CENTRAL SEQUENCES IN C*–ALGEBRAS AND STRONGLY PURELY INFINITE ALGEBRAS

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Abstract. If A is a separable unital C*–algebra and if the relative commutant $A^c := A' \cap A_\omega$ is simple, then either $A^c = \mathbb{C} \cdot 1_A$ and $A \cong M_n$, or A and A^c are both simple and purely infinite. In particular, $A \cong A \otimes \mathcal{O}_\infty$ if A^c is simple and $A^c \neq \mathbb{C} \cdot 1_A$. A version of this result for non-unital A is given if $A^c/\mathrm{Ann}(A, A_\omega)$ is simple.

The converse holds in the nuclear case: If A is simple, purely infinite, separable, nuclear and unital, then A^c is simple (and purely infinite).

We show that $Q^c = \mathbb{C} \cdot 1$ for the Calkin algebra $Q := \mathcal{L}/\mathcal{K}$, in contrast to the separable case.

We introduce an invariant $\mathrm{cov}(B) \in \mathbb{N} \cup \{\infty\}$ of unital C*–algebras B with $\mathrm{cov}(B) \leq \mathrm{cov}(C)$ if there is a unital *-homomorphism from C into B.

If B is nuclear and has no finite-dimensional quotient then $\mathrm{cov}(B) \leq \mathrm{dr}(B) + 1$ for the decomposition rank $\mathrm{dr}(B)$ of B. In particular, $\mathrm{cov}(\mathcal{Z}) = 2$ for the Jian–Su algebra $\mathcal{Z}$, because $\mathrm{dr}(\mathcal{Z}) = 1$.

It is shown for (non-simple) separable C*–algebras A that A is strongly purely infinite in the sense of [12] if A does not admit a non-trivial lower semi-continuous 2-quasi-trace, $\mathrm{cov}(A^c/\mathrm{Ann}(A_\omega, A)) < \infty$ and if there is an image of $C^*((0,1], M_2)$ that generates a full hereditary C*-subalgebra of $A^c/\mathrm{Ann}(A_\omega, A))$.

It follows that A is strongly purely infinite if $A^c/\mathrm{Ann}(A)$ contains a simple C*–algebra B unitally such that $\mathrm{cov}(B) < \infty$. In particular, $A \otimes \mathcal{Z}$ is strongly purely infinite if A_+ admits no non-trivial lower semi-continuous 2-quasi-trace.

1. The case of simple $A^c/\mathrm{Ann}(A)$

We suppose that A is a *separable C*–algebra.* Let ω a *free* ultra-filter on $\mathbb{N}$. We also denote by ω the related *character* on $\ell_\infty(\mathbb{N})$ with $\omega(c_0(\mathbb{N})) = \{0\}$. Recall that $\lim_\omega \alpha_n$ means the complex number $\omega(\alpha_1, \alpha_2, \ldots)$ for $(\alpha_1, \alpha_2, \ldots) \in \ell_\infty(\mathbb{N})$. Then $A_\omega := \ell_\infty(A)/c_\omega(A)$ with $c_\omega(A) := \{(a_1, a_2, \ldots) \in \ell_\infty(A) ;, \lim_\omega \|a_n\| = 0\}$. The natural epimorphism from $\ell_\infty(A)$ onto A_ω is denoted by π_ω. Sometimes we say that $(a_1, a_2, \ldots) \in \ell_\infty(A)$ is a *representing sequence* for $b \in A_\omega$ if $\pi_\omega(a_1, a_2, \ldots) = b$. We consider A as a C*–subalgebra of A_ω by the diagonal embedding

$$a \mapsto \pi_\omega(a, a, \ldots) = (a, a, \ldots) + c_\omega(A),$$

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and let $A^c := A' \cap A_\omega$ the *algebra of (ω–) central sequences in A*. The (two-sided) *annihilator*

$$\mathrm{Ann}(A) := \mathrm{Ann}(A, A_\omega) := \{b \in A_\omega\,;\ bA = \{0\} = Ab\}$$

of A in A_ω is contained in A^c, but $\mathrm{Ann}(A)$ does not carry much information about A.

The below mentioned (or later needed) basic facts on A^c are proved in Section 3 (Appendix). $\mathrm{Ann}(A)$ is a closed ideal of A^c, and $A^c/\mathrm{Ann}(A)$ is a *unital* C*-algebra. $\mathrm{Ann}(A) = \{0\}$ if and only if A is unital. There is a natural *-homomorphism

$$\rho\colon (A^c/\mathrm{Ann}(A)) \otimes^{\max} A \to A_\omega$$

given by $\rho((d + \mathrm{Ann}(A)) \otimes b) := db$ for $d \in A^c$ and $b \in A$. It holds $\rho(1 \otimes b) = b$ for $b \in A$ (*cf.* (A.1)).

Let $\mathcal{K}$ denote the compact operators on $\ell_2(\mathbb{N})$. $\mathcal{K}^c$ is huge, but $\mathcal{K}^c/\mathrm{Ann}(\mathcal{K}) \cong \mathbb{C} = \mathbb{C}_\omega$. More generally, if p is a full projection of A then $A^c/\mathrm{Ann}(A)$ is naturally isomorphic to $(pAp)^c \subset (pAp)_\omega \cong p(A_\omega)p$ (*cf.* (A.1) of the Appendix.).

A is simple if $A^c/\mathrm{Ann}(A)$ is simple. A is simple and unital if A^c is simple, *cf.* (A.2). To get the main result Theorem 1.8 of this section, we have to improve here (in the case where A is simple) some of the general results on A^c and $A^c/\mathrm{Ann}(A)$ in the Appendix.

Remark 1.1. Let A a σ-unital C*–algebra. The closed ideal J_A of A_ω generated by A is simple, if and only if, either A is simple and purely infinite or A is isomorphic to the compact operators $\mathcal{K}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. If $A \ncong \mathcal{K}(\mathcal{H})$, then A_ω and is simple and purely infinite. If $A \cong \mathcal{K}(\mathcal{H})$, then $J_A \cong \mathcal{K}(\mathcal{H}_\omega)$ (and $J_A \neq A_\omega$ if $\mathrm{Dim}(\mathcal{H}) = \infty$).

Proof. It is easy to see (with help of representing sequences) that for $b, c \in (A_\omega)_+$ there is a contraction $d \in (A_\omega)_+$ with $\|c\|d^*bd = \|b\|c$ if A is simple and purely infinite.

Conversely, suppose that J_A is simple. Clearly, A is simple. Suppose that $A \ncong \mathcal{K}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$, i.e. that A is antiliminary. Let $b, c \in (J_A)_+$ with $\|b\| = \|c\|$. Since A is antiliminary, by (A.10) there exists a *-monomorphism $\psi\colon C_0((0,1], \mathcal{K}) \hookrightarrow A_\omega$ with $b\psi(f) = f$ for every $f \in C_0((0,1], \mathcal{K})$. Let D denote the hereditary C*–subalgebra of A_ω generated by the image of ψ. D is non-zero, stable and satisfies $bg = g = gb$ for all $g \in D$. In particular, $D \subset J_A$. Since J_A is simple and D is stable, there is $d \in J_A$ with $d^*d = c$ and $dd^* \in D$. Thus $d^*bd = d^*d = c$. It follows that A is purely infinite, because we can take $b, c \in A$ and find a representing sequence $(d_1, d_2, \ldots) \in \ell_\infty(A)$ for d with $d^*bd = c$ in A_ω. □

Lemma 1.2. *Suppose that A is a separable unital C*–algebra, such that 1_A is properly infinite. Then A_ω contains a non-zero C*-subalgebra D such that $AD + DA \subset D$ and $A \cap D = \{0\}$.*

In particular, $A^c \neq \mathbb{C} \cdot 1_A$.

Proof. We find a faithful unital *-representation $\varphi\colon A \to \mathcal{L}(\mathcal{H})$ over a separable Hilbert space $\mathcal{H}$ and a faithful normal state μ on $\mathcal{L}(\mathcal{H})$.

