Flows on a separable C^*-algebra

A. Kishimoto

Motivations

- Unbounded derivations
  (as a continuation of the theory of derivations)
- Applications to Physics
  (KMS states, ground states, --)
- Actions of $\mathbb{R}$
  (instead of locally compact abelian groups)

Why flows ?

easy to construct
easy to perturb

References

- G. K. Pedersen "C^*-algebras and their automorphism groups" 1979
- O. Bratteli, D. W. Robinson "Operator Algebras and Quantum Statistical Mechanics" I, II, 1979, 1981
- S. Sakai "Operator Algebras in Dynamical Systems" 1991

I mostly study flows in the spirit of Pedersen's book.

Perturbations

Let α be a flow on A,

$$\left(\begin{array}{ll} t \in \mathbb{R} \to \alpha_t \in \mathrm{Aut}(A) & \text{homomorphism} \\ t \mapsto \alpha_t(x) \text{ cont. for } x \in A & \end{array} \right)$$

Let δ_α be the generator of α : $\alpha_t = e^{t\delta_\alpha}$

- For $h = h^* \in A$, $\alpha^{(h)}$ denotes the flow generated by $\delta_\alpha + \mathrm{ad}\, ih$, called an <u>inner perturbation</u> of α.

- Let u be an α-cocycle

$$u : \mathbb{R} \longrightarrow \mathcal{U}(A) \quad (\text{or } \mathcal{U}(M(A)))$$

$$u_s \alpha_s(u_t) = u_{s+t}$$

Then $t \mapsto \mathrm{Ad}\, u_t \alpha_t$ is a flow, called a <u>cocycle perturbation</u> of α

Suppose $A \ni 1$.

If the α-cocycle u is differentible and $i \cdot h = \left.\frac{d u_t}{dt}\right|_{t=0}$, then

$$\mathrm{Ad}\, u\, \alpha = \alpha^{(h)}$$

In general the α-cocycle u is given as

$$u_t = w\, v_t\, \alpha_t(w^*)$$

where $w \in \mathcal{U}(A)$

v : differentible α-cocycle

Hence $\mathrm{Ad}\, u\, \alpha$ is of the form

$$\mathrm{Ad}\, w \circ \alpha^{(h)} \circ \mathrm{Ad}\, w^*$$

Hence in the unital case there is not much difference between inner perturbations and cocycle perturbations.

But in the non-unital case I do not know.

(In this case $t \mapsto u_t$ is cont. in the strict topology.)

- In general we want to study properties which are invariant under cocycle perturbations.

Properties for flows

- AI (= approximately inner)
- Rohlin
- Spectra

  e.g. Connes spectrum
- KMS states (as a bundle over the set of temperatures)

....

Remark

- I do not know if AI is invariant under cocycle perturbations
- KMS states should be replaced by KMS positive functionals (to make it invariant under cocycle perturbations)

Covariant Representations

(A, α)

(π, U) $\qquad U_t \pi(x) U_t^* = \pi\alpha_t(x), \quad x \in A$

Problems :

Are there many covariant representation?

What is the Connes spectrum for $\bar{\alpha}$:

$\bar{\alpha}_t = \mathrm{Ad}\, U_t$ on $\pi(A)''$?

--- This can be answered by adapting Glimm's theorem for antiliminary C^*-algebras.

Theorem

A : seprable prime (antiliminary)

α : flow on A, the Connes spectrum $\mathbb{R}(\alpha) \neq 0$.

Then the following are equivalent:

(1) $\exists$ faithful family of cov. irreducible rep.

(2) $\exists$ faithful cov. irreducible rep (π, U)

$\ker(\pi \times U)$: invariant under $\hat{\alpha} | \mathbb{R}(\alpha)$

(3) $\forall D = \bigotimes_{n=1}^{\infty} M_{k_n}$ $\quad \forall \varepsilon > 0$

$\forall \gamma_t = \otimes \mathrm{Ad}\, e^{ith_n}$ $\quad Sp(\gamma) \subset \mathbb{R}(\alpha)$

$\exists B \subset A$, $h \in A_{sa}$, $q \in A^{**}$ closed

$\|h\| < \varepsilon$

$\alpha_t^{(h)}(B) = B \qquad \alpha_t^{(h)}(q) = q$

$qAq = Bq$

$(Bq, \alpha^{(h)} | Bq) \simeq (D, \gamma)$

$\forall x \in A \quad x\, c(q) = 0 \Rightarrow x = 0$

(where $c(q)$ the central support of q in A^{**})

Note: For compact group actions, a similar result is obtained without perturbations (Bratteli-Robinson-K)

Problem Can we generalize this to actions of $\mathbb{R}^2$?

