M_\infty(A) = M_\infty(\mathbb{C}) \hat{\otimes} A \cong \varinjlim_k M_k(A)$$

Another standard locally convex algebra is the algebra $\mathcal{K}$ of “smooth compact operators” consisting of all $\mathbb{N}\times\mathbb{N}$-matrices (a_{ij}) with rapidly decreasing matrix elements $a_{ij} \in \mathbb{C}$, $i, j = 0, 1, 2 \dots$. The topology on $\mathcal{K}$ is given by the family of norms p_n, $n = 0, 1, 2 \dots$, which are defined by

$$p_n\big((a_{ij})\big) = \sum_{i,j} |1+i|^n |1+j|^n \, |a_{ij}|$$

Thus, $\mathcal{K}$ is isomorphic to the projective tensor product $s \hat{\otimes} s$, where s denotes the space of rapidly decreasing sequences $a = (a_i)_{i\in\mathbb{N}}$.

Definition 10. *A functor $E : \mathrm{LCA} \to \mathrm{Ab}$ is called M_∞-stable ($\mathcal{K}$-stable), if the natural inclusion $A \to M_\infty A$ ($A \to \mathcal{K} \hat{\otimes} A$) induces an isomorphism $E(A) \to E(M_\infty A)$ ($E(A) \to E(\mathcal{K} \hat{\otimes} A)$) for each locally convex algebra A.*

We introduce the dual suspension $\hat{S}A$ of a locally convex algebra A as the kernel of the natural map $A[z, z^{-1}] \to A$, that maps z to 1 and abbreviate $\hat{S}\mathbb{C}$ to $\hat{S}$. The dual cone $\hat{C}A$ is defined as the kernel of the canonical homomorphism $\mathcal{T}^{\mathrm{alg}} \hat{\otimes} A \to A$ that maps v to 1 and $\hat{C}\mathbb{C}$ is abbreviated to $\hat{C}$.

$$\begin{tikzcd}
0 \arrow[r] & \hat{S}A \arrow[r] & A[z,z^{-1}] \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & \hat{C}A \arrow[r] \arrow[u] & \mathcal{T}^{\mathrm{alg}} \hat{\otimes} A \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & M_\infty A \arrow[r] \arrow[u] & M_\infty A \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The terminology "dual cone" and "dual suspension" is motivated by the following

Proposition 11. *For every half-exact and M_∞-stable diffotopy functor E and for every locally convex algebra A , we have $E(\hat{C}A) = 0$.*

Proof (Sketch of proof). In the terminology explained in 3.3 there is a quasi-homomorphism $(\varphi, \bar{\varphi}) : \hat{C}A \to M_\infty \hat{C}A[0,1]$ such that $E(\varphi_0, \bar{\varphi}_0) = E(j)$ while $E(\varphi_1, \bar{\varphi}_1) = 0$ (for a complete proof see [8], 8.1, 8.2).

We obtain the dual cone extension

$$0 \to M_\infty A \to \hat{C}A \to \hat{S}A \to 0$$

Applying the long exact sequence, one immediately gets

Proposition 12. *For every half-exact and M_∞-stable diffotopy functor E and for every locally convex algebra A there is a natural isomorphism $\beta : E(S\hat{S}A) \to E(A)$.*

Proof. $\beta : E(S\hat{S}A) \to E(M_\infty A) \cong E(A)$ is given by the boundary map in the long exact sequence for the dual cone extension.

Remark 13. It is clear that the dual suspension is very closely related to the construction of negative K-theory by Bass.

1.5 A dual boundary map

In this subsection we assume throughout that E is a diffotopy invariant, half-exact and M_∞-stable functor. Let $\alpha : A \to B$ be a continuous homomorphism between locally convex algebras and $0 \to M_\infty B \to \hat{C}B \xrightarrow{\pi} \hat{S}B \to 0$ the dual cone extension for B. We define the dual mapping cone by

$$\hat{C}_\alpha = \{(x,y) \in \hat{S}A \oplus \hat{C}B \,|\, \hat{S}\alpha(x) = \pi(y)\}$$

There is a natural extension $0 \to M_\infty B \to \hat{C}_\alpha \to \hat{S}A \to 0$.

Consider now an extension $0 \to I \xrightarrow{j} A \to B \to 0$ and the dual mapping cone $\hat{C}_j$. There are two natural extensions

$$0 \to M_\infty A \to \hat{C}_j \xrightarrow{\pi_1} \hat{S}I \to 0$$

and

$$0 \to \hat{C}I \to \hat{C}_j \xrightarrow{\pi_2} M_\infty B \to 0$$

The second extension and the fact that $E(\hat{C}I) = 0$ for any locally convex algebra I, shows that $E(\pi_2) : E(\hat{C}_j) \to E(M_\infty B) \cong E(B)$ is an isomorphism.

Setting $\delta = E(\pi_1) \circ E(\pi_2)^{-1}$, we obtain the following commutative diagram

$$\begin{tikzcd}
E(I) \arrow[r] & E(A) \arrow[r] \arrow[dr, bend right, "E(\kappa)"'] & E(B) \arrow[dr, bend left, "\delta"] & \\
& & E(\hat{C}_j) \arrow[u, "E(\pi_2)"] \arrow[r, "E(\pi_1)"] & E(\hat{S}I)
\end{tikzcd}$$

where $E(\kappa)$ is induced by the natural inclusion $\kappa : A \to M_\infty A \subset \hat{C}_j$.

Since $A \cong \kappa(A)$ is isomorphic under E to the kernel $M_\infty A$ of the natural surjection $\pi_1 : \hat{C}_j \to \hat{S}I$, it can be shown easily that the two sequences

$$E(I) \longrightarrow E(A) \longrightarrow E(B) \xrightarrow{\delta} E(\hat{S}I)$$

and

$$E(I) \longrightarrow E(A) \xrightarrow{\kappa} E(\hat{C}_j) \xrightarrow{E(\pi_1)} E(\hat{S}I)$$

obtained from this diagram are exact and can in fact be continued indefinitely to the right (this is the dual mapping cone sequence discussed in [5]). We mention that the surjective map $\hat{C}_j \xrightarrow{\pi} \hat{S}I \oplus B$ with $\pi = \pi_1 \oplus \pi_2$ is exactly analogous to the dual inclusion map $SB \oplus I \longrightarrow C_q$ that has been used in the construction of the boundary map ∂ in 5.

We show now that, after identification by the Bott isomorphism β, the boundary maps ∂ and δ coincide up to a sign.