By assumption, there are isometries $s_1, s_2 \in A$ with $s_1^* s_2 = 0$. Let $a_1, a_2, \ldots$ a sequence that is dense in the positive contractions of A and $c_1 := \sum_{n\geq 1}(s_2)^n s_1 a_n s_1^*(s_2^*)^n$. Then A is generated (as a C*–algebra) by the five self-adjoint elements

$$c_1,\ c_2 := (s_1^* + s_1)/2,\ c_3 := (s_1^* - s_1)/2i,\ c_4 := (s_2^* + s_2)/2,\ c_5 := (s_2^* - s_2)/2i$$

of norm ≤ 1, and there is a unital *-epimorphism $h\colon C^*(F_5) \to A$ given by $h(g_j) := e^{ic_j}$. Here F_5 denotes the free group on 5 generators $g_1, \ldots, g_5$, and $C^*(F_5)$ the full C*–group algebra.

Let $l(w) \in \mathbb{N}$ denote the reduced word-length of an element $w \in F_5$. Then (obviously) $l(w_1 w_2) \leq l(w_1) + l(w_2)$ and one can easily see that $R(n) := \sharp\{w \in F_5\,;\, l(w) = n\}$ tends to ∞ for $n \to \infty$ and $R(n) \leq 10^n$. Thus

$$0 < \gamma := \sum_{w\in F_5} 20^{-l(w)} = \sum_{n=0}^{\infty} 20^{-n} R(n) < \infty$$

and $\nu(a) := \gamma^{-1}\sum_{w\in F_5} 20^{-l(w)}\mu\circ\varphi(h(w^{-1})ah(w))$ is a *faithful* state on A. ν satisfies $\nu(h(v)^* a h(v)) \leq 20^{l(v)}\nu(a)$ for all $a \in A_+$ and all $v \in F_5$.

We define a state ν_ω on A_ω by $\nu_\omega(b) := \omega - \lim_n \nu(b_n)$ for $b \in A_\omega$ and $(b_1, b_2, \ldots) \in \ell_\infty(A)$ with $\pi_\omega(b_1, b_2, \ldots) = b$. Let $L \subset A_\omega$ the closed left ideal of elements $b \in A_\omega$ with $\nu_\omega(b^*b) = 0$. Since $\nu_\omega(h(v)^* b^* b h(v)) \leq (20)^{l(v)}\nu_\omega(b^*b)$, we get $Lh(v) \subset L$ for all $v \in F_5$. It follows that $LA \subset A$. Thus $D := L^* \cap L$ satisfies $AD + DA \subset D$. $A \cap D \subset A \cap L = \{0\}$, because $0 = \nu_\omega(a^*a) = \nu(a^*a)$ implies $a = 0$.

By (A.6) and (A.5), there exists a non-scalar positive element in A^c. □

Lemma 1.3. *If A is separable (and non-zero) and $A^c/\mathrm{Ann}(A) \cong \mathbb{C}$ then $A \otimes \mathcal{K} \cong \mathcal{K}$.*

Proof. A is simple by (A.2) and the closed ideal J_A of A_ω generated by A must be simple by (A.6). By Remark 1.1, either $A \otimes \mathcal{K} \cong \mathcal{K}$ or A is purely infinite.

Suppose that A is purely infinite, then A contains a non-zero projection $p \in A$ and p is properly infinite, i.e. the unital algebra pAp has a properly infinite unit element. By (A.1), $(pAp)^c \cong A^c/\mathrm{Ann}(A) \cong \mathbb{C}$, which contradicts that $(pAp)^c$ is not isomorphic to $\mathbb{C}$ by Lemma 1.2. □

Lemma 1.4. *Suppose that A is simple.*

(i) *Then for every non-zero positive contraction $b \in A^c/\mathrm{Ann}(A)$ there is a positive contraction $d \in A^c/\mathrm{Ann}(A)$ with $\|d\| = 1$ and $db = bd = \|b\|d$.*

(ii) *If* $e \in (A^c/\mathrm{Ann}(A))_+$ *is not invertible, then there exists non-zero* $d \in (A^c/\mathrm{Ann}(A))_+$ *with* $de = 0$.

(iii) *Every maximal family of orthogonal positive contractions in* $A^c/\mathrm{Ann}(A)$ *is either un-countable, or is finite and has a invertible sum.*

Proof. Ad(i): We can suppose that $\|b\| = 1$. Then there is a contraction $c \in A^c_+$ with $b = c + \mathrm{Ann}(A)$. Let $a \in A_+$ a strictly positive contraction with $\|a\| = 1$.

By (A.1), $\rho\colon (A^c/\mathrm{Ann}(A)) \otimes^{\max} A \to A_\omega$ induces an *isomorphism* from $C^*(1,b) \otimes^{\min} A$ onto $C^*(A, cA) \subset A_\omega$ with $\rho(b \otimes a) = ca$, because A is simple, $C^*(1,b) \subset A^c/\mathrm{Ann}(A)$ is nuclear and $\rho(u \otimes v) = 0$ implies $u = 0$ or $v = 0$. In particular, $\|ca\| = \|b \otimes a\| = 1$.

Thus, there is a character μ on $C^*(a, ca^n\,;\, n = 1, 2, \ldots)$ with $\mu(ca) = 1$.

By (A.3) there exists $g \in (A_\omega)_+$ with $\|g\| = 1$ and $cag = g$. It follows $cg = g$ and $ag = g = ga$, because $ca \leq c \leq 1$ and $ca \leq a \leq 1$. In particular, $\mathrm{Ann}(A)g = \{0\}$. By (A.8) there is a positive contraction $d_1 \in A^c$ with $d_1 c = d_1$ and $d_1 g = g$. Thus $d := d_1 + \mathrm{Ann}(A) \in A^c/\mathrm{Ann}(A)$ satisfies $db = d$, $\rho(d \otimes a)g = d_1 ag = g$ and $1 \geq \|d\| \geq \|\rho(d \otimes a)\| \geq 1$.

Ad(ii): Then $b := 1 - \|e\|^{-1}e$ has norm $\|b\| = 1$. By (i), there is positive $d \in A^c/\mathrm{Ann}(A)$ with $\|d\| = 1$ and $db = d$. d is orthogonal to e.

Ad(iii): If $e_1, e_2, \ldots \in A^c/\mathrm{Ann}(A)$ is a sequence of pairwise orthogonal positive contractions, and $e := \sum 2^{-n} e_n$. If e is invertible, then $e_n = 0$ for $n \leq n_0$. If e is not invertible, then there exists non-zero $d \in (A^c/\mathrm{Ann}(A))_+$ with $ed = 0$ by (ii). Thus $e_n d = 0$ for all $n \in \mathbb{N}$. □

Lemma 1.5. *If* $A^c/\mathrm{Ann}(A)$ *is simple and stably finite, then* $A^c/\mathrm{Ann}(A) = \mathbb{C} \cdot 1$ *and* $A \otimes \mathcal{K} \cong \mathcal{K}$.

Proof. A is simple by (A.2) and the unital simple C*–algebra $A^c/\mathrm{Ann}(A)$ has a non-zero finite 2-quasi-trace that is necessarily faithful.

If A is simple and $A^c/\mathrm{Ann}(A)$ admits a faithful bounded quasi-trace, then every maximal family of non-zero mutually orthogonal positive contractions in $A^c/\mathrm{Ann}(A)$ is finite by Lemma 1.4(iii). It follows that every (maximal) commutative C*-subalgebra of $A^c/\mathrm{Ann}(A)$ must be of finite dimension. Thus $A^c/\mathrm{Ann}(A)$ is of finite dimension ($\leq$ square of the dimension of any maximal commutative C*–subalgebra).

Hence $A^c/\mathrm{Ann}(A) \cong M_n$ for some $n \in \mathbb{N}$. By (A.9) holds $M_n \otimes M_n \subset A^c$. Thus, $n = 1$.