So the problem on covariant representations reduces to the one on the UHF flow γ, which is easy.

Note in the above situation
$\forall \varepsilon > 0$, $\exists\, h = h^* \in A$, $\|h\| < \varepsilon$
the set of $\alpha^{(h)}$-invariant states include the state space of a UHF algebra.

Note the condition
$\exists$ faithful family of cov. irreducible representations
seems to follow or is required below.

- $\forall \pi_1, \pi_2$: irreducible rep of separable A s.t. $\mathrm{Ker}\, \pi_1 = \mathrm{Ker}\, \pi_2$

  $\exists \gamma$: (asymptotically inner) automorphism of A

  $\pi_1 \gamma \sim \pi_2$. (due to Ozawa, Sakai, K)

Problem Can we generalize it to (A, α)

Symmetry group for α

$$G_\alpha = \{ \gamma \in \mathrm{Aut}\, A \mid \gamma \alpha \gamma^{-1} : \text{cocycle perturb. of } \alpha \}$$

Theorem

A separable prime C^*-alg

α flow $R(\alpha) \neq 0$

$\forall\ (\pi_1, U_1), (\pi_2, U)$ cov. irr. rep

$\mathrm{Ker}\ \pi_1 \times U_1 = \mathrm{Ker}\ \pi_2 \times U_2$

$\Rightarrow$ $\gamma \in \mathrm{Aut}(A)$

$\gamma\alpha\gamma^{-1}$: cocycle perturbation of α

$\pi_1 \gamma \sim \pi_2$.

Remark The above γ extends to an asymptotically inner aut. $\bar{\gamma}$ of $A \rtimes_{\alpha} \mathbb{R}$ s.t. $(\pi_1 \times U_1)\bar{\gamma} \sim \pi_2 \times U_2$.

Remark : If π is the GNS rep. associated with a KMS state, then $\pi\gamma \simeq \pi$

[Fannes, Vanheuverzwijn, Verbeure]

AI (= approximately inner)

There is no good characterization except for the definition.

- If A is unital and has a tracial state, then
  $\exists$ KMS states for all temperatures.

- (π, U) cov. rep on a separable H_π
  $\pi(A)''$ type I
  $\exists\,(h_n)$ $\quad \operatorname{Ad} e^{ith_n}(x) \to \alpha_t(x), \quad x \in A$
  $$\pi(e^{ith_n}) \to U_t \quad \text{strongly}$$

Theorem

A : separable antiliminary

$\exists\, \alpha$: AI flow, $\mathbb{R}(\alpha) = \mathbb{R}$.

Remark "The Connes spectrum $\mathbb{R}(\alpha) = \mathbb{R}$" can be replaced by

$\forall\, \mathcal{O} \subset \mathbb{R}$ non-empty open

$\exists\, (z_n)$ in $A^{\alpha}(\mathcal{O})$ central sequence
non-trivial in the sense
$\lim_n \| x z_n \| = 0 \Rightarrow x = 0 \quad (\forall x \in A)$

Problem

Are there many cocycle conjugacy classes of flows on A ?

Rohlin flows

<u>Definition</u> α has the Rohlin property if $\forall\, p \in \mathbb{R}$ $\exists\, (u_n)$ in $\mathcal{U}(M(A))$

$$\| \alpha_t(u_n) - e^{ipt} u_n \| \to 0$$

$$\| [u_n, x] \| \to 0 , \quad x \in A$$

Example

$$A_\theta = C^*(u, v), \quad \text{where } uv = e^{2\pi i \theta} vu$$

$$\theta \notin \mathbb{Q}$$

$$\alpha_t(u) = e^{2\pi i t} u$$

$$\alpha_t(v) = e^{2\pi i p t} v$$

$\Rightarrow$ If ~~$p \notin \mathbb{Q}$~~, α has the Rohlin property.
$1, p$ are linearly independent over $\mathbb{Z} + \theta\mathbb{Z}$

$\rightsquigarrow$ generalized to simple AT algebras of RR0 (thanks to Elliott's classification th.)