Proposition 14. *Let $\partial : E(SB) \to E(I)$ be the boundary map for the extension $0 \to I \to A \to B \to 0$ and $\delta : E(SB) \to E(\hat{S}SI)$ the dual boundary map for the suspended extension $0 \to SI \to SA \to SB \to 0$. Then $\partial = -\beta \circ \delta$*

Proof. Consider the mapping cones C_φ, C_π and C_q for the natural surjections $\varphi : \hat{C}I \to \hat{S}I$, $\pi : \hat{C}_j \to \hat{S}I \oplus B$ and $q : A \to B$ together with the associated inclusion maps in the Puppe sequence $S\hat{S}I \to C_\varphi$, $S\hat{S}I \to C_\pi$, $SB \to C_\pi$ and $SB \to C_q$. We identify $S\hat{S}I$ with $\hat{S}SI$.

We obtain the following diagram in which the upper half and the lower half commute:

$$\begin{tikzcd}
 & S\hat{S}I \arrow[rr, "\kappa_1"] \arrow[dr, "\alpha_1"'] & & C_\varphi \arrow[dl, "\sim"'] & \\
S\hat{C}_j \arrow[ur, bend left, "\psi_1"] \arrow[dr, bend right, "\psi_2"'] & & C_\pi & & I \arrow[ll, "e_3"'] \arrow[ul, bend right, "e_1"'] \arrow[dl, bend left, "e_2"] \\
 & SB \arrow[rr, "\kappa_2"'] \arrow[ur, "\alpha_2"] & & C_q \arrow[ul, "\sim"] &
\end{tikzcd}$$

We have $E(e_1)^{-1}E(\kappa_1) = \beta$ and $E(e_2)^{-1}E(\kappa_2) = \partial$.

Moreover, $E(\alpha_1 \circ \psi_1) + E(\alpha_2 \circ \psi_2) = 0$, since $\alpha_1 \circ \psi_1 + \alpha_2 \circ \psi_2$ is the composition of the maps $S\hat{C}_j \to S\hat{S}I \oplus SB \to C_\pi$ in the mapping cone sequence for π. Thus

$$0 = E(e_3)^{-1}(E(\alpha_1 \circ \psi_1) + E(\alpha_2 \circ \psi_2)) = \beta E(\psi_1) + \partial E(\psi_2)$$

Since, by definition $\delta = E(\psi_1)E(\psi_2)^{-1}$, the assertion follows.

2 The categories kk^{alg} and kk

From now on we will describe our constructions in the category kk^{alg}. Since kk^{alg} acts on every diffotopy invariant, half-exact functor which is also $\mathcal{K}$-stable in the sense of 10, statements derived in kk^{alg} will pass to any functor with these properties. Some of the statements that we prove in the kk^{alg}-setting could also be proved for functors which are just M_∞-stable, rather than $\mathcal{K}$-stable. This slight loss of generality could easily be recovered by the interested reader wherever necessary. We mention at any rate that the arguments below depend in general only on some formal properties of the theory kk^{alg} and work just as well for other functors or bivariant theories satisfying the same conditions.

Explicitly, kk_*^{alg} is defined as

$$kk_n^{\mathrm{alg}}(A,\, B\,) = \varinjlim_k [J^{k-n}A,\, \mathcal{K}\hat{\otimes}S^kB\,]$$

where, given two locally convex algebras C and D, $[C, D]$ denotes the set of diffotopy classes of homomorphisms from C to D, see [8].

Here are some properties of kk^{alg} which are essential for our constructions:

- Every continuous homomorphism $\alpha : A \to B$ determines an element $kk(\alpha)$ in $kk_0^{\mathrm{alg}}(A, B)$. Given two homomorphisms α and β, we have $kk(\alpha \circ \beta) = kk(\beta)kk(\alpha)$.
- Every extension $\mathcal{E} :\ 0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$ determines canonically an element $kk(\mathcal{E})$ in $kk_{-1}^{\mathrm{alg}}(B, I)$. The class of the cone extension

$$0 \to SA \to A(0,1] \to A \to 0$$

is the identity element in $kk_0^{\mathrm{alg}}(A,A) = kk_{-1}^{\mathrm{alg}}(A,SA)$.
If

$$\begin{array}{ccccccccc}
(\mathcal{E}):\ 0 \to & A_1 & \to & A_2 & \to & A_3 & \to 0 \\
 & \downarrow\alpha & & \downarrow & & \downarrow\beta & \\
(\mathcal{E}'):0 \to & B_1 & \to & B_2 & \to & B_3 & \to 0
\end{array}$$

is a morphism of extensions (a commutative diagram where the rows are extensions), then $kk(\mathcal{E})kk(\alpha) = kk(\beta)kk(\mathcal{E}')$.

Moreover, for each fixed locally convex algebra D, the functor $A \mapsto kk_0^{\mathrm{alg}}(D,A)$ is covariant, half-exact, diffotopy invariant and M_∞-stable, while $A \mapsto kk_0^{\mathrm{alg}}(A,D)$ is a contravariant functor with the same properties. Thus both of these functors have long exact sequences where the boundary maps are given by the construction described in section 1.2. We refer to [8] for more details.

In [9] we considered the category $kk_*^{\mathcal{L}^p}$ defined by $kk_*^{\mathcal{L}^p}(A,B) = kk_*^{\mathrm{alg}}(A, B\hat{\otimes}\mathcal{L}^p)$ where $\mathcal{L}^p$ denotes the Schatten ideal of p-summable operators for $1 \leq p < \infty$. We showed that it follows from a result in [6] that $kk_*^{\mathcal{L}^p}$ does not depend on p. Let us denote $kk_*^{\mathcal{L}^p}$ by kk_* (this notation was not used in [9]). As shown in [9], a big advantage of the resulting theory kk_* is that its coefficient ring can be determined as $kk_0(\mathbb{C},\mathbb{C}) = \mathbb{Z}$ and $kk_1(\mathbb{C},\mathbb{C}) = 0$. Otherwise, the theory kk_* has the same good formal properties as kk_*^{alg} and there is a natural functor from the category kk_*^{alg} to kk_*. Thus all identities proved in the category kk_*^{alg} carry over to kk_*.