$A \otimes \mathcal{K} \cong \mathcal{K}$ follows from $A^c/\mathrm{Ann}(A) \cong \mathbb{C}$ by Lemma 1.3. □

Lemma 1.6. *If* $A^c/\mathrm{Ann}(A)$ *is simple and is not stably finite, then* A *is simple and purely infinite.*

Proof. Then there is $n \in \mathbb{N}$ such that $M_n(A^c/\mathrm{Ann}(A))$ contains a copy of $\mathcal{O}_\infty$ unitally, because $A^c/\mathrm{Ann}(A)$ is unital and simple. It implies that the ultrapower $D_\omega \subset A_\omega$ contains a properly infinite projection $p \in \rho(\mathcal{O}_\infty \otimes^{\min} E) \subset D_\omega$ for every "n-stable" hereditary C*–subalgebra $D \cong M_n \otimes E$ of A. Here we naturally embed $\mathcal{O}_\infty \otimes^{\min} E$ into $(A^c/\mathrm{Ann}(A)) \otimes^{\max} (M_n \otimes E)$, and use that $(A^c/\mathrm{Ann}(A)) \otimes^{\max} D$ is a subalgebra of $(A^c/\mathrm{Ann}(A)) \otimes^{\max} A$.

By the semi-projectivity of the relations for infinite projections, D contains a copy of $\mathcal{O}_\infty$ (non-unitally). Since every non-zero hereditary C*–subalgebra of A contains a non-zero n-homogenous element, A is purely infinite. □

Lemma 1.7. *If* $A^c/\mathrm{Ann}(A)$ *is simple and is not stably finite, then* $A^c/\mathrm{Ann}(A)$ *is purely infinite and* $A \cong A \otimes \mathcal{O}_\infty$.

Proof. We split the proof into steps (α)–(ϵ):

(α) If $A^c/\mathrm{Ann}(A)$ is simple, $\neq \mathbb{C} \cdot 1_A$ and B is a separable C*–subalgebra of $A^c/\mathrm{Ann}(A)$, then the commutant $B' \cap A^c/\mathrm{Ann}(A)$ is not sub-homogenous, because it contains a copy of every separable simple unital C*-subalgebra of $A^c/\mathrm{Ann}(A)$ unitally by (A.9).

(β) If $A^c/\mathrm{Ann}(A)$ is simple and is not stably finite, then there is $n \in \mathbb{N}$ such that $M_n(A^c/\mathrm{Ann}(A))$ contains a copy of $\mathcal{O}_\infty$ unitally, and, for every $a \in (A^c/\mathrm{Ann}(A))_+ \setminus \{0\}$ there exists $m(a) \in \mathbb{N}$ such that $M_{m(a)}(\overline{a(A^c/\mathrm{Ann}(A))a})$ contains a copy of $\mathcal{O}_\infty$ (non-unitally).

(γ) Let $a \in (A^c/\mathrm{Ann}(A))_+ \setminus \{0\}$. We find a unital simple separable C*–subalgebra B of $A^c/\mathrm{Ann}(A)$ such that B contains a and the matrix-entries of the generators of $\mathcal{O}_\infty$ in $M_{m(a)}(\overline{a(A^c/\mathrm{Ann}(A))a})$. It follows, that the image of every non-zero *-homomorphism from $C_0((0,1], M_{m(a)}) \otimes \overline{aBa}$ into $A^c/\mathrm{Ann}(A)$ contains a non-zero stable C*-subalgebra of $A^c/\mathrm{Ann}(A)$.

(δ) Since $B' \cap A^c/\mathrm{Ann}(A)$ is not sub-homogenous, by the Glimm halving lemma [15, lem. 6.7.1] there is a non-zero *-homomorphism h_0 from $C_0((0,1], M_{m(a)})$ into $B' \cap A^c/\mathrm{Ann}(A)$.

Then the natural *-homomorphism $h\colon C_0((0,1], M_{m(a)}) \otimes B \to A^c/\mathrm{Ann}(A)$ with $h(f \otimes b) = h_0(f)b$ is non-zero, because $1 \in B$. Since B is simple, the restriction of h to $C_0((0,1], M_{m(a)}) \otimes \overline{aBa}$ is also non-zero. The image is contained in the hereditary C*-subalgebra of $A^c/\mathrm{Ann}(A)$ generated a. Thus, $A^c/\mathrm{Ann}(A)$ is locally purely infinite by (γ).

Hence $A^c/\mathrm{Ann}(A)$ is purely infinite. In particular, its unit element is properly infinite, i.e. there is a copy of $\mathcal{O}_\infty$ unitally contained in $A^c/\mathrm{Ann}(A)$.

(ϵ) A is simple and purely infinite by Lemma 1.6. So A is unital or it contains a non-zero projection p such that $A \cong (pAp)\otimes\mathcal{K}$ by Zhang dichotomy for simple σ-unital purely infinite C*–algebras.

Let $p \in A$ a non-zero projection (it should be the unit element of A in the case where A is unital). Then

$$b \in A^c/\mathrm{Ann}(A) \mapsto \rho(b \otimes p) \in p(A_\omega)p \cong (pAp)_\omega$$

is a unital *-homomorphism from $A^c/\mathrm{Ann}(A)$ into $(pAp)^c$. Thus $(pAp)^c$ contains a unital copy of $\mathcal{O}_\infty$. It implies $pAp \cong pAp \otimes \mathcal{O}_\infty$ by [12], because pAp is separable. Thus $A \otimes \mathcal{O}_\infty \cong A$. □

Theorem 1.8. *Suppose that A is a separable C*–algebra. Then $A^c/\mathrm{Ann}(A)$ is unital and A is unital if $\mathrm{Ann}(A) = \{0\}$.*

If $A^c/\mathrm{Ann}(A)$ is simple, then, either $A^c/\mathrm{Ann}(A) \cong \mathbb{C}$ and A is stably isomorphic to $\mathcal{K}(\ell_2(\mathbb{N}))$, or $A^c/\mathrm{Ann}(A)$ is purely infinite. If $A^c/\mathrm{Ann}(A)$ is purely infinite, then $A \cong A \otimes \mathcal{O}_\infty$ and A_ω is simple and purely infinite.

Note that A is simple and purely infinite if A_ω is simple by Remark 1.1.

Proof. $A^c/\mathrm{Ann}(A)$ is unital by (A.1). If $\mathrm{Ann}(A) = \{0\}$, then A is unital by (A.1).

If $A^c/\mathrm{Ann}(A)$ is simple and stably finite, then $A^c/\mathrm{Ann}(A) = \mathbb{C}\cdot 1$ by Lemma 1.5. It is the case if and only if $A \otimes \mathcal{K} \cong \mathcal{K}$ by Lemma 1.3.

Thus, if $A^c/\mathrm{Ann}(A)$ is simple and A is not stably isomorphic to $\mathcal{K}(\ell_2)$, then A is not stably finite. It follows that $A^c/\mathrm{Ann}(A)$ is purely infinite and $A \cong \mathcal{O}_\infty \otimes A$ by Lemma 1.7.

A is simple (and purely infinite) by Lemma 1.6. A_ω is simple and purely infinite by Remark 1.1, if A is purely infinite. □

Now we consider the nuclear case. It suffices to consider the unital case because a simple and purely infinite C*–algebra A contains a non-zero projection $p \in A$ and $A^c/\mathrm{Ann}(A) \cong (pAp)^c$ by (A.1).

Proposition 1.9. *A^c is simple and purely infinite if A is simple, purely infinite, separable, unital and nuclear.*

Proof. If separable unital A is purely infinite, simple and nuclear, then, for $b \in A^c$ with $0 \le b \le 1$, $\|b\| = 1$, there is an isometry $S \in A_\omega$ with $S^*bS = 1$ and $S^*aS = a$ for all $a \in A$. To get S, recall that the nuclear c.p. map $f \to f(1)$ from $C_0(\mathrm{Spec}(b), A) \cong C^*(b, 1) \otimes A \cong C^*(b, A)$ into $A \subset A_\omega$ is approximately one-step inner (in A_ω). Then use (A.4).

It follows $SS^* \in A^c$ and $S \in A^c$.

$A^c \ncong \mathbb{C}$ by Lemma 1.2.