Lemma : Let α be a Rohlin flow on a unital C^*-algebra A and $L > 0$.
Let γ denote the flow on $C(\mathbb{R}/L\mathbb{Z})$ by translations. Then

$$\exists (\phi_n) : A \otimes C(\mathbb{R}/L\mathbb{Z}) \to A$$

approximate homomorphism

$$\phi_n(a \otimes 1) = a, \quad a \in A$$

$$\| \phi_\gamma \circ \alpha_t \otimes \gamma_t (x) - \alpha_t \phi_n(x) \| \to 0$$

for $x \in A \otimes C(\mathbb{R}/L\mathbb{Z})$

Proof Let (u_n) in $\mathcal{U}(A)$ be such that

$$\| [u_n, a] \| \to 0$$

$$\| \alpha_t(u_n) - e^{i\frac{t}{L}} u_n \| \to 0$$

Then ϕ_n is defined by

$$\phi_n(a \otimes z) = a u_n, \quad a \in A$$

Theorem:

If α has the Rohlin property (for a unital A), then an α-cocycle is almost a coboundary

$\ell^\infty(A)$ bounded sequences in A

$C_0(A)$ " converging to 0

$A^\infty = \ell^\infty(A)/C_0(A)$

$\ell^\infty_\alpha(A)$: α-continuous part of $\ell^\infty(A)$

$A^\infty_\alpha = \ell^\infty_\alpha(A)/C_0(A)$

More precisely

If $u : \mathbb{R} \to \mathcal{U}(A' \cap A^\infty_\alpha)$ is an α-cocycle

then $\exists\, v \in \mathcal{U}(A' \cap A^\infty_\alpha)$

$$u_t = v\,\alpha_t(v^*).$$

Prop. Let A be a separable purely infinite simple C^*-algebra and α a Rohlin flow on A. Then for any $\varepsilon > 0$

$\exists\, h = h^* \in A$

$\exists\, \varphi$ pure state of A

$\exists\, (e_n)$ decreasing sequence of projections in A

$\alpha_t^{(h)}(e_n) = e_n$

$\lim e_n$ = the support projection of φ

Prop. Let A be a separable purely infinite simple C^*-algebra and α a Rohlin flow on A. Then $A \times_\alpha \mathbb{R}$ is purely infinite and simple.

Proof

A simple $\Rightarrow$ $A \times_\alpha \mathbb{R}$ simple

because $\hat{\alpha}_p$ is approximately inner

so any ideal of $A \times_\alpha \mathbb{R}$

is $\hat{\alpha}$-invariant

To show "purely infinite" we mimic the proof for the case of Cuntz algebra

$\mathcal{O}_n = C^*(s_1, \cdots, s_n)$ Cuntz algebra

$$\sum_{i=1}^{n} s_i s_i^* = 1$$

$$s_i^* s_i = 1.$$

For $(p_1, \cdots, p_n)$ in $\mathbb{R}$ define a flow α on $\mathcal{O}_n$ by

$$\alpha_t(s_i) = e^{ip_i t} s_i, \quad i=1, \cdots, n.$$

Then

$\mathcal{O}_n$: separable purely infinite simple nuclear

α : non AI flow for $(p_1, \cdots, p_n) \neq 0$

Theorem

Let α be the quasi-free flow on $\mathcal{O}_n$ corresponding to $p_1, p_2, \cdots, p_n$. Then the following are equivalent.

(1) α has the Rohlin property.

(2) $\mathcal{O}_n \times_\alpha \mathbb{R}$ is purely infinite & simple

(3) $p_1, \cdots, p_n$ generate $\mathbb{R}$ as a closed subsemigroup.

In this case all such α are cocycle conjugate with each other.

i.e., if α and β are as above,

$\exists \phi \in \mathrm{Aut}\, \mathcal{O}_n$, $\exists u$: α-cocycle

$\mathrm{Ad}\, u\, \alpha = \phi \beta \phi^{-1}$

Proof (3) ⇒ (1)

- For $p \in \mathbb{R}$, find $u \in \mathcal{U}(\mathcal{O}_n)$ of the form $\sum_i S_{I_i} S_{J_i}^*$, where I_i, J_i are finite sequences in $\{1, 2, \cdots, n\}$ s.t.

$$\alpha_t(u) \simeq e^{ipt} u$$

- $\exists (v_k)$ in $\mathcal{U}(\mathcal{O}_n)$

$$\alpha_t(v_k) = v_k$$

$$(v_k S_i)_k \ : \ \text{central sequence}$$

(Thus (v_k) must satisfy:

$$v_k \lambda(v_k^*) \simeq \sum_{i,j=1}^{n} S_i S_j S_i^* S_j^*$$

where $\lambda(\cdot) = \sum_{i=1}^{n} S_i \cdot S_i^*$)

Define an endomorphism ϕ_k of $\mathcal{O}_n$ by

$$\phi_k(S_i) = v_k S_i, \qquad i=1,\dots,n.$$

Then $\phi_k \alpha_t = \alpha_t \phi_k$

and for the unitary u with $\alpha_t(u) \simeq e^{itp} u$,

$\phi_k(u)$ satisfies

$$\alpha_t(\phi_k(u)) \simeq e^{ipt} \phi_k(u)$$

$(\phi_k(u))$ central

Note :

$$U = \sum_{i,j=1}^{n} S_i S_j^* S_i^* S_j^* \in M_n \otimes M_n \subset \bigotimes_{\mathbb{N}} M_n \subset \mathcal{O}_n$$