3 Abstract Kasparov modules

Baum and Douglas consider K-homology elements in $K^0(A)$ for a C^*-algebra A which are represented by an even Kasparov module. Such a Kasparov module consists of a pair (φ, F), where φ is a homomorphism from A into the algebra $\mathcal{L}(H)$ of bounded operators on a $\mathbb{Z}/2$-graded Hilbert space $H = H_+ \oplus H_-$ and F is a (self-adjoint) element of $\mathcal{L}(H)$ such that φ is even, F is odd, and such that for all $x \in A$ the following expressions lie in the algebra $\mathcal{K}(H)$ of compact operators on H

$$\varphi(x)(F - F^2), \quad [\varphi(x), F]$$

In the direct sum decomposition of H, F and φ correspond to matrices of the form

$$F = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix} \qquad \varphi = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$$

The corresponding K-homology element can be described by an associated quasihomomophism (see below).

Higson considers also the case of K-homology elements in $K^1(A)$ represented by an odd Kasparov module. Such a module consists again of a pair (φ, F), where φ is a homomorphism from A into the algebra $\mathcal{L}(H)$ of

bounded operators on a Hilbert space H (which is this time trivially graded) and F is a (self-adjoint) element of $\mathcal{L}(H)$ such that for all $x \in A$ we have $\varphi(x)(F - F^2)$, $[\varphi(x), F] \in \mathcal{K}(H)$.
In this case the K-homology element defined by (φ, F) is the one associated with the extension

$$0 \to \mathcal{K}(H) \to D \to A \to 0$$

where D is the subalgebra of $A \oplus \mathcal{L}(H)$ generated by products of $x \oplus \varphi(x),\ x \in A$ together with elements of the algebra generated by $1 \oplus F$.

We will now describe kk^{alg}-elements associated with a Kasparov module in an abstract setting.

3.1 Morphism extensions

Let A and B be locally convex algebras. A morphism extension from A to B will be a diagram of the form

$$\begin{tikzcd} & & & A \arrow[d, "\varphi"] & \\ 0 \arrow[r] & K \arrow[r] & D \arrow[r, "q"] & B \arrow[r] & 0 \end{tikzcd}$$

where φ is a homomorphism and the row is an extension.

We can encode the information contained in a morphism extension in a single (pull back) extension in the following way.

Define D' as the subalgebra of $A \oplus D$ consisting of all elements (a, d) such that $\varphi(x) = q(d)$. The natural homomorphism $\pi : D' \to D$ defined by $\pi((a, d)) = d$ gives the following morphism of extensions

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & D' \arrow[r] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & K \arrow[r] & D \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

If $\mathcal{E}$ is the original extension in the second row and $\mathcal{E}'$ the pull back extension, then $kk(\mathcal{E}') = kk(\varphi)kk(\mathcal{E})$. We say that this element $kk(\mathcal{E}') = kk(\varphi)kk(\mathcal{E})$ is the element of $kk^{\mathrm{alg}}_{-1}(A, K)$ associated with the given morphism extension and denote it by $kk(\mathcal{E}, \varphi)$.

3.2 Abstract odd Kasparov modules

Definition 15. *Let A be a locally convex algebra and $0 \to K \to D \to D/K \to 0$ an extension of locally convex algebras where D is unital. An abstract odd Kasparov (A, K)-module relative to D is a pair (φ, P) where*

- *φ is a continuous homomorphism from A into D.*

- *P is an element in D such that the following expressions are in K for all $x \in A$:*
$$[P, \varphi(x)], \quad \varphi(x)(P - P^2)$$

With an odd Kasparov module we can associate the following morphism extension

$$\begin{tikzcd} & & & A \arrow[d, "\tau"] & \\ 0 \arrow[r] & K \arrow[r] & D \arrow[r, "q"] & D/K \arrow[r] & 0 \end{tikzcd}$$

where q is the quotient map and $\tau(x) = q(P\varphi(x)P)$. We denote by $kk(\varphi, P)$ the element of $kk^{\mathrm{alg}}_{-1}(A, K)$ associated with this morphism extension.

3.3 Quasihomomorphisms

Let α and $\bar{\alpha}$ be two homomorphisms $A \to D$ between locally convex algebras. Assume that B is a closed subalgebra of D such that $\alpha(x) - \bar{\alpha}(x) \in B$ and $\alpha(x)B \subset B$, $B\alpha(x) \subset B$ for all $x \in A$. We call such a pair $(\alpha, \bar{\alpha})$ a quasihomomorphism from A to B relative to D and denote it by $(\alpha, \bar{\alpha}) : A \to B$.

We will show that $(\alpha, \bar{\alpha})$ induces a homomorphism $E(\alpha, \bar{\alpha}) : E(A) \to E(B)$ in the following way. Define $\alpha', \bar{\alpha}' : A \to A \oplus D$ by $\alpha'(x) = (x, \alpha(x))$, $\bar{\alpha}' = (x, \bar{\alpha}(x))$ and denote by D' the subalgebra of $D \oplus A$ generated by all elements $\alpha'(x),\ x \in A$ and by $0 \oplus B$. We obtain an extension with two splitting homomorphisms α' and $\bar{\alpha}'$:
$$0 \to B \to D' \to A \to 0$$
where the map $D' \to A$ by definition maps $(x, \alpha(x))$ to x and $(0, b)$ to 0. The map $E(\alpha, \bar{\alpha})$ is defined to be $E(\alpha') - E(\bar{\alpha}') : E(A) \to E(B) \subset E(D')$ (this uses split-exactness). Note that $E(\alpha, \bar{\alpha})$ is independent of D in the sense that we can enlarge D without changing $E(\alpha, \bar{\alpha})$ as long as B maintains the properties above.

Proposition 16. *The assignment $(\alpha, \bar{\alpha}) \to E(\alpha, \bar{\alpha})$ has the following properties:*

(a) $E(\bar{\alpha}, \alpha) = -E(\alpha, \bar{\alpha})$
(b) If the linear map $\varphi = \alpha - \bar{\alpha}$ is a homomorphism and satisfies $\varphi(x)\bar{\alpha}(y) = \bar{\alpha}(x)\varphi(y) = 0$ for all $x, y \in A$, then $E(\alpha, \bar{\alpha}) = E(\varphi)$.
(c) Assume that α is diffotopic to α' and $\bar{\alpha}$ is diffotopic to $\bar{\alpha}'$ via diffotopies φ, $\bar{\varphi}$ such that $\varphi_t(x) - \alpha(x), \bar{\varphi}_t(x) - \bar{\alpha}(x) \in B$ for all $x \in A$ (we denote this situation by $\alpha \sim_B \alpha'$, $\bar{\alpha} \sim_B \bar{\alpha}'$). Then $E(\alpha, \bar{\alpha}) = E(\alpha', \bar{\alpha}')$.