□

Question 1.10. Let A a simple, purely infinite, unital, exact and separable C*–algebra. Is A^c simple if $A \cong A \otimes \mathcal{O}_2$?

Let $\mathcal{A}$ denote the reduced free product C*-algebra considered in [6]. $\mathcal{A}$ is unital, simple and purely infinite, but $\mathcal{A}^c$ does not contain $\mathcal{O}_\infty$. Thus $\mathcal{A}^c$ can not be simple.

There are unital non-separable purely infinite C*–algebras (e.g. the Calkin algebra) A with $A^c \cong \mathbb{C}$ by Corollary 1.13. This comes from the following Lemma and from Voiculescu's description of the neutral element of $\mathrm{Ext}(B)$ for separable B (*cf.* proof of Proposition 1.12).

Lemma 1.11. *Let B a separable unital C*-algebra. There exist a unital C*-algebra D, a unital *-monomorphism $\eta\colon B \to D$ and a projection $p \in D$ such that*

$$\|(1-p)\eta(b)p\| = \|p\eta(b) - \eta(b)p\| = \mathrm{dist}(b, \mathbb{C} \cdot 1)$$

for every $b \in B$.

Proof. Let $D := B * E$ the unital full free C*–algebra product of B and of $E := C^*(1, p = p^2 = p^*) \cong \mathbb{C} \oplus \mathbb{C}$. Then $\eta\colon b \mapsto b * 1$ and $\theta\colon e \to 1 * e$ are unital *-monomorphisms from B (respectively from E) into D. We identify $e \in E$ with $\theta(e)$. Note that, for all $b \in B$,

$$\max(\|(1-p)\eta(b)p\|, \|p\eta(b)(1-p)\|) = \|p\eta(b) - \eta(b)p\| \leq \mathrm{dist}(b, \mathbb{C} \cdot 1).$$

Let $b \in B \setminus \mathbb{C} \cdot 1$, i.e. $\mathrm{dist}(b, \mathbb{C} \cdot 1) > 0$. Since $|z| \leq \|b - z1\| + \|b\|$, there exists $z_0 \in \mathbb{C}$ with $|z_0| \leq 2\|b\|$ such that $\|b - z_0 1\| = \mathrm{dist}(b, \mathbb{C} \cdot 1)$. $\mathrm{dist}(b, \mathbb{C} \cdot 1)$ is the norm of $b + \mathbb{C} \cdot 1$ in $B/\mathbb{C} \cdot 1$. Thus, there exists a linear functional φ on B with $\varphi(1) = 0$, $\|\varphi\| = 1$ and $\varphi(b - z_0 1) = \|b - z_0 1\|$. With help of the polar-decomposition $\varphi = |\varphi|(u\cdot)$ of φ in $B^* = (B^{**})_*$, *cf.* [15, prop. 3.6.7], we can see that there are a unital *-representation $\lambda\colon B \to \mathcal{L}(\mathcal{H})$ and vectors $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ such that $\varphi(c) = \langle \lambda(c)x, y \rangle$ for all $c \in B$. It follows $x \perp y$ and $\lambda(b - z_0 1)x = \|b - z_0 1\| y$. Let $q \in \mathcal{L}(\mathcal{H})$ denote the orthogonal projection onto $\mathbb{C}x$. Then $(1-q)\lambda(b)qx = \|b - z_0 1\| y$. Thus

$$\mathrm{dist}(b, \mathbb{C} \cdot 1) \leq \|(1-q)\lambda(b)q\| \leq \|(1-p)\eta(b)p\|$$

because there is a unital *-homomorphism $\kappa\colon D \to \mathcal{L}(\mathcal{H})$ with $\kappa(p) = q$ and $\kappa(\eta(b)) = \lambda(b)$. $\square$

Proposition 1.12. *For every separable unital C*–subalgebra B of the Calkin algebra $Q = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ (on $\mathcal{H} \cong \ell_2(\mathbb{N})$) there is a projection $P \in Q$ with $\|Pb - bP\| = \mathrm{dist}(b, \mathbb{C}\cdot 1)$ for all $b \in B$.*

Proof. Let D, $\eta\colon B \to D$ and $p \in D$ as Lemma 1.11. D can be unitally and faithfully represented on $\mathcal{H} := \ell_2(\mathbb{N})$ such that $D \cap \mathcal{K} = \{0\}$. Let $s_1, s_2 \in \mathcal{L}(\mathcal{H})$ two isometries with $s_1 s_1 * + s_2 s_2^* = 1$, $\pi\colon t \in \mathcal{L}(\mathcal{H}) \mapsto t + \mathcal{K} \in Q$ denotes the quotient map. There is a unitary $U \in Q$ with $U^* b U = \pi(s_1) b \pi(s_1)^* + \pi(s_2 \eta(b) s_2^*)$ for $b \in B$, by the generalized Weyl–von-Neumann theorem of Voiculescu, *cf.* [1]. Thus $P := U\pi(s_2 p s_2^*)U^*$ is a projection in Q that satisfies $\|Pb - bP\| = \mathrm{dist}(b, \mathbb{C}\cdot 1)$ for all $b \in B$. $\square$

Proposition 1.12 implies:

Corollary 1.13. $Q^c = \mathbb{C}\cdot 1$.

Proof. Let $b = \pi_\omega(b_1, b_2, \ldots) \in Q_\omega$ for $(b_1, b_2, \ldots) \in \ell_\infty(Q)$, B the unital C*–subalgebra generated by $b_1, b_2, \ldots$ and $P \in Q$ as in Proposition 1.12. Then $Pb - bP = \pi_\omega(Pb_1 - b_1P, Pb_2 - b_2P, \ldots)$ and $\|Pb - bP\| = \omega - \lim_n \mathrm{dist}(b_n, \mathbb{C}\cdot 1)$. It follows $b \in \mathbb{C}\cdot 1 \cong (\mathbb{C}\cdot 1)_\omega$ if $Pb = bP$. $\square$

2. Other properties of A^c and its implications

We consider separable C*–algebras A (not necessarily simple or unital). The really interesting case seems to be where $A^c/\mathrm{Ann}(A)$ contains a full simple C*–algebra B of dimension $\mathrm{Dim}(B) > 1$. We show below that in this case A is strongly purely infinite if A is weakly purely infinite, and we study a condition on $A^c/\mathrm{Ann}(A)$ that implies weak pure infiniteness if A has no non-trivial lower semi-continuous 2-quasi-trace.

The next considerations are concerned with a sufficient condition on $A^c/\mathrm{Ann}(A)$ that allows to derive that A is weakly purely infinite if every lower semi-continuous 2-quasi-trace on A_+ takes only the values 0 and ∞ (*cf.* 2.5).

Definition 2.1. $X \subset B_+$ is *full* if the ideal of B generated by X is dense in B. We say: $a \in B_+$ is full if $X := \{a\}$ is full. A *-homomorphism $h\colon C \to B$ is full if $h(C_+)$ is full in B.

An element $a \in B_+$ is *k-homogenous* if there is a *-homomorphism $h\colon C_0((0,1]) \otimes M_k \to B$ such that $h(f_0 \otimes 1_k) = a$. Here $f_0(t) := t$ for $t \in (0,1]$. (0 is k-homogenous for every $k \in \mathbb{N}$ by definition.)

We define for a unital C*–algebra B a number $\mathrm{cov}(B, m)$ as the minimum in $\mathbb{N} \cup \{+\infty\}$ of the numbers $n \in \mathbb{N}$ such that there are $a_1, \ldots, a_n \in B_+$ and $d_1, \ldots, d_n \in B$ such that $\sum_j d_j^* a_j d_j = 1$ and a_j is the sum $a_j = \sum_{i=1}^{l_j} a_{j,i}$ of mutually orthogonal $k_{j,i}$-homogenous elements $a_{j,i} \in B_+$ with $k_{j,i} \geq m$ for $j = 1, \ldots, n$ and $i = 1, \ldots, l_j$. (The minimum of an empty subset of $\mathbb{N}$ is considered as $+\infty$.) In other words: $\mathrm{cov}(B, m) \leq n < \infty$, if and only if, there are finite-dimensional C*–algebras $F_1, \ldots, F_n$, *-homomorphisms $h_j \colon C_0((0, 1]) \otimes F_j \to B$ and $d_1, \ldots, d_j$ such that every irreducible representation of F_j is of dimension $\geq m$ and $1 = \sum_j d_j^* h_j(f_0 \otimes 1) d_j$ for $j = 1, \ldots, n$.

