I. A Functorial approach to assembly map (joint with R. Meyer)

G - locally compact, s.c.

KK^G - the category of separable G-C^*algebras, with $\mathrm{Mor}(A, B) := KK^G(A, B)$

CC - the full subcategory of compactly contractible algebras

$$A \in CC \iff \begin{cases} A = 0 \text{ in } KK^H \\ \text{for every compact} \\ \text{subgroup } H \subset G \end{cases}$$

BC - relating two functors:

$$\begin{tikzcd} & (Ab) \arrow[d, phantom, "\| \left(\begin{smallmatrix}\text{cat. of}\\ \text{abelian}\\ \text{groups}\end{smallmatrix}\right)"] \\ KK^G \arrow[ur, "K^G_{top}(\cdot)"] \arrow[dr, "K(\cdot \rtimes_{red} G)"'] & \\ & (Ab) \end{tikzcd}$$

Two basic observations

$$\left.\begin{array}{l} A \text{ proper} \\ G\text{-algebra} \end{array}\right) \Rightarrow \left(K^G_{top}(A) = K(A \rtimes G) \right)$$

(Kasparov)

(f. els. $C_0(G/H)$
H compact)

$$A \in CC \implies K^G_{top}(A) = 0$$

(Chabert, Echterhoff
Oyono - Oyono)

This suggests following. Form "quotient" KK^G/CC. Then we get

$$\begin{tikzcd}
KK^G/CC \arrow[drr, "K^G_{top}"] & & \\
\{\text{proper actions}\} \arrow[u] \arrow[rr, "="] & & (Ab) \\
KK^G \arrow[u, phantom, "\bigcap"] \arrow[urr, "K(\cdot \rtimes_{red} G)"'] & &
\end{tikzcd}$$

Abstract nonsense

KK^G is not abelian, but has enough structure to form "quotient"

KK^G is triangulated:

- Has suspension (Bott periodicity)
- Closed under direct sums
- Has exact triangles (of the form:

$$\begin{tikzcd} A \arrow[rr, "\phi"] & & B \arrow[dl, "{[1]}"] \\ & C_\phi \arrow[ul] & \end{tikzcd}$$

$[1] \leftarrow$ suspension

(Cone of ϕ:

$\{(f, a) \mid f \in C_0(]0,1], B),\ f(1) = \phi(a$

+ some axioms (Verdier)

(Basic: any KK^G-element can be represented by a $*$-homomorphism)

This allowes to form quotient category by defining

$(KK^G/_{CC})(A, B) \ni \Phi$ iff

Φ is of the form

$$\begin{tikzcd} & C \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ A & & B \end{tikzcd}$$

$\alpha \in KK^G(C, B)$ and $C_\beta \in CC$

Then homological algebra says that for any homological functor F (like $K(\cdot \rtimes_{red} G)$) on KK^G there exists left-derived functor $\mathbb{L}F$ on $KK^G/_{CC}$ and a natural transformation of functors

$$\mathbb{L}F \longrightarrow F .$$

Need better description.

- $A \in KK^G$ is projective if $\forall B \in CC$, $KK^G(A, B) = 0$

  $\mathcal{P}$ – the set of projectives

- $\exists$ projective approximation, i.e. $\forall A \in KK^G$ $\exists D_A \in KK^G(P_A, A)$ with P_A projective and $\forall Q \in \mathcal{P}$

  $$KK^G(Q, P_A) \xrightarrow{\cdot D} KK^G(Q, A)$$

  is an isomorphism.

- Any proper G-algebra is projective

- For any homological F

  $$\mathbb{L}F(A) = F(P_A) \xrightarrow{F(D)} F(A)$$

<u>Main points</u> $\quad H \subset G$ compact

$KK^G \to KK^H$ restriction

$KK^H \to KK^G$ induction

are adjoint functors, i.e.

$$KK^G(\mathrm{Ind}_H^G A, B) \simeq KK^H(A, B)$$

—— . ——

Let $\langle CI \rangle$ be the subcategory of KK^G (triangulated) generated by compactly induced actions.

—— . ——

- Any element of $\langle CI \rangle$ is projective
- $\left.\begin{array}{l} A \in \langle CI \rangle \\ B \in KK^G \end{array}\right\} \Longrightarrow A \otimes B \in \mathcal{P}$

—— . ——

Hence it is enough to construct projective approximation $P \in \langle CI \rangle$

DIRAC $\quad D \in KK^G(P, \mathbb{C})$ for $\mathbb{C}$.