Proof. (a) This is obvious from the definition. (b) This follows $\varphi + \bar{\alpha} = \alpha$ and the fact that $E(\varphi + \bar{\alpha}) = E(\varphi) + E(\bar{\alpha})$. (c) follows from the definition of $E(\alpha, \bar{\alpha})$ and diffotopy invariance of E.

Choosing $E(?) = kk_0^{\mathrm{alg}}(A,?)$ produces in particular an element $kk(\alpha,\bar\alpha)$ in $kk_0^{\mathrm{alg}}(A,B)$ (obtained by applying $E(\alpha,\bar\alpha)$ to the unit element 1_A in $kk_0^{\mathrm{alg}}(A,A)$) .

3.4 Abstract even Kasparov modules

Definition 17. *Let A, K and D be locally convex algebras. Assume that D is unital and contains K as a closed ideal. An abstract even Kasparov (A,K)-module relative to D is a triple $(\alpha,\bar\alpha,U)$ where*

- *$\alpha,\bar\alpha$ are continuous homomorphisms from A into D.*
- *U is an invertible element in D such that $U\bar\alpha(x) - \alpha(x)U$ is in K for all $x \in A$.*

¿From an even Kasparov module we obtain a quasihomomorphism $(\alpha, \mathrm{Ad}\,U \circ \bar\alpha) : A \to K$. We write $kk(\alpha,\bar\alpha,U)$ for the element of $kk_0^{\mathrm{alg}}(A,K)$ associated with this quasihomomorphism. More generally, if E is a half-exact diffotopy functor we write $E(\alpha,\bar\alpha,U)$ for the morphism $E(A) \to E(K)$ obtained from this quasihomomorphism.

Remark 18. The connection with Kasparov's definition in the C^*-algebra/Hilbert space setting mentioned at the beginning of section 3 is obtained by setting

$$U = \begin{pmatrix} \sqrt{f_1} & v \\ -v^* & \sqrt{f_2} \end{pmatrix}$$

where $f_1 = 1 - vv^*$ and $f_2 = 1 - v^*v$, and by replacing $\alpha,\bar\alpha$ by $\alpha \oplus 0, 0 \oplus \bar\alpha$.

This corresponds to replacing the Kasparov module (H,F) by the inflated module (H',F') where H' is the $\mathbb{Z}/2$-graded Hilbert space $H \oplus H$ with $H = H_+ \oplus H_-$ and F by

$$F' = \begin{pmatrix} 0 & U \\ U^{-1} & 0 \end{pmatrix}$$

3.5 Special abstract even Kasparov modules

Let $\varphi : A \to D$ be a homomorphism of locally convex algebras where D is unital and contains a closed ideal K. Assume that D contains elements v, v^* such that the expressions

$$[\varphi(x), v], [\varphi(x), v^*], \varphi(x)(vv^* - 1), \varphi(x)(v^*v - 1)$$

are in K for all $x \in A$.

If we assume moreover that, in D, there are square roots for the elements $f_1 = 1 - vv^*$ and $f_2 = 1 - v^*v$, we can form an abstract even Kasparov module (relative to M_2D) by choosing

$$\alpha = \bar{\alpha} = \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} \qquad U = \begin{pmatrix} v & \sqrt{f_1} \\ \sqrt{f_2} & -v^* \end{pmatrix}$$

If we suppose in this case moreover that there is a continuous linear splitting $D/K \to D$, we can associate with this even Kasparov module (α, α, U) also a morphism extension

$$\begin{tikzcd} & & & \hat{S}A \arrow[d, "\rho"] & \\ 0 \arrow[r] & K \arrow[r] & D \arrow[r, "\pi"] & D/K \arrow[r] & 0 \end{tikzcd}$$

by defining $\rho(\sum x_i z^i) = \pi(\sum \alpha(x_i) U^i)$.

It can be checked that the element in $kk^{\mathrm{alg}}_{-1}(\hat{S}A, K)$ defined by this morphism extension corresponds to the element $kk(\alpha, \alpha, U)$ constructed above under the Bott isomorphism $kk^{\mathrm{alg}}_{-1}(\hat{S}A, K) \cong kk^{\mathrm{alg}}_{0}(A, K)$.

We will later consider the case where $v^*v = 1$ and thus $f_1 = 1 - vv^*$ is an idempotent.

Remark 19. In [9] we had considered a different notion of an even Kasparov module. This notion is closely related to the situation considered here.

4 The boundary map in the Baum-Douglas situation

Baum and Douglas consider an extension $0 \to I \to A \to B \to 0$ of C^*-algebras and obtain for K-homology elements, that are realized by a special kind of Kasparov modules, a formula for the boundary map $K^0 I \to K^1 B$ in K-homology in the long exact sequence associated with the extension $0 \longrightarrow I \xrightarrow{j} A \xrightarrow{q} B \longrightarrow 0$. The basic assumption on the given element in $K^0 I$ for which the boundary is determined is that it should be realized by a Kasparov module (φ, F) for which φ extends to a homomorphism such that $\varphi(A)$ still commutes with F modulo compacts (however $\varphi(x)F^2 = \varphi(x)$ holds only for $x \in I$ and not necessarily for $x \in A$). Higson gives a similar formula under analogous conditions for the boundary map $K^1 I \to K^0 B$.

It turns out that the condition imposed by Baum-Douglas on the Kasparov module means exactly that the Kasparov module for $\hat{S}I$ extends to a Kasparov module for the dual mapping cone $\hat{C}_j$, while Higson's condition in the odd case means that the corresponding Kasparov module extends to one for the ordinary mapping cone C_q.

With this observation and the preliminaries explained in the previous sections it is completely straightforward to deduce explicit formulas for the images of the given K-homology elements under the boundary map.

4.1 The odd case

Higson has stated and proved a formula, for the image under the boundary map of certain odd K-homology elements, which is analogous to the Baum-Douglas formula for even elements, [10], [11]. In this subsection we give a simple proof for this formula. As in the even case, our proof carries over verbatim to the case of C^*-algebras and thus to the case considered by Higson.
Let $0 \to I \to A \xrightarrow{q} B \to 0$ be an extension of locally convex algebras with a continuous linear splitting s.
Assume, we are given an odd Kasparov (I, K)-module (φ, P), where φ is a continuous homomorphism into a locally convex algebra D, P is an element of D and K is a closed ideal in D. Thus by definition $[\varphi(I), P] \subset K$ and $\varphi(I)(P - P^2) \subset K$.