We define $\mathrm{cov}(B) := \sup_m \mathrm{cov}(B, m)$.

Remark 2.2. It follows easily from the definitions that for unital B holds:

(i) $\mathrm{cov}(B, m) \leq \mathrm{cov}(B, m + 1)$,
(ii) $\mathrm{cov}(C, m) \leq \mathrm{cov}(B, m)$ if there exist a unital *-homomorphism from B into C, in particular $\mathrm{cov}(\mathcal{O}_2, m) = 1$ for all $m \in \mathbb{N}$.
(iii) $\mathrm{cov}(B, m) = \inf_n \mathrm{cov}(B_n, m)$ if B is an inductive limit of unital C*-algebras $B_1, B_2, \ldots$, because $C_0((0, 1], F)$ is projective for C*–algebras F of finite dimension.
(iv) It follows $\mathrm{cov}(B) = \sup_m \inf_n \mathrm{cov}(B_n, m)$.
(v) If 1_B is finite, then $\mathrm{cov}(B) = 1$ if and only if there are for every $m \in \mathbb{N}$ a C*–algebra A_m of finite dimension and a unital *-homomorphism $h_m \colon A_m \to B$, such that every irreducible representation of A_m has dimension $\geq m$.
(vi) $\mathrm{cov}(\mathcal{O}_\infty) = 1$ because $\mathrm{cov}(\mathcal{O}_2) = 1$. Thus $\mathrm{cov}(B) = 1$ if 1_B is properly infinite.

Proposition 2.3. *If a unital nuclear separable C*–algebra B has decomposition rank $\mathrm{dr}(B) < \infty$ (cf.* [13, def. 3.1]*) and if B has no irreducible representation of finite dimension, then* $\mathrm{cov}(B) \leq \mathrm{dr}(B) + 1$.

Proof. This follows easily from the definition of the decomposition rank [13, def. 3.1] by [13, prop. 5.1], which implies that the c.p. contractions $\varphi_{r_i} \colon M_{r_i} \to B$ of strict order zero arising in n-decomposable c.p. approximations $\varphi \colon \bigoplus_{i=1}^s M_{r_i} \to B$ and $\psi \colon B \to \bigoplus_{i=1}^s M_{r_i}$ of [13, def. 3.1] can be chosen such that (eventually) $\min r_1, \ldots, r_s \geq q$ if $\psi \circ \varphi \to \mathrm{id}_B$ (in point-norm) and B has no irreducible representation of dimension $\leq q$.

Indeed, suppose that $\varphi_n \colon C_n \oplus D_n \to B$ and $\psi_n \colon B \to C_n \oplus D_n$ are completely positive contractions with suitable C*-algebras C_n and D_n such that $\varphi_n \circ \psi_n$ tends to id_B in point-norm, the curvatures $\|\psi_n(b^*b) - \psi_n(b^*)\psi_n(b)\|$ tend to zero for every $b \in B$, ψ_n is unital and every irreducible representation of C_n has dimension $\leq q$. Then the ultrapower $C := \prod_\omega \{C_1, C_2, \ldots\}$ has only irreducible representations of dimension $\leq q$ and the restriction to B of the ultrapower $U \colon B_\omega \to C$ of the completely positive contractions $p_1 \circ \psi_n \colon B \to C_n$ is a unital *-homomorphism from B into C. The latter contradicts that B has no irreducible representation of dimension $\leq q$. $\square$

Remark 2.4. A quasi-trace $\tau \colon A_+ \to [0, \infty]$ is called trivial if it takes only the values 0 and $+\infty$. Suppose that every lower semi-continuous 2-quasi-trace on A_+ is trivial. Then, for every $n \in \mathbb{N}$, $a \in A_+ \setminus \{0\}$ and $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there are $d_1, \ldots, d_n$ in $M_k \otimes A$ such that $d_i^*(1_k \otimes a)d_j = \delta_{i,j}(1_k \otimes (a-\varepsilon)_+)$.

(The latter is a reformulation of [11, prop. 5.7].)

Proposition 2.5. *If* $\operatorname{cov}(A^c/\operatorname{Ann}(A)) < \infty$ *and if every lower semi-continuous* 2-*quasi-trace on* A_+ *is trivial, then* A *is weakly purely infinite.*

Proof. Let $m := \operatorname{cov}(A^c/\operatorname{Ann}(A))$ and $n := 2n$. Below we show that, for $a \in A_+$ and $\varepsilon > 0$, there exists a matrix $V = [v_{j,q}]_{m,n} \in M_{m,n}(A_\omega)$ such that $V^*(a \otimes 1_m)V = (a-\varepsilon)_+ \otimes 1_n$. It follows that A is pi-m in the sense of [12, def. 4.3] (use representing sequences and $M_{m,n}(A_\omega) \cong (M_{m,n}(A))_\omega$). Thus A is weakly purely infinite.

Let $k_0 \in \mathbb{N}$ as in Remark 2.4 for $a \in A_+$ and $\varepsilon > 0$. We find finite-dimensional C*-algebras $F_1, \ldots, F_m$, *-homomorphisms $h_j \colon C_0((0,1]) \otimes F_j) \to A^c/\operatorname{Ann}(A)$ and elements $g_j \in A^c/\operatorname{Ann}(A)$ such that $\sum_j g_j^* b_j g_j = 1$ for $b_j := h_j(f_0 \otimes 1_{F_j})$, and that F_j has only irreducible representations of dimension $\geq k_0$ for $j = 1, \ldots, m$. (We allow $b_j = 0$ for $\operatorname{cov}(A^c/\operatorname{Ann}(A), k_0) \leq j \leq m$, to simplify notation.)

For every $j = 1, \ldots, m$ we find by Remark 2.4 $d_{j,1}, \ldots, d_{j,n} \in F_j \otimes A$ such that, for $1 \leq j \leq m$ and $1 \leq p, q \leq n$

$$d_{j,p}^*(1_{F_j} \otimes a)d_{j,q} = \delta_{p,q}(1_{F_j} \otimes (a-\varepsilon)_+)\,.$$

We define, for $j = 1, \ldots, m$ and $q = 1, \ldots, n = 2m$,

$$v_{j,q} := \rho(h_j \otimes \mathrm{id}_A(f_0 \otimes d_{j,q})(g_j \otimes 1))$$

(Note here that $g_j \otimes 1$ is a multiplier of $(A^c/\operatorname{Ann}(A)) \otimes A$.)

A straight calculation shows that $V := [v_{j,q}]_{m,n}$ is as desired, because

$$v_{j,p}^* a v_{j,q} = \delta_{p,q}\rho\left(g_j^* b_j g_j \otimes (a-\varepsilon)_+\right)\,.$$

$\square$

Now we study situations where we can deduce strong pure infiniteness from weak pure infiniteness.

Lemma 2.6. *If* A *is purely infinite and* $A^c/\operatorname{Ann}(A_\omega, A))$ *contains two orthogonal full hereditary* C*-*subalgebras, then* A *is strongly purely infinite.*

Proof. Let $a, b \in A_+$ and $\varepsilon > 0$, $\delta := \varepsilon/2$. If $E_1, E_2 \subset A^c/\mathrm{Ann}(A)$ are orthogonal full hereditary C*–subalgebras, there are $e_i \in (E_i)_+$ and $g_j, h_k \in A^c/\mathrm{Ann}(A)$ ($i = 1, 2$, $j = 1, \ldots, m$, $k = 1, \ldots, n$) such that $1 = \sum_j g_j^*(e_1)^2 g_j$ and $1 = \sum_k h_k^*(e_2)^2 h_k$. Thus, $a^2 = \rho(1 \otimes a^2)$ (respectively b^2) is in the ideal of A_ω generated by $\rho(e_1 \otimes a)$ (respectively $\rho(e_2 \otimes b)$), because, *e.g.* $1 \otimes a^2$ is in the ideal of $(A^c/\mathrm{Ann}(A)) \otimes^{\max} A$ generated by $e_1 \otimes a$. Let $u_i \in (A^c)_+ \subset A_\omega$ with $e_i = u_i + \mathrm{Ann}(A)$. Then $u_1 a b u_2 = \rho(e_1 e_2 \otimes ab) = 0$ and a^2 (respectively b^2) is in the closed ideal of A_ω generated by $u_1 a^2 u_1 = \rho((e_1)^2 \otimes a^2)$ (respectively $u_2 b^2 u_2$).

