

# I. A Functorial approach to assembly map (joint work R. Meyer)

$G$  - locally compact, s.c.

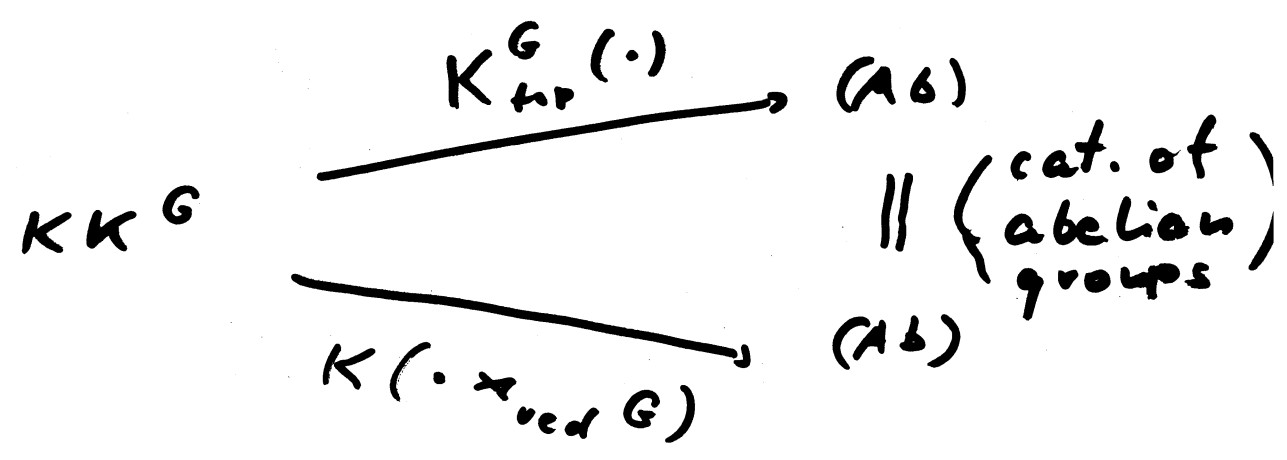
$KK^G$  - the category of separable  $G$ - $C^*$  algebras, with

$$Mor(A, B) := KK^G(A, B)$$

$CC$  - the full subcategory of compactly contractible algebras

$$A \in CC \iff \begin{cases} A = 0 & \text{in } KK^H \\ \text{for every compact} \\ \text{subgroup } H \subset G \end{cases}$$

$BC$  - relating two functors:

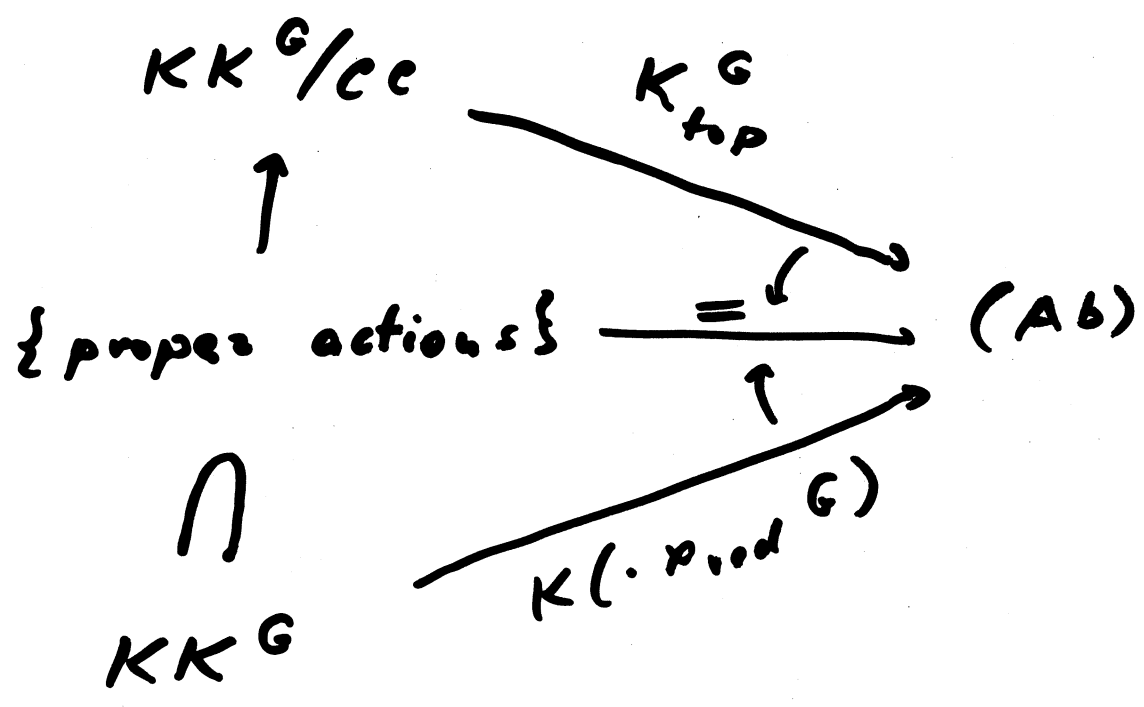


# Two basic observations

$$\begin{aligned}
 & \left( \begin{array}{l} A \text{ proper} \\ G\text{-algebra} \end{array} \right) \Rightarrow \left( K_{top}^G(A) = K(A \rtimes G) \right) \\
 & \left( \begin{array}{l} \text{f. els. } C_0(G/H) \\ H \text{ compact} \end{array} \right) \qquad \qquad \qquad \text{(Kasparov)}
 \end{aligned}$$

$$\begin{aligned}
 A \in CC & \Rightarrow K_{top}^G(A) = 0 \\
 & \text{(Chabert, Echterhoff)} \\
 & \text{Oyano - Oyano}
 \end{aligned}$$

This suggests following. Form "quotient"  $KK^G/CC$ . Then we get

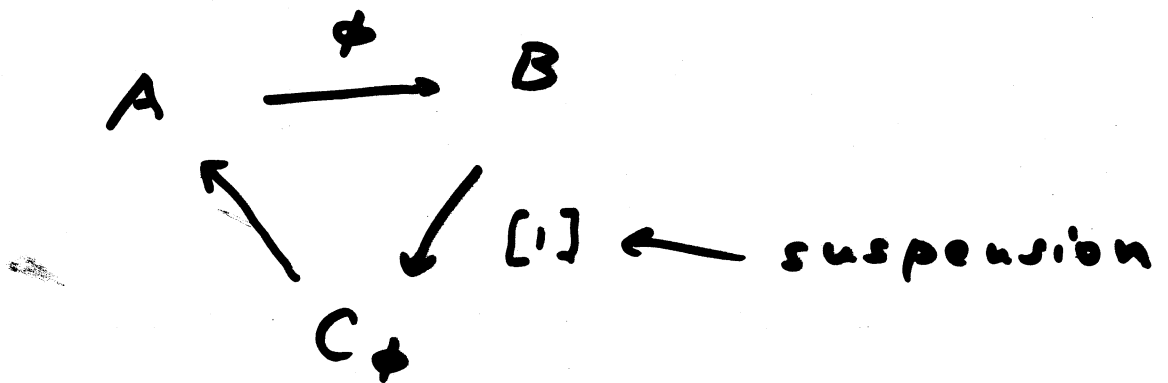


## Abstract nonsense

$KKG$  is not abelian, but has enough structure to form "quotient"

$KKG$  is triangulated:

- Has suspension (Bott periodicity)
- Closed under direct sums
- Has exact triangles (of the form:



(Cone of  $\phi$ :

$$\{ (f, a) \mid f \in C_0([0, 1], B), f(1) = \phi a \}$$

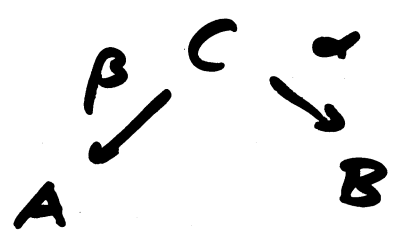
+ some axioms (Verdier)

(Basic: any  $KKG$ -element can be represented by a  $*$ -homomorphism)

This allows to form quotient category by defining

$$(KK^G / CC)(A, B) \ni \mathbb{F} \text{ iff}$$

$\mathbb{F}$  is of the form



$$\alpha \in KK^G(C, B) \text{ and } C, \beta \in CC$$


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Then homological algebra says that for any homological functor  $F$  (like  $K(\cdot \text{ mod } G)$ ) on  $KK^G$  there exists left-derived functor  $\mathbb{L}F$  on  $KK^G / CC$  and a natural transformation of functors  $\mathbb{L}F \rightarrow F$ .