Suppose now that φ extends to a homomorphism $\varphi : A \to D$ such that also $[\varphi(A), P] \subset K$.
Consider the morphism extension $(\mathcal{E}, \tau)$

$$\begin{tikzcd} & & & I \arrow[d, "\tau"] & \\ 0 \arrow[r] & K \arrow[r] & D \arrow[r, "\pi"] & D/K \arrow[r] & 0 \end{tikzcd}$$

associated to (φ, P) as in 3.2.

Proposition 20. *Let $\partial : kk_{-1}^{\mathrm{alg}}(I, K) \longrightarrow kk_{-1}^{\mathrm{alg}}(S^{\mathrm{alg}}B, K)$ be the boundary map. Then $\partial(kk(\mathcal{E}, \tau))$ is represented by the morphism extension $(\mathcal{E}, \psi)$ given by the diagram*

$$\begin{tikzcd} & & & S^{\mathrm{alg}}B \arrow[d, "\psi"] & \\ 0 \arrow[r] & K \arrow[r] & D \arrow[r, "\pi"] & D/K \arrow[r] & 0 \end{tikzcd}$$

where π is the quotient map and ψ is defined by $\psi(\sum b_i t^i) = \pi(\sum \varphi(sb_i)P^i)$.

Proof. We define a homomorphism $\rho : C_q^{\mathrm{alg}} \to D/K$ as follows. Let $(x, f) \in C_q^{\mathrm{alg}}$ where $f = \sum b_i t^i$ is in $C^{\mathrm{alg}}B$ and $f(1) = q(x)$. We set $\rho((x, f)) = \pi(\varphi(x - sq(x)) + \sum \varphi(sb_i)P^i)$.

We have now three morphism extensions defined by the extension $0 \to K \to D \to D/K \to 0$ together with the homomorphisms $I \to D/K$, $S^{\mathrm{alg}}B \to D/K$ and $C_q^{\mathrm{alg}} \to D/K$. Consider the three extensions $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ associated to these morphism extensions as in 3.1. We obtain the following commutative diagram of extensions

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & D_1 \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, "e"] & 0 \\
0 \arrow[r] & K \arrow[r] & D_3 \arrow[r] & C_q^{\mathrm{alg}} \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u] & D_2 \arrow[r] \arrow[u] & S^{\mathrm{alg}}B \arrow[r] \arrow[u, "\kappa"] & 0
\end{tikzcd}$$

where the first row is $\mathcal{E}_1$, the last one $\mathcal{E}_2$ and the extension in the middle is $\mathcal{E}_3$. It follows that $kk(\mathcal{E}_3)kk(\kappa) = kk(\mathcal{E}_2)$ and $kk(\mathcal{E}_3)kk(e) = kk(\mathcal{E}_1)$. Since $kk(e)$ is invertible we conclude $kk(\mathcal{E}_2) = kk(\mathcal{E}_1)kk(e)^{-1}kk(\kappa)$. But $kk(e)^{-1}kk(\kappa) = \partial$ by 5.

4.2 The even case

This subsection contains the proof of the Baum-Douglas formula for the image, under the boundary map, of certain even K-homology elements. Our proof is extremely short. It uses only a small part of the discussion above, namely the description of the boundary map in subsection 1.3.

Proposition 21. *Let $0 \to I \to A \to B \to 0$ be an extension of locally convex algebras and $s : B \to A$ a continuous linear splitting.*
Let $(\alpha, \bar{\alpha}, U)$ be an even (I, K)-Kasparov module relative to D and $z = kk(\alpha, \bar{\alpha}, U)$ the corresponding element of $kk_0^{\mathrm{alg}}(I, K)$. Then $\partial z \in kk_1^{\mathrm{alg}}(B, K) = kk_0^{\mathrm{alg}}(JB, K)$ is given by $kk((\alpha \oplus 0) \circ \gamma_s,\ \mathrm{Ad}\, U(\bar{\alpha} \oplus 0) \circ \gamma_s)$ where $\gamma_s : JB \to I$ is the classifying map.

Proof. This follows immediately from 9 applied to $E(?) = kk_0^{\mathrm{alg}}(?, K)$ and using the identification $kk_0(JB, K) \cong kk_0(SB, K)$ via $E(\psi_B)$.

In order to transform this formula for ∂z into a more usable form we need the following trivial lemma.

Lemma 22. *Let B, D, K be locally convex algebras such that K is a closed ideal in D. Let $\rho, \bar{\rho} : B \to D$ be continuous linear maps such that for the induced maps $\gamma_\rho, \gamma_{\bar{\rho}} : JB \to D$ we have $\gamma_\rho(x) - \gamma_{\bar{\rho}}(x) \in K$ for all $x \in JB$. Assume moreover that $\rho', \bar{\rho}' : B \to D$ is another pair of continuous linear maps which are congruent to $\rho, \bar{\rho} : B \to D$ in the sense that $\rho(x) - \rho'(x) \in K$ and $\bar{\rho}(x) - \bar{\rho}'(x) \in K$ for all x in B. Then the quasihomomorphism $(\gamma_\rho, \gamma_{\bar{\rho}}) : JB \to K$ is diffotopic to $(\gamma'_\rho, \gamma'_{\bar{\rho}})$ in the sense of 16 (c).*

Proof. Let $\sigma, \bar{\sigma} : B \to D[0, 1]$ denote the linear maps defined by $\sigma_t(x) = t\rho(x) + (1 - t)\rho'(x)$ and $\bar{\sigma}_t(x) = t\bar{\rho}(x) + (1 - t)\bar{\rho}'(x)$. Then $(\gamma_\sigma, \gamma_{\bar{\sigma}})$ defines a diffotopy between $(\gamma_\rho, \gamma_{\bar{\rho}})$ and $(\gamma'_\rho, \gamma'_{\bar{\rho}})$.

Consider again the situation of 21. We will assume now that the unital algebra D admits a "2×2-matrix decomposition" (i.e. D is a direct sum of subspaces D_{ij}, $i,j=1,2$, with $D_{ij}D_{jk}\subset D_{ik}$) and that $\alpha,\bar\alpha$ and U are of the form

$$\alpha(x)=\begin{pmatrix}\alpha_0(x) & 0\\ 0 & 0\end{pmatrix}\qquad \bar\alpha(x)=\begin{pmatrix}0 & 0\\ 0 & \bar\alpha_0(x)\end{pmatrix}\qquad U=\begin{pmatrix}e & v\\ -v^* & \bar e\end{pmatrix}$$

where $v\in D_{12}, v^*\in D_{21}$ are elements such that $e=1_{D_{11}}-vv^*$ and $\bar e=1_{D_{22}}-v^*v$ are idempotents. We also assume that the ideal K is compatible with this 2×2-matrix decomposition. Recall that this is a typical form in which Kasparov modules arise in applications.