Since A is purely infinite, A_ω is again purely infinite, *cf.* [11].

It follows that there are $f_1, f_2 \in A_\omega$ such that $f_1 u_1 a^2 u_1 f_1 = (a^2 - \delta)_+$ and $f_2 u_2 b^2 u_2 f_2 = (b^2 - \delta)_+$.

With $v_i := f_i u_i$ holds $\|v_1^* a^2 v_1 - a^2\| < \varepsilon$, $\|v_2^* b^2 v_2 - b^2\| < \varepsilon$ and $v_1^* a b v_2 = 0$ in A_ω. With help of representing sequences for v_1 and v_2 in $\ell_\infty(A)$ we find $d_1, d_2 \in A$ with $\|d_1^* a^2 d_1 - a^2\| < \varepsilon$, $\|d_2^* b^2 d_2 - b^2\| < \varepsilon$ and $\|d_1^* a b d_2\| < \varepsilon$. This means that A is strongly purely infinite, *cf.* [3], [12]. $\square$

Lemma 2.7. *If $A^c/\mathrm{Ann}(A)$ contains a full 2-homogenous element, then A has the global Glimm halving property of* [2] *(cf. also* [3]*).*

If, in addition, A is weakly purely infinite, then A is strongly purely infinite.

Proof. Let $a \in A_+$, $\varepsilon \in (0, 1)$, $\delta := \varepsilon^2/2$ and $D := \overline{aAa}$. By assumption, there exists $b \in A^c/\mathrm{Ann}(A)$ and $d_1, \ldots, d_n \in A^c/\mathrm{Ann}(A)$ with $b^2 = 0$ and $\sum_j d_j^* b^* b d_j = 1$.

Let $e_j := \rho(d_j \otimes a^{1/2})$, $c \in A^c$ with $b = c + \mathrm{Ann}(A)$ and $f := ca = \rho(b \otimes a^{1/2})$. Then $f^2 = 0$ and $a^2 = \sum_j e_j f^* f e_j$. f and $e_1, \ldots, e_n$ are in the hereditary C*–subalgebra of A_ω generated by a, in particular they are in D_ω. Let $h = (h_1, h_2, \ldots) \in \ell_\infty(D)$ self-adjoint with $\pi_\omega(h) = f^* f - f f^*$, $g = (g_1, g_2, \ldots) \in \ell_\infty(D)$ with $\pi_\omega(g) = f$, and let $u_k := (h_k)_-^{1/k} g_k (h_k)_+^{1/k}$ for $k := 1, 2, \ldots$.

Then $u_k \in D$, $u_k^2 = 0$ and $\pi_\omega(u_1, u_2, \ldots) = f$.

With help of representing sequences in $\ell_\infty(D)$ for $e_1, \ldots, e_n \in D_\omega$ one can see that there exists $k \in \mathbb{N}$ and $v_1, \ldots, v_n \in D$ such that $\|a^2 - \sum_j v_j^* u_k^* u_k u_k v_j\| < \delta$.

By [12, lem. 2.2] there is a contraction $z \in A$ such that $\sum_j w_j^* u_k^* u_k w_j = (a - \varepsilon)_+$ for $w_j := v_j z h(a)$ with $h(t) := \max(0, t - \varepsilon)^{1/2} / \max(0, t^2 - \delta)^{1/2}$ on $[0, \infty]$.

It follows that $(a - \varepsilon)_+$ is in the ideal generated by u_k. Thus A has the global Glimm halving property of [2].

By [3] (and [2]) A is purely infinite if and only if A is weakly purely infinite and has the global Glimm halving property.

Thus, A is strongly purely infinite, because Lemma 2.6 applies. □

Theorem 2.8. *If A has no non-trivial lower semi-continuous 2-quasi-trace and if $A^c/\mathrm{Ann}(A)$ contains a simple C^*-subalgebra B with $1 \in B$ and*

$$\mathrm{cov}(B \otimes^{\max} B \otimes^{\max} \cdots) < \infty,$$

then A is strongly purely infinite.

Proof. Since $\mathrm{cov}(B\otimes^{\max} B\otimes^{\max}\cdots) < \infty$, it follows $B \neq \mathbb{C}$ and, by (A.9) and Remark 2.2(ii), that and $\mathrm{cov}(A^c/\mathrm{Ann}(A)) < \infty$. Thus Proposition 2.5 applies and A is weakly purely infinite. By the Glimm halving lemma (*cf.* [15, lem. 6.7.1]), Lemma 2.7 applies and A is strongly purely infinite. □

Lemma 2.9. $\mathrm{cov}(\mathcal{I}(m,n), \min(n,m)) \leq 2$*, and* $\mathrm{cov}(\mathcal{Z}) = 2$ *for the Jian–Su algebra* $\mathcal{Z}$*.*

Here $\mathcal{I}(m,n) \subset C([0,1], M_{mn})$ denotes the dimension-drop algebra given by the subalgebra of continuous functions $f\colon [0,1] \to M_m \otimes M_n$ with $f(0) \in M_m \otimes 1_n$ and $f(1) \in 1_m \otimes M_n$. One can use Proposition 2.3 for a proof because $\mathrm{dr}(\mathcal{I}(m,n), \min(n,m)) \leq 2$, but we give an independent proof.

Proof. Let $a \in C([0,1], M_{mn})_+$ the contraction given by $a(t) = t1_{mn}$. Then $a \in \mathcal{I}(m,n)$, $a^{1/3}$ is n-homogenous and $(1-a)^{1/3}$ is m-homogenous in $\mathcal{I}(m,n)$. $1 = d_1^* a^{1/3} d_1 + d_2^*(1-a)d_2$ for $d_1 = a^{1/3}$ and $d_2 = (1-a)^{1/3}$.

If $k \leq n, m \in \mathbb{N}$ and n, m are relative prime, then $\mathcal{I}(m,n) \subset \mathcal{Z}$ (unitally) and $\mathrm{cov}(\mathcal{I}(m,n), k) \leq 2$. Thus $\mathrm{cov}(\mathcal{Z}, k) \leq 2$ for all $k \in \mathbb{N}$. $\mathrm{cov}(\mathcal{Z}, 2) > 1$, because $1_{\mathcal{Z}}$ is finite and is not 2-homogenous. Hence $\mathrm{cov}(\mathcal{Z}, k) = 2$ for $k = 2, 3, \ldots$. □

Corollary 2.10. $A \otimes \mathcal{Z}$ *is strongly purely infinite if A has no non-trivial lower semi-continuous 2-quasi-trace.*

Proof. Since $\mathcal{Z} \cong \mathcal{Z}\otimes\mathcal{Z}\otimes\cdots$, $(A\otimes\mathcal{Z})^c/\mathrm{Ann}(A\otimes\mathcal{Z})$ contains a copy of $\mathcal{Z}$ unitally. □

Corollary 2.11. *If A is simple, and is neither stably finite nor purely infinite, then A^c can not contain a sequence of unital copies of $\mathcal{I}(m_k, n_k)$ for $\min(m_k, n_k) \to \infty$.*

Proof. Follows from $\mathrm{cov}(\mathcal{I}(m_k, n_k), n) \leq 2$ for $n \leq \min(m_k, n_k)$. □

Remarks 2.12. Let $D \not\cong \mathbb{C}$ a unital separable C*–algebra such that $\eta_1\colon d \in D \mapsto d \otimes 1 \in D \otimes^{\min} D$ and $\eta_2\colon d \in D \mapsto 1 \otimes d \in D \otimes^{\min} D$ are approximately unitarily equivalent in $D \otimes^{\min} D$. (We use here only the minimal C*-tensor product. It would be enough that η_1 and η_2 are equivalent in $\mathcal{D} := D^{\otimes\infty} := D \otimes D \otimes^{\min} \cdots$ for our considerations. Even that is not trivial for the algebras listed in (iv).)