Need better description.

- $A \in KK^G$  is projective if  
 $\forall B \in CC, KK^G(A, B) = 0$   
 $\mathcal{P}$  - the set of projectives

- $\exists$  projective approximation, i.e.

$\forall A \in KK^G \exists P_A \in \mathcal{P}$  projective and  
 $D_A \in KK^G(P_A, A)$

$\forall Q \in \mathcal{P}$

$KK^G(Q, P_A) \xrightarrow{D_A} KK^G(Q, A)$

is an isomorphism.

- Any proper  $G$ -algebra is projective

- For any homological  $F$

$\ll F(A) = F(P_A) \xrightarrow{F(D_A)} F(A)$

Main points

$H \subset G$  compact

$KK^G \rightarrow KK^H$  restriction

$KK^H \rightarrow KK^G$  induction

are adjoint functors, i.e.

$KK^G(\text{Ind}_H^G A, B) \cong KK^H(A, B)$

Let  $\langle CI \rangle$  be the subcategory of  $KK^G$  (triangulated) generated by compactly induced actions.

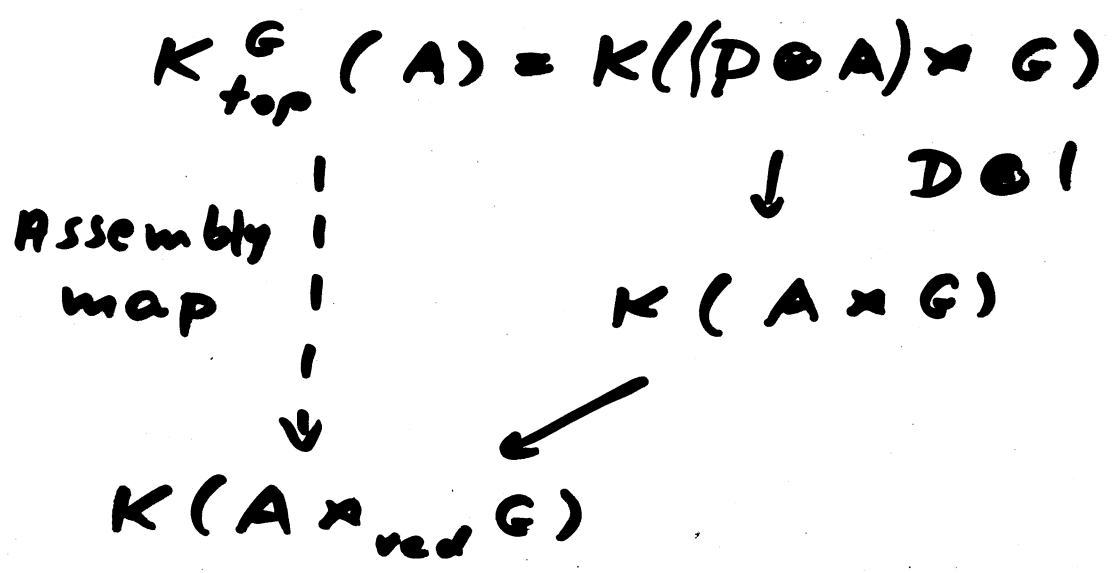
• Any element of  $\langle CI \rangle$  is projective

$\left. \begin{matrix} A \in \langle CI \rangle \\ B \in KK^G \end{matrix} \right\} \Rightarrow A \otimes B \in \mathcal{P}$

Hence it is enough to construct projective approximation  $P \in \langle CI \rangle$ .

DIRAC  $D \in KK^G(\mathcal{P}, \mathcal{C})$  for  $\mathcal{C}$ .

Then



Dual Dirac :  $\partial \in KK^G(\mathbb{C}, \mathcal{T})$

$$\partial \circ D = \mathbb{1}_P$$

$\gamma$ -element  $\gamma = D \circ \partial$

If  $\gamma$ -element exists  $KK^G$  splits.