Theorem 23. *Let $0\to I\to A\to B\to 0$ be an extension of locally convex algebras, $s:B\to A$ a continuous linear splitting and $\partial : kk_0^{\rm alg}(I,K)\to kk_{-1}^{\rm alg}(B,K)$ the associated boundary map.*
Let $(\alpha,\bar\alpha,U)$ as above represent an even (I,K)-module relative to D.
Assume that $\alpha_0,\bar\alpha_0 : I\to D$ extend to homomorphisms, still denoted by $\alpha_0,\bar\alpha_0$, from A to D_{11}, resp. to D_{22}, and assume moreover that the elements $v\bar\alpha_0(x)-\alpha_0(x)v$, $\bar\alpha_0(x)v^*-v^*\alpha_0(x)$ are in K for all $x\in A$. Let $z=kk(\alpha,\bar\alpha,U)\in kk_0^{\rm alg}(I,K)$. Then ∂z is represented by $kk(\gamma_\tau)-kk(\gamma_{\bar\tau})$ where $\tau,\bar\tau : B\to D$ are given by $\tau(x)=e\,\alpha_0 s(x)\,e$ and $\bar\tau(x)=\bar e\,\bar\alpha_0 s(x)\,\bar e$ and $\gamma_\tau,\gamma_{\bar\tau} : JB\to K$ are the corresponding homomorphisms.

Proof. We have $\alpha\circ\gamma_s=\gamma_{\alpha\circ s}$, ${\rm Ad}U\circ\bar\alpha\circ\gamma_s=\gamma_{{\rm Ad}U\circ\bar\alpha\circ s}$ (here we use the fact that α and $\bar\alpha$ extend to A!). Therefore, from 21, the element ∂z is represented by the quasihomomorphism $(\gamma_\rho,\gamma_{\bar\rho}) : JB\to K$ where $\rho(x)=\alpha(sx)$ and $\bar\rho(x)=U\bar\alpha(sx)U^{-1}$.
Writing $e^\perp=1-e=vv^*$ we have $v\bar\alpha_0(sx)v^*-e^\perp\alpha_0(sx)e^\perp\in K$ and $\alpha_0(sx)-e^\perp\alpha_0(sx)e^\perp-\tau(x)\in K$ for all $x\in B$. Therefore, setting $\rho_0(x)=\alpha_0(sx)$, $\bar\rho_0(x)=\bar\alpha_0(sx)$ we have the following congruences

$$\rho\simeq\begin{pmatrix}e^\perp\rho_0e^\perp+\tau & 0\\ 0 & 0\end{pmatrix}\qquad \bar\rho\simeq\begin{pmatrix}v\bar\rho_0v^* & 0\\ 0 & \bar\tau\end{pmatrix}$$

Thus, by Lemma 22,

$$\gamma_\rho\sim\begin{pmatrix}\gamma_{v\rho_0v^*}+\gamma_\tau & 0\\ 0 & 0\end{pmatrix}\qquad \gamma_{\bar\rho}\sim\begin{pmatrix}\gamma_{v\rho_0v^*} & 0\\ 0 & \gamma_{\bar\tau}\end{pmatrix}$$

whence $kk(\gamma_\rho,\gamma_{\bar\rho})=kk(\gamma_\tau,\gamma_{\bar\tau})=kk(\gamma_\tau)-kk(\gamma_{\bar\tau})$ (here it is important that the homomorphisms $\gamma_\tau,\gamma_{\bar\tau}$ themselves and not only their difference map into K).

4.3 The dual boundary map in the even case

There is an alternative way of describing the boundary map in the Baum-Douglas situation using the dual mapping cone construction described in 1.5.

We limit our discussion here to the case of special (see 3.5) even Kasparov modules.

Let $0 \to I \xrightarrow{j} A \to B \to 0$ be an extension of locally convex algebras with a continuous linear splitting s. Assume that $\varphi : I \to D$ and $v, v^* \in D$ satisfy the conditions in 3.5 and thus, by the construction in 3.5, define an even Kasparov (I,K)-module relative to D. We now assume that it satisfies the Baum-Douglas condition that φ extends from I to a homomorphism still denoted φ from A to D such that $[\varphi(A), v], [\varphi(A), v^*] \subset K$. Assume moreover that $v^*v = 1$ and denote by e the idempotent $e = 1 - vv^*$.

Consider the morphism extension $(\mathcal{E}, \tau)$

$$\begin{tikzcd} & & & \hat{S}I \arrow[d, "\tau"] & \\ 0 \arrow[r] & K \arrow[r] & D \arrow[r, "\pi"] & D/K \arrow[r] & 0 \end{tikzcd}$$

associated to (φ, v) as in 3.5.

Proposition 24. *Let $\partial : kk^{\mathrm{alg}}_{-1}(\hat{S}I, K) \longrightarrow kk^{\mathrm{alg}}_{-1}(B, K)$ be the boundary map (where we identify $kk^{\mathrm{alg}}_{-1}(S\hat{S}B, K)$ with $kk^{\mathrm{alg}}_{-1}(B, K)$ via Bott periodicity). Then $\partial(kk(\mathcal{E}, \tau))$ is represented by the morphism extension $(\mathcal{E}, \psi)$ given by the diagram*

$$\begin{tikzcd} & & & B \arrow[d, "\psi"] & \\ 0 \arrow[r] & K \arrow[r] & D \arrow[r, "\pi"] & D/K \arrow[r] & 0 \end{tikzcd}$$

where π is the quotient map and ψ is defined by $\psi(b) = \pi(e\varphi(sb)e)$.

Proof. The proof is exactly analogous to the proof of 20. We define a homomorphism $\rho : \hat{C}_j \to D/K$ as follows. Let $f \in \hat{C}_j \subset \hat{C}A$ where $f = \sum a_i z^i$. We set $\rho(f) = \pi(\sum \varphi(a_i) v^i)$ (where we use the convention that $v^{-k} = (v^*)^k$ for $k \in \mathbb{N}$).