(i) D is simple, nuclear and has at most one tracial state (by an observation of E. Effros and J. Rosenberg).

(ii) $\mathcal{D} := D^{\otimes\infty}$ is a unital simple nuclear C*–algebra such that $\eta_1 \colon a \in \mathcal{D} \mapsto a \otimes 1 \in \mathcal{D} \otimes \mathcal{D}$ is approximately unitarily equivalent to a *-isomorphism from $\mathcal{D}$ onto $\mathcal{D} \otimes \mathcal{D}$.

(iii) Conversely, if $\mathcal{D}$ is a separable unital C*-algebra such that $\eta_1 \colon a \in \mathcal{D} \mapsto a \otimes 1 \in \mathcal{D} \otimes \mathcal{D}$ is approximately unitarily equivalent to a *-isomorphism from $\mathcal{D}$ onto $\mathcal{D} \otimes \mathcal{D}$, then (using an observation of G. Elliott) even $\eta_{1,\infty} \colon a \in \mathcal{D} \mapsto a \otimes 1 \otimes \cdots \in \mathcal{D}^{\otimes\infty}$ is approximately unitarily equivalent to a *-isomorphism from $\mathcal{D}$ onto $\mathcal{D}^{\otimes\infty}$.
It follows immediately that every unital *-endomorphism of $\mathcal{D}$ is approximately inner. In particular, the flip automorphism of $\mathcal{D} \otimes \mathcal{D}$ is approximately inner.

(iv) Examples of $\mathcal{D}$ in (iii) are $\mathcal{O}_2$, $\mathcal{O}_\infty$, M_{p^∞}, $\mathcal{Z}$ and (infinite) tensor products $\mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \ldots$. Up to tensoring with $\mathcal{O}_\infty$ this list exhausts all $\mathcal{D}$ in the UCT class (see below).

An example of D with $D \ncong \mathcal{D}$ is $D = P_\infty$ the unique p.i.s.u.n. algebra in the UCT class with $K_0(P_\infty) = 0$ and $K_1(P_\infty) \cong \mathbb{Z}$. It holds $\mathcal{D} = D^{\otimes\infty} \cong \mathcal{O}_2$. More generally, let D any separable simple C*–algebra that contains a copy of $\mathcal{O}_2$ unitally, then η_1 and η_2 are approximately unitarily equivalent in $D \otimes D$ (M. Rørdam gave an example of a simple nuclear C*–algebra D that contains a copy of $\mathcal{O}_2$ unitally and is not purely infinite.)

(v) With the methods of [9] one can show that $A \cong \mathcal{D} \otimes A$ if and only if $\mathcal{M}(A)$ and $A^c/\mathrm{Ann}(A)$ contain copies of $\mathcal{D}$ unitally. It follows (essentially by applications of (A.1),(A.3) and (A.9)) that the property $A \otimes^{min} \mathcal{K} \cong \mathcal{D} \otimes^{min} A \otimes \mathcal{K}$ has nice permanence properties as *e.g.* invariance under extensions, inductive limits, passage to hereditary subalgebras, quotients, and tensor products.

(vi) If, in addition, $u^* \otimes u \in \mathcal{U}_0(\mathcal{D} \otimes \mathcal{D})$ for every unitary $u \in \mathcal{U}(\mathcal{D})$ (equivalently: $uvu^*v^* \in \mathcal{U}_0(\mathcal{D})$ for all $u, v \in \mathcal{U}(\mathcal{D})$)), then the technics of [8] applies, and one can show that $A \cong A \otimes \mathcal{D}$, if and only if, the quotient of $A' \cap C_b(\mathbb{R}_+, A)/C_0(\mathbb{R}_+, A)$ by the annihilator of A in $C_b(\mathbb{R}_+, A)/C_0(\mathbb{R}_+, A)$ contains $\mathcal{D}$ unitally. (The point is to construct a continuous path in $\mathrm{End}(\mathcal{D})$ from η_1 to η_2.)
It follows, *e.g.* (if one let $A = \mathcal{D}$) that every unital endomorphism of $\mathcal{D}$ is unitarily homotopic to the identity map on $\mathcal{D}$, and that, for general separable A, $A \otimes^{min} \mathcal{K} \cong \mathcal{D} \otimes^{min} A \otimes \mathcal{K}$ implies $A \cong \mathcal{D} \otimes^{min} A$. (The latter result and the permanences of (v) have been also obtained recently by W. Winter and A. Toms under the assumption that $\mathcal{U}(\mathcal{D}) = \mathcal{U}_0(\mathcal{D})$ and with different methods.)

(vii) Let $\mathcal{D}$ purely infinite (=not stably finite here). Since $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(D) \to K_1(\mathcal{D})$ is an isomorphism (*cf.* [5]), we get that $uvu^*v^* \in \mathcal{U}_0(\mathcal{D})$ for all $u, v \in \mathcal{U}(\mathcal{D})$. Thus every unital *-endomorphism of $\mathcal{D}$ is unitarily homotopic to the identity of $\mathcal{D}$. It allows to define a natural isomorphism from $K_0(\mathcal{D})$ into $KK(\mathcal{D}, \mathcal{D})$ such that the class of a *-morphism $\psi \colon \mathcal{D} \otimes \mathcal{K} \to \mathcal{D} \otimes \mathcal{K}$ corresponds to $[\psi(1 \otimes p_{1,1})] \in K_0(\mathcal{D})$.

If $\mathcal{D}$ is in the UCT class, then this implies that $L \in \mathrm{End}_{\mathbb{Z}}(K_*(\mathcal{D}))$ must be the identity of $K_*(\mathcal{D})$ if $L([1_{\mathcal{D}}]) = [1_D]$. This implies that $K_1(\mathcal{D})$ must be zero, and $\mathrm{End}_{\mathbb{Z}}(K_0(\mathcal{D}))$ is a commutative ring with additive group isomorphic to $K_0(\mathcal{D})$.

From $K_1(\mathcal{D}) = K_1(\mathcal{D} \otimes \mathcal{D}) = 0$ we get by Künneth theorem on tensor products that that $Tor(K_0(\mathcal{D}), K_0(\mathcal{D})) = 0$. Thus $K_0(\mathcal{D})$ is torsion-free. The natural isomorphism $K_0(\mathcal{D}) \otimes_{\mathbb{Z}} K_0(\mathcal{D}) \cong K_0(\mathcal{D} \otimes \mathcal{D}) \cong \mathcal{K}_0(\mathcal{D})$ defines a unital ring with unit $[1_{\mathcal{D}}]$ that is the same as the ring induced by the additive isomorphism from $K_0(\mathcal{D})$ onto $KK(\mathcal{D}, \mathcal{D})$. In particular, every group endomorphism (i.e. $\mathbb{Z}$-module endomorphism) is also a ring endomorphism of the commutative ring. Moreover $K_0(\mathcal{D} \otimes M_2 \otimes M_3 \otimes \cdots) \cong K_0(\mathcal{D}) \otimes \mathbb{Q}$ has the same properties, because $\mathcal{D} \otimes M_2 \otimes M_3 \otimes \cdots$ satisfies also the condition in (iii). Thus the $\mathbb{Q}$-vector space $K_0(\mathcal{D}) \otimes \mathbb{Q}$ is one-dimensional (over $\mathbb{Q}$), i.e. there is a natural monomorphism from $K_0(D)$ into $\mathbb{Q}$. All this together happens if and only if $K_0(\mathcal{D})$ is a subring of $\mathbb{Q}$ or is zero. It follows that the (infinite) tensor products of the examples in (iv) exhaust all purely infinite $\mathcal{D}$ of (iii) in the UCT class by the classification theory for simple p.i.s.u.n. algebras.