$\lambda \upharpoonright \bigotimes_{\mathbb{N}} M_n$: one-sided shift

To show that there is such a (v_k)

with $\| v_k \lambda(v_k^*) - U \| \to 0$

we show

<u>Lemma</u> $B = \bigotimes_{\mathbb{Z}} M_n$

$$\gamma : \mathbb{T}^{n-1} \to \mathrm{Aut}(B)$$

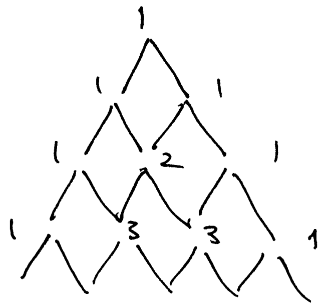
$$z \mapsto \bigotimes_{\mathbb{Z}} \mathrm{Ad}\begin{pmatrix} 1 & & & 0 \\ & z_1 & & \\ & & \ddots & \\ 0 & & & z_{n-1} \end{pmatrix}$$

$A = B^{\gamma}$

σ : shift on B

$\Rightarrow$ $\sigma | A$ has the one-cocycle property.

A is an AF algebra, whose Bratteli diagram looks like Pascal's triangle for $n=2$ and a higher dimensional version for a general n.



A has n characters $\phi_1, \dots, \phi_n$.

"$\sigma | A$ has the one-cocycle property"

means

$\forall\ u = (u_n) \in B' \cap A^\omega$ unitary

$\phi_i(u) = \lim \phi_i(u_n) = 1$

$\exists\ v \in \mathcal{U}(B' \cap A^\omega)$

$u = v\sigma(v^*)$.

The shift σ on $B = \bigotimes_{\mathbb{Z}} M_n$ has the one-cocycle property (or Rohlin property) [Bratteli - Rørdam - Størmer - K]. Since B has a unique tracial state, some qualitative arguments are enough to conclude this, based on the result for type II_1 factors due to Connes (which yield an "approximate Rohlin property"). Since the factorial tracial state space of A is a $(n-1)$-dim. simplex, some qualitative arguments are required to prove "the one-cocycle property for $\sigma|A$".

For $n = 2$

this is obtained from the CAR algebra formalism (like the BRSK arguments for $\bigotimes_{\mathbb{Z}} M_2$)

For $n > 2$ By Induction

$$\bigotimes_{1}^{L} M_n \hookrightarrow \bigotimes_{\mathbb{Z}} M_{n+1}$$

For a free ultrafilter ω on $\mathbb{N}$

$$C_\omega(A) = \{ x \in \ell^\infty(A) \mid \lim_\omega \|x_n\| = 0 \}$$

$$A^\omega = \ell^\infty(A) / C_\omega(A)$$

$$A^\omega_\alpha = \ell^\infty_\alpha(A) / C_\omega(A)$$

Then $A' \cap A^\omega$ purely infinite and simple

if A is unital separable nuclear & purely infinite simple

Based on this result and classification result due to Kirchberg and Phillips

we show :

Theorem

A : unital sep. nuc. purely inf. simple C^*-algebra satisfying UCT.

α : flow on A.

Then the following are equivalent:

(1) α has the Rohlin property

(2) $(A' \cap A^\omega_\alpha)^\alpha$ purely infinite simple

$K_0((A' \cap A^\omega_\alpha)^\alpha) \cong K_0(A' \cap A^\omega)$

$Sp(\alpha \mid A' \cap A^\omega_\alpha) = \mathbb{R}$

(3) $A \times_\alpha \mathbb{R}$ purely infinite and simple

$\hat{\alpha}$: Rohlin flow

(4) $A \times_\alpha \mathbb{R}$ purely infinite and simple

Each α_t is α-invariantly approximately inner

$$\left(\begin{array}{l} \forall t \in \mathbb{R}, \quad \exists (u_n) \text{ in } \mathcal{U}(A) \\ \alpha_t = \lim \operatorname{Ad} u_n \\ \| \alpha_s(u_n) - u_n \| \to 0 \end{array} \right)$$

In this case

$$K_1((A' \cap A^\omega_\alpha)^\alpha) \cong K_1(A' \cap A^\omega)$$

also follows.

