PROPERTIES OF ALGEBRAS DEFINED BY CENTRAL SEQUENCES

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ABSTRACT:
We consider the quotient $A^c/\mathrm{Ann}(A)$ of the algebra $A^c := A' \cap A_\omega$ of (ω-)central sequences by the two-sided annihilator $\mathrm{Ann}(A) := \mathrm{Ann}(A, A_\omega)$ of A in in the ultrapower A_ω of a *separable* C*–algebra A. $A^c/\mathrm{Ann}(A)$ is always unital and is a stable invariant of A in the sense that $D^c/\mathrm{Ann}(D) \cong A^c/\mathrm{Ann}(A)$ if D is a full hereditary C*–subalgebra of A. Thus $\mathcal{K}^c/\mathrm{Ann}(\mathcal{K}) \cong \mathbb{C}$, but $\mathrm{Ann}(\mathcal{K})$ is huge. We discuss some properties of A if $A^c/\mathrm{Ann}(A)$ contains certain subalgebras.
We introduce an invariant $\mathrm{cov}(B) \in \mathbb{N} \cup \{\infty\}$ of unital C*–algebras B, (a non-commutative covering number) with the property $\mathrm{cov}(B) \leq \mathrm{cov}(C)$ if there is a unital *-homomorphism from C into B. It holds $\mathrm{cov}(\mathcal{O}_2) = \mathrm{cov}(\mathcal{O}_\infty) = 1$ and $\mathrm{cov}(C) = \infty$ for all Abelian C*–algebras C. This invariant satisfies $\mathrm{cov}(B) \leq \mathrm{dr}(B) + 1$ for the decomposition rank $\mathrm{dr}(B)$ of B if B is nuclear and has no finite-dimensional quotient. (In particular, $\mathrm{cov}(\mathcal{Z}) = 2$ for the Jian–Su algebra $\mathcal{Z}$, because $\mathrm{dr}(\mathcal{Z}) = 1$.) We get that A is strongly purely infinite if $A^c/\mathrm{Ann}(A)$ contains a simple C*–algebra B unitally with $\mathrm{cov}(B) < \infty$ and every lower semi-continuous 2-quasi-trace on A_+ takes only the values 0 and ∞. (In particular, $A \otimes \mathcal{Z}$ is strongly purely infinite if A is exact and A_+ admits no non-trivial lower semi-continuous trace.)

Some questions on the simple algebras contained in $A^c/\mathrm{Ann}(A)$ will be considered, *e.g.* What happens if $A^c/\mathrm{Ann}(A)$ itself is simple? The answer is:
If A is a separable unital C*–algebra and if the relative commutant $A^c := A' \cap A_\omega$ is simple, then either $A^c = \mathbb{C} \cdot 1_A$ and $A \cong M_n$, or A and A^c are both simple and purely infinite. In particular, $A \cong A \otimes \mathcal{O}_\infty$ if A^c is simple and $A^c \neq \mathbb{C} \cdot 1_A$. A version of this result for non-unital A is given if $A^c/\mathrm{Ann}(A)$ is simple.
The converse holds in the nuclear case: If A is simple, purely infinite, separable, nuclear and unital, then A^c is simple (and purely infinite).
$Q^c = \mathbb{C} \cdot 1$ for the Calkin algebra $Q := \mathcal{L}/\mathcal{K}$, in contrast to the separable case.

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