We have now three morphism extensions defined by the extension $0 \to K \to D \to D/K \to 0$ together with the homomorphisms $\hat{S}I \to D/K$, $B \to D/K$ and $\hat{C}_j \to D/K$. Consider now the three pull back extensions $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ associated to these morphism extensions as in 3.1. We obtain the following commutative diagram of extensions

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & D_1 \arrow[r] & \hat{S}I \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r] \arrow[u] \arrow[d] & D_3 \arrow[r] \arrow[u] \arrow[d] & \hat{C}_j \arrow[r] \arrow[u, "\pi_1"] \arrow[d, "\pi_2"] & 0 \\ 0 \arrow[r] & K \arrow[r] & D_2 \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

where the first row is $\mathcal{E}_1$, the last one $\mathcal{E}_2$ and the extension in the middle is $\mathcal{E}_3$. It follows that $kk(\mathcal{E}_3) = kk(\pi_1)kk(\mathcal{E}_1)$ and $kk(\mathcal{E}_3) = kk(\pi_2)kk(\mathcal{E}_2)$. Since $kk(\pi_2)$ is invertible we conclude $kk(\mathcal{E}_2) = kk(\pi_2)^{-1}kk(\pi_1)kk(\mathcal{E}_1)$. But $kk(\pi_2)^{-1}kk(\pi_1) = \delta$ and δ corresponds to ∂ after identifying by Bott periodicity, see 14.

5 The case of C^*-algebras

We can carry through the construction of bivariant K-theory described above in the case of locally convex algebras in many other categories of algebras and in particular in the category of C^*-algebras. We have to replace the basic ingredients by the appropriate constructions in that category. Thus we replace:

the algebras of functions such as $A[a,b], A(a,b), SA, CA$ by the corresponding algebras of continuous (rather than smooth) functions, and diffotopy by homotopy
the locally convex algebra $\mathcal{K}$ of smooth compact operators by the C^*-algebra $\mathcal{K}$ of compact operators
the projective tensor product by the C^*-tensor product
the smooth Toeplitz algebra by the well known Toeplitz C^*-algebra
and - most importantly - the tensor algebra TA by the tensor algebra in the category of C^*-algebras described in the next paragraph.

Let A be a C^*-algebra. To construct the tensor algebra TA in the category of C^*-algebras consider as before the algebraic tensor algebra

$$T_{alg}A = A \oplus A \otimes A \oplus A^{\otimes^3} \oplus \ldots$$

with product given by concatenation of tensors and let σ denote the canonical linear map $\sigma : A \to T_{alg}A$.

Equip $T_{alg}A$ with the C^*-norm given as the sup over all C^*-seminorms of the form $\alpha \circ \varphi$, where φ is any homomorphism from $T_{alg}A$ into a C^*-algebra B such that $\varphi \circ \sigma$ is completely positive contractive on A, and α is the C^*-norm on B. Let TA be the completion of $T_{alg}A$ with respect to this C^*-norm. TA has the following universal property:

for every contractive completely positive map $s : A \to B$ where B is a C^*-algebra, there is a unique homomorphism $\varphi : TA \to B$ such that $s = \varphi \circ \sigma$.

The tensor algebra extension:

$$0 \to JA \to TA \xrightarrow{\pi} A \to 0$$

is (uni)versal in the sense that, given any extension $0 \to I \to E \to B \to 0$ admitting a completely positive splitting, and any continuous homomorphism $\alpha : A \to B$, there is a morphism of extensions

$$\begin{array}{ccccccccc} 0 \to & JA & \to & TA & \to & A & \to 0 \\ & \downarrow \gamma & & \downarrow \tau & & \downarrow \alpha & \\ 0 \to & I & \to & E & \to & B & \to 0 \end{array}$$

We can now define

$$KK_n(A, B) = \varinjlim_k [J^{k-n}A, \mathcal{K} \otimes S^k B]$$

This definition is exactly analogous to the definition of kk^{alg} in section 2. By the same arguments as in the case of kk^{alg} it is seen that this functor has a product, long exact sequences associated to extensions with cp splitting and is Bott-periodic and $\mathcal{K}$-stable. In fact, it is universal with these properties and this shows that KK gives an alternative construction for Kasparov's KK-functor.

The description of the boundary maps given above now carry over basically word by word. In particular, for an extension

$$0 \to I \to A \to B \to 0$$

of C^*-algebras with a completely positive splitting, the associated element in $KK_{-1}(B, I)$ is represented by the classifying map $\gamma_s : JB \to I$ and the boundary map is given by composition with this homomorphism.

6 The index theorem of Baum-Douglas-Taylor

For completeness we briefly recall the argument by Baum-Douglas-Taylor in [2].

Let M be a compact $\mathcal{C}^\infty$-manifold. Then a neighbourhood of M in T^*M has a complex structure and, considering the ball bundle with sufficiently small radius, B^*M can be considered as a strongly pseudoconvex domain with boundary S^*M. We obtain an extension of C^*-algebras

$$\mathcal{E}_{B^*M} : \quad 0 \to \mathcal{C}_0(T^*M) \to \mathcal{C}(B^*M) \to \mathcal{C}(S^*M) \to 0$$

On B^*M, there is an operator $D =: V \to V$, where V denotes the space of differential forms in $\Lambda^{0,*}$ on B^*M , satisfying a natural boundary condition on S^*M, such that the restriction of D to T^*M defines the Dolbeault operator $\bar\partial + \bar\partial^*$. Denote by H the L^2-completion of V.

Let $\bar D$ denote the maximal extension of D defined on the completion of V with respect to the norm $\|f\| = \|f + Df\|_2$. Then 0 is an isolated point in the spectrum of $\bar D$, the range of $\bar D$ is closed and its cokernel is finite-dimensional.

With respect to the decomposition of $\Lambda^{0,*}$ into even and odd forms H splits into $H = H_+ \oplus H_-$.

The Kasparov $\mathcal{C}_0(T^*M)$-$\mathcal{K}$-module (φ, F), where $F = \bar D/\sqrt{\bar D^2}$ and φ denotes the representation of $\mathcal{C}_0(T^*M)$ by multiplication operators, describes the Dolbeault element $[\bar\partial_{T^*M}]$. With respect to the $\mathbb{Z}/2$-grading of H and modulo compact operators F is of the form

$$\begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}$$

where $v^*v = 1$ and $P = 1 - vv^*$ is the projection onto the Bergman space of 0-forms, thus functions u, for which $\bar\partial u = 0$ (i.e. holomorphic L^2-functions). Moreover, since D extends to B^*M, it satisfies the conditions required in 23 for $I = \mathcal{C}_0(T^*M)$, $A = \mathcal{C}(B^*M)$ and $B = \mathcal{C}(S^*M)$.

Therefore by the Baum-Douglas formula in 23 the image of the element $[\bar\partial_{T^*M}] \in KK_0(\mathcal{C}_0(T^*M), \mathbb{C})$ in $KK_0(\mathcal{C}(S^*M), \mathbb{C})$ is represented by the extension

$$0 \to \mathcal{K} \to D \to \mathcal{C}(S^*M) \to 0 \tag{1}$$

where D denotes the subalgebra of $\mathcal{L}(PH)$ generated by $\mathcal{K} = \mathcal{K}(PH)$ together with $P\varphi(\mathcal{C}(S^*M))P$.

Boutet de Monvel [4], [3] constructs a unitary operator G mapping $L^2(M)$ to $PL^2(T^*M)$. It has the property that G^*T_fG is a pseudodifferential operator with symbol $f|_{S^*M}$ for a Toeplitz operator of the form $T_f = PfP$, $f \in \mathcal{C}(B^*M)$.

Therefore G conjugates the extension (1) of Toeplitz operators into the extension

$$\mathcal{E}_\Psi : \quad 0 \to \mathcal{K} \to \Psi \to \mathcal{C}(S^*M) \to 0$$

where Ψ denotes the C^*-completion of the algebra of pseudodifferential operators of order ≤ 0 and $\mathcal{K}$ the completion of the algebra of operators of order < 0.

In conclusion we get the theorem that

$$KK(\mathcal{E}_{B^*M})[\bar\partial_{T^*M}] = KK(\mathcal{E}_\Psi)$$

where $[\bar\partial_{T^*M}]$ is the K-homology element in $KK(\mathcal{C}_0(T^*M), \mathbb{C})$ defined by (φ, F) and where $\mathcal{E}_\Psi$, $\mathcal{E}_{B^*M}$ are the two natural extensions of $\mathcal{C}(S^*M)$.

The proof of the theorem that we outlined above works verbatim in the setting of locally convex algebras. The Baum-Douglas-Taylor theorem then reads as

$$kk(\mathcal{E}_\Psi) = kk(\mathcal{E}_{B^*M}) \cdot [\bar\partial_{T^*M}]$$

where $kk(\mathcal{E}_{B^*M})$, $kk(\mathcal{E}_\Psi), [\bar\partial_{T^*M}]$ are the elements in kk_* determined by the corresponding extensions of locally convex algebras (using algebras of $\mathcal{C}^\infty$-functions and replacing the ideal $\mathcal{K}$ by a Schatten ideal).

We briefly sketch the connection of this theorem to the index theorems of Kasparov and of Atiyah-Singer. Kasparov's theorem determines the K-homology class $[P] \in KK(\mathcal{C}M, \mathbb{C})$ determined by an elliptic operator P by the formula

$$[P] = [[\sigma(P)]] \cdot [\bar\partial_{T^*M}]$$

Here $[[\sigma(P)]] = [[\Sigma(P)]] \cdot KK(\mathcal{E}_{B^*M})$ and $[[\Sigma(P)]] \in KK(\mathcal{C}M, \mathcal{C}(S^*M))$ is a naturally defined bivariant class associated with the symbol of P. Kasparov's formula is (in the non-equivariant case) a consequence of the Baum-Douglas-Taylor theorem, since - basically by definition - we have $[P] = [[\Sigma(P)]] \cdot KK(\mathcal{E}_\Psi)$.

The Atiyah-Singer theorem determines the index of P as an element of $KK(\mathbb{C},\mathbb{C}) = \mathbb{Z}$ by

$$\operatorname{ind} P = \operatorname{ind}_t[\sigma(P)]$$

where ind_t is the "topological index" map. This formula is a consequence of Kasparov's formula since, by definition, $\operatorname{ind} P = [1] \cdot [P]$, $[\sigma(P)] = [1] \cdot [[\sigma(P)]]$, and since one can check that $\operatorname{ind}_t(x) = x \cdot [\bar{\partial}_{T^*M}]$ for each x in $KK(\mathbb{C}, \mathcal{C}_0(T^*M))$.

Finally, we note that, by construction, the Baum-Douglas-Taylor theorem of course also gives a formula for the index of Toeplitz operators on strictly pseudoconvex domains. This formula is also discussed in [12].

References

1. Paul Baum and Ronald G. Douglas. Relative K homology and C^* algebras. *K-Theory*, 5(1):1–46, 1991.
2. Paul Baum, Ronald G. Douglas, and Michael E. Taylor. Cycles and relative cycles in analytic K-homology. *J. Differential Geom.*, 30(3):761–804, 1989.
3. L. Boutet de Monvel and V. Guillemin. *The spectral theory of Toeplitz operators*, volume 99 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1981.
4. Louis Boutet de Monvel. On the index of Toeplitz operators of several complex variables. *Invent. Math.*, 50(3):249–272, 1978/79.
5. J. Cuntz and G. Skandalis. Mapping cones and exact sequences in KK-theory. *J. Operator Theory*, 15(1):163–180, 1986.
6. Joachim Cuntz. Bivariante K-Theorie für lokalkonvexe Algebren und der Chern-Connes-Charakter. *Doc. Math.*, 2:139–182 (electronic), 1997.
7. Joachim Cuntz. A general construction of bivariant K-theories on the category of C^*-algebras. In *Operator algebras and operator theory (Shanghai, 1997)*, volume 228 of *Contemp. Math.*, pages 31–43. Amer. Math. Soc., Providence, RI, 1998.
8. Joachim Cuntz. Bivariant K-theory and the Weyl algebra. *K-Theory*, to appear.
9. Joachim Cuntz and Andreas Thom. Algebraic K-theory and locally convex algebras. *preprint*, 2005.
10. Nigel Higson. On the relative K-homology theory of Baum and Douglas. *preprint*, 1988.
11. Nigel Higson. C^*-algebra extension theory and duality. *J. Funct. Anal.*, 129(2):349–363, 1995.
12. Nigel Higson and John Roe. *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.
13. G. G. Kasparov. Relative K-homology and K-homology of an ideal. *K-Theory*, 5(1):47–49, 1991.

14. François Trèves. *Topological vector spaces, distributions and kernels.* Academic Press, New York, 1967.