If  $\gamma = 1$ ,  $KK^G = \mathcal{P}$  ( $\mathbb{C} \otimes 0!$ )

In this case  $\llcorner F = F$  for any homological functor on  $KK^G$

( $K$ -homology of crossed products, local cyclic cohomology, ...)

Computability of  $\llcorner F$  spectral sequence

# II. G locally compact quantum group

Need "strong regularity".

joint w/ R. Meyer, S. Vaes

Basic O.K.

Thm.  $KK^G$  is a triangulated category.

PROBLEM - no obvious analogue of CC

- one can choose the same as before, replacing compact subgroups by compact quantum subgroups
- On the other hand, there exist "homogeneous proper actions"



We will choose first possibility.  
In this case the constructions work.

Two "differences"

- Tensor products of  $G$ -algebras.

For  $G$  not cocommutative,

$A, B \in KK^G$ ,  $A \otimes B$  has no  $G$ -action.

However, there is a natural replacement.

Take example

$$G = C(H) \quad \hat{G} = C^*(H)$$

with  $H$  a locally compact group. Then  $\hat{G}$  acts on  $G$  via conjugation.

Suppose  $A, B$  are  $G$ -algebras and  $A$  has a  $\hat{G}$ -coaction compatible with  $\hat{G}$  action on  $G$

In the example.  $G = C(H)$

(10)

$G$ -action is the same as  $\hat{A}$ -grading

on  $A = \bigoplus_g A_g$ ,  $A_g A_h \subseteq A_{gh}$

compatible  $\hat{G}$  coaction is

$$\alpha : H \rightarrow \text{Aut}(A)$$

satisfying

$$\alpha_g : A_h \rightarrow A_{ghg^{-1}}$$

### Definition

$A \otimes_g B$  is the ( $C^*$ -algebra)

generated by  $a \in A$ ,  $b \in B$

satisfying the relation

$$b_g a = \alpha_g(a) b$$

(One can construct both the universal and minimal version)

$$\Delta : a_g b_h \rightarrow a_g b_h \otimes gh$$

is a "tensor product"

action of  $G$

$$\Delta : A \otimes_g B \rightarrow A \otimes_g B \otimes \hat{G}$$

(2)

For general quantum groups  
conjugation action of  $\hat{G}$  on  
 $G$  still makes sense.

Observation For  $A \in KK^G$ ,

$A \rtimes G \rtimes \hat{G}$  is a  $G$ -algebra  
with a compatible  $\hat{G}$  coaction

Assume that  $G$  is strongly  
regular, i.e.

$$A \rtimes G \rtimes \hat{G} \simeq A \otimes K(L^2(G))$$

Then

- twisted tensor products  
exist in  $KK^G$
- they satisfy the same  
functoriality properties  
as in the commutative  
case

"THEOREM"

Assume that  $G$  is discrete,  
hence  $G \subset K(L^2(G))$

Then the Dirac element  
exists.

• Proof is "by hand" by  
constructing explicitly both  
 $P$  and  $D \in KK^G(P, \mathbb{C})$

•  $P$  comes with simplicial  
structure, hence one can  
write a spectral sequence  
computing

$$K_+(A \otimes_s P) \rtimes G$$

• This gives both

$$K_{top}^G(A) := K_+(A \otimes_s P) \rtimes G$$

and the assembly map

$$K_{top}^G(A) \rightarrow K_+(A \rtimes G)$$

"\_"

refers to the choice of the category  $\mathcal{C}$  based on compact quantum subgroups

Example again

$G = C(H)$   $H$  compact, locally connected

Then proper homogeneous actions of  $G$  coincide with

$C^*(K)$   $K$  open in  $H$

Hence  $C^*(H_0)$  is the minimal proper homogeneous action; coming from

a subgroup  $C(H/H_0)$  both

choices for  $\mathcal{C}$  are equivalent.

In this case  
via Baum-Skandalis duality

①

Baum-Connes conjecture holds

for  $G = C(H)$



$\forall A \in KK^0$

$$K_+(A \rtimes H_0) = 0 \Rightarrow K_+(A) = 0$$

② This is known for Lie groups.

③ There exists a spectral sequence converging to  $K_+(A \rtimes G)$  with

$$E_2 \cong H(BG, K_+(A))$$

classifying space of the representation ring of  $H$ .