Proof

$$(1) \Rightarrow (4) \Rightarrow (3) \xRightarrow{\text{Takesaki's duality}} (1)$$

and

$$(1) \Leftrightarrow (2)$$

Problem

Is "$A \times_\alpha \mathbb{R}$ purely inf & simple" enough to conclude the Rohlin property ?

(1) $\Rightarrow$ (4)

Fix t

$\exists (u_n)$ $\quad \alpha_t = \lim \mathrm{Ad}\, u_n$

$\left(s \mapsto u_n^* \alpha_s(u_n) \right)$: central seq of α-cocycles

$\exists (v_n) \in A' \cap A^\omega$

$$\| u_n^* \alpha_s(u_n) - v_n \alpha_s(v_n^*) \| \to 0$$

Then

$$\alpha_t = \lim \mathrm{Ad}\,(u_n v_n)$$

$$\| \alpha_s(u_n v_n) - u_n v_n \| \to 0$$

(4) $\Rightarrow$ (3) $\quad$ If $w_n = u_n v_n$ as above

$(w_n^* \lambda(t))$ central in $A \rtimes_\alpha \mathbb{R}$

$$\hat{\alpha}_p (w_n^* \lambda(t)) = e^{ipt} w_n^* \lambda(t)$$

Suppose that α has the Rohlin property.

Proof of $K_0((A' \cap A^\omega_\alpha)^\alpha) \to K_0(A' \cap A^\omega)$ surjective

$\because$ $e \in A' \cap A^\omega$ proj.

$e = (e_n)$

$T \gg 0$ Define $E_n \in (A' \cap C[0,T] \otimes A)^\omega$ by

$$E_n(t) = \alpha_t(e_n)$$

- $\exists (U_n) \in A' \cap (C[0,T] \otimes A)^\omega$

$$\mathrm{Ad}\, U_n(t)\big(E_n(0)\big) = E_n(t)$$

- $\exists (V_n) \in A' \cap (C[0,T] \otimes A)^\omega$

$$V_n(0) = 1$$
$$V_n(T) = U_n(T)^*$$
$$\| V_n(s_2) - V_n(s_1) \| < C|s_2 - s_1| / T$$

(due to Nakamura based on K-P's

$\mathcal{O}_\infty \hookrightarrow A$)

Define $F_n \in C(\mathbb{R}/T\mathbb{Z}) \otimes A$ by

$$F_n(t) = \alpha_{t-T}(V_n(t))\,\alpha_t(e_n)\,\alpha_{t-T}(V_n(t)^*)$$
$$= \mathrm{Ad}\left(\alpha_{t-T}(V_n(t))\,U_n(t)\right)(e_n)$$
$$\left(\Rightarrow F_n(0) = e_n = F_n(T)\right)$$

If γ is the flow on $C(\mathbb{R}/T\mathbb{Z})$ by translations, then

$$\| \gamma_t \otimes \alpha_t(F_n) - F_n \| \leq 2C\,|t|/T$$

By using the approximate homomorphism (ϕ_k) of $C(\mathbb{R}/T\mathbb{Z}) \otimes A$ into A with $\phi_k \circ \gamma_t \otimes \alpha_t \simeq \alpha_t \circ \phi_k$, we define $f = \left(\phi_{k_n}(F_n)\right)$, which belongs to $(A' \cap A_\alpha^\omega)^\alpha$ and satisfies

$$f \sim e$$

Proof of $K_0((A' \cap A^\omega_\alpha)^\alpha) \to K_0(A' \cap A^\omega)$ injective

$e, f \in (A' \cap A^\omega_\alpha)^\alpha$ proj.

$e \sim f$ in $A' \cap A^\omega$

We may assume that

$e = (e_n)$ e_n proj

$f = (f_n)$ f_n proj.

$\delta_\alpha(e_n) = 0$

$\delta_\alpha(f_n) = 0$

$\exists\, u = (u_n) \in A' \cap A^\omega$

$u_n^* u_n = e_n$

$u_n u_n^* = f_n$

Let $z_n(t) = u_n^* \alpha_t(u_n) \in e_n A e_n$

z_n : $\alpha | e_n A e_n$ - cocycle

$\exists\, y_n \in \mathcal{U}(e_n A e_n)$

$\| z_n(t) - y_n \alpha_t(y_n^*) \| \to 0$

$(y_n) \in A' \cap A^\omega$

Then $(u_n y_n) \in (A' \cap A^\omega_\alpha)^\alpha$

and

$$(u_n y_n)^* (u_n y_n) = e_n$$

$$(u_n y_n)(u_n y_n)^* = f_n$$

i.e., $e = (e_n) \sim f = (f_n)$ in $(A' \cap A^\omega_\alpha)^\alpha$