(viii) Suppose now that $\mathcal{D}$ has a tracial state. Then $\mathcal{D}$ has the Dixmier property. $\mathcal{D}$ and $\mathcal{D} \otimes \mathcal{Z}$ have the same KK-class and same ordered K_0. The tracial state gives an order and ring isomorphism from K_0 into $\mathbb{Q}$ if $\mathcal{D}$ is in the UCT class. One does not know whether $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$ or not, even if $\mathcal{D}$ is in the UCT class. From recent results of M. Rørdam [14] it follows that $\mathcal{D} \otimes \mathcal{Z}$ has stable rank one and $\mathcal{D} \otimes \mathcal{Z}$ has real rank zero if $K_0(D) \ncong \mathbb{Z}$.

(ix) Since $E := (M_2 \oplus M_3) \otimes (M_2 \oplus M_3) \otimes \ldots$ contains a simple unital AF-algebra of infinite dimension (by an observation of M. Rørdam), E contains also a copy of $\mathcal{Z}$ (in fact $E \otimes \mathcal{Z} \cong E$). Thus, by (v), (vi) and (A.9) $A \cong \mathcal{Z} \otimes A$ if there is a unital *-homomorphism from $M_2 \oplus M_3$ into $A^c/\mathrm{Ann}(A)$.

The Remarks 2.12 lead to the following questions.

Questions 2.13. Let $\mathcal{D}$ as in part (iii) of 2.12.

(i) Is $\mathrm{cov}(\mathcal{D}) < \infty$? (The answer is positive if 1_D is infinite.)

(ii) Is $\mathcal{Z}$ unitally contained in $\mathcal{D}$? (Then $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$ and $\mathrm{cov}(\mathcal{D}) \leq 2$. This is the case if $\mathcal{D}$ has no tracial state.)

(iii) Is always $K_1(\mathcal{D}) = 0$? Is $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \to K_1(\mathcal{D})$ an isomorphism? ($K_1(\mathcal{D}) = 0$ holds if $\mathcal{D}$ is in the UCT–class. $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \to K_1(\mathcal{D})$ is an isomorphism if $\mathcal{D}$ has no tracial state, or if $\mathcal{D}$ has stable rank one.)

(iv) Is $u^* \otimes u \in \mathcal{U}_0(\mathcal{D} \otimes \mathcal{D})$ for every unitary $u \in \mathcal{U}(\mathcal{D})$? (Is the case if $\mathcal{D}$ is purely infinite by an old result of J. Cuntz, or if $\mathcal{D}$ has stable rank one.)

(v) Is (non-unital !) A approximately divisible if there is a unital *-homomorphism from $M_2 \oplus M_3$ into $A^c/\mathrm{Ann}(A)$? (We believe that the answer is negative, and that the conclusion $A \otimes \mathcal{Z} \cong A$ is the best possible.)

(vi) Does there exist a C*-algebra A such that A is stably projection-less and that the flip automorphism of $A \otimes^{\min} A$ is approximately inner (by unitaries in $\mathcal{M}(A \otimes^{\min} A)$).

(vii) Let D be as at the beginning of Remarks 2.12. Is D stably finite if 1_D is finite? Is the flip automorphism of $D \otimes D$ then approximately inner? (The answers are positive for $\mathcal{D}$.)

Remark 2.14. The families of relations for the definition of $\mathrm{cov}(B, m)$ are *semi-projective*, because we can suppose that the $d_1, \ldots, d_n$ and $h_j \colon C_0((0,1]) \otimes F_j \to B$ of Definition 2.1 satisfy in addition $d_1^* d_1 + \ldots + d_n^* d_n = 1$ and $h_j(f_0 \otimes 1)d_j = d_j$ for $j = 1, \ldots, n := \mathrm{cov}(B)$.

Indeed, let $h_j \colon C_0((0,1]) \otimes F_j \to B$ and $d_1, \ldots, d_n$ such that $1 = \sum_j d_j^* h_j(f_0 \otimes 1) d_j$ (where F_j is finite-dimensional and every irreducible representation of F_j is of dimension $\geq m$ for $j = 1, \ldots, n$). We $\delta \in (0,1)$ such that $1/2 \geq g := \sum_j d_j^* h_j((f_0 - \delta)_+ \otimes 1) d_j \leq 1$. Let $\tilde{d}_j := h_j((f_0 - \delta)_+ \otimes 1)^{1/2} d_j g^{-1/2}$ then $\tilde{d}_1^* \tilde{d}_1 + \ldots + \tilde{d}_n^* \tilde{d}_n = 1$. There is a unique *-monomorphism $\psi \colon C_0(0,1] \to C_0(0,1]$ with $\psi(f_0) = g_\delta$ where $g_\delta(t) := \min(t/\delta, 1)$. Let $\tilde{h}_j := h_j \circ (\psi \otimes \mathrm{id}_{F_j})$, then $\tilde{h}_j(f_0 \otimes 1)\tilde{d}_j = \tilde{d}_j$.

3. Appendix: Elementary properties of A^c

The following list of properties of A^c are of elementary nature. Sometimes we only sketch the proofs. We suppose in general that A is separable, but in (A.1) we need only that A and $D \subset A$ are σ-unital. More details on the given arguments can be found in the preliminaries (or in the technical chapters) of [9], [12]. Recall that $\pi_\omega \colon \ell_\infty(A) \to A_\omega$ denotes the natural quotient map.

(A.1) The (two-sided) annihilator $\mathrm{Ann}(A) := \mathrm{Ann}(A, A_\omega)$ of A in A_ω is a closed ideal of A^c, and $A^c/\mathrm{Ann}(A)$ is a *unital* C*-algebra. $\mathrm{Ann}(A) = \{0\}$ *if and only if A is unital.*

If $d \in A_+$ is a positive contraction that is full in A then $\|b + \mathrm{Ann}(A)\| = \sup_n \|b d^{1/n}\|$ for all $b \in A^c$. There is a natural *-homomorphism

$$\rho \colon (A^c/\mathrm{Ann}(A)) \otimes^{\max} A \to A_\omega$$

given by $\rho((b + \mathrm{Ann}(A)) \otimes c) := bc$ for $b \in A^c$ and $c \in A$. (Thus $\rho(1 \otimes c) = c$ for $c \in A$.)

Let $D \subset A$ a full hereditary C*–subalgebra of A. There is a natural *-isomorphism ι from $A^c/\mathrm{Ann}(A)$ onto $D^c/\mathrm{Ann}(D, D_\omega)$ with $\rho_A(b\otimes d) = \rho_D(\iota(b)\otimes d)$ for $b \in A^c/\mathrm{Ann}(A)$ and $d \in D$. (ι is determined by the values $\rho_D(\iota(b) \otimes d)$ for a fixed full element $d \in D$.)

In particular, $A^c/\mathrm{Ann}(A) \cong (pAp)^c \subset pA_\omega p \cong (pAp)_\omega$ *if* p *is a full projection in* A.

Proof. If $Ab = \{0\} = bA$ then $Acb = \{0\} = cbA$ and $Abc = \{0\} = bcA$ for $c \in A^c$. Clearly, $\mathrm{Ann}(A) = \{0\}$ if A is unital. Conversely, if A is not unital and if $a \in A_+$ is a strictly positive contraction with $\|a\| = 1$, then there exists a sequence $\alpha_1 > \alpha_2 > \ldots$ in $\mathrm{Spec}(a) \setminus \{0\}$ with $\lim_n \alpha_n = 0$. Let $f_n(t) := \min(\alpha_{n+1}^{-1}t, 1) - \min(\alpha_n^{-1}t, 1)$. Then $f_n(a) \geq 0$, $\|f_n(a)\| = 1$ and $\|f_n(a)a\| \leq \alpha_n$. $c := \pi_\omega(f_1(a), f_2(a), \ldots)$