Then

$$\begin{tikzcd}
K^G_{top}(A) = K((\mathcal{P}\otimes A)\rtimes G) \arrow[dd, dashed, "\text{Assembly map}"'] \arrow[d, "D\otimes 1"] \\
K(A\rtimes G) \arrow[d] \\
K(A\rtimes_{red} G)
\end{tikzcd}$$

<u>Dual Dirac</u> : $\partial \in KK^G(\mathbb{C}, \mathcal{P})$

$$\partial \circ D = 1_{\mathcal{P}}$$

<u>γ-element</u> $\gamma = D \circ \partial$

If γ-element exists KK^G splits.

If $\gamma = 1$, $KK^G = \mathbb{P}$ ($CC = 0$!)

In this case $\mathbb{L}F = F$ for any homological functor on KK^G (K-homology of crossed product, local cyclic cohomology, ...)

$\boxed{\text{Computability of } \mathbb{L}F}$ spectral sequence

II. G locally compact quantum group

Need "strong regularity".
joint w/ R. Meyer, S. Vaes

Basic O.K.

<u>Thm.</u> KK^G is a triangulated category.

PROBLEM – no obvious analogue of CC

– one can choose the same as before, replacing compact subgroups by compact quantum subgroups

– On the other hand, there exist "homogeneous proper actions"

We will choose first possibility. In this case the constructions work.

Two "differences"

- Tensor products of G-algebras.

For G not cocommutative, $A, B \in KK^G$, $A \otimes B$ has no G-action.

However, there is a natural replacement.

Take example

$G = C(H) \quad \hat{G} = C^*(H)$

with H a locally compact group. Then $\hat{G}$ acts on G via conjugation.

Suppose A, B are G-algebras and A has a $\hat{G}$-coaction compatible with $\hat{G}$ action on G

In the example. $G = C(H)$

G- action is the same as ~~A~~ grading on $A = \bigoplus_g A_g \quad A_g A_h \subseteq A_{gh}$

compatible $\hat{G}$ coaction is $\alpha : H \longrightarrow \mathrm{Aut}(A)$

satisfying
$$\alpha_g : A_h \longrightarrow A_{ghg^{-1}}$$

<u>Definition</u>

$A \otimes_s B$ is the (C^* - algebra) generated by $a \in A$, $b \in B$ satisfying the relation

$$b_g \, a = \alpha_g(a) \, b$$

(One can construct both the universal and minimal version)

$$\Delta : a_g b_h \longrightarrow a_g b_h \otimes g^h$$

is a "tensor product"

action of G
$$\Delta : A \otimes_s B \rightarrow A \otimes_s B \otimes \hat{G}$$

For general quantum groups conjugation action of $\hat{G}$ on G still makes sense.

<u>Observation</u> For $A \in KK^G$, $A \rtimes G \rtimes \hat{G}$ is a G-algebra with a compatible $\hat{G}$ coaction

Assume that G is strongly regular, i.e.

$$A \rtimes G \rtimes \hat{G} \simeq A \otimes K(L^2(G))$$

Then

- twisted tensor products exist in KK^G
- they satisfy the same functoriality properties as in the cocommutative case

"THEOREM"

Assume that G is discrete, hence $G \subset \mathcal{K}(L^2(G))$

Then the Dirac element exists.

- Proof is "by hand" by constructing explicitly both $\mathcal{P}$ and $D \in KK^G(\mathcal{P}, \mathbb{C})$
- $\mathcal{P}$ comes with simplicial structure, hence one can write a spectral sequence computing

$$K_*((A \otimes_s \mathcal{P}) \rtimes G)$$

- This gives both

$$K^G_{top}(A) := K_*((A \otimes_s \mathcal{P}) \rtimes G)$$

and the assembly map

$$K^G_{top}(A) \longrightarrow K_*(A \rtimes G)$$

"_"

referes to the choice of the category CC based on compact quantum subgroups

Example again

$G = C(H)$ H compact, locally connected

Then proper homogeneous actions of G coincide with $C^*(K)$ K open in H

Hence $C^*(H_0)$ is the minimal proper homogeneous action; coming from a subgroup $C(H/H_0)$ both choices for CC are equivalent.

In this case via Baaj – Skandalis duality

(1) Baum – Connes conjecture holds for $G = C(H)$

$$\Updownarrow$$

$\forall A \in KK^{H_0}$

$$K_*(A \rtimes H_0) = 0 \Rightarrow K_*(A) = 0$$

(2) This is known for Lie groups.

(3) There exists a spectral sequence converging to $K_*(A \rtimes G)$ with

$$E_2 \simeq H(BG, K_*(A))$$

classifying space of the representation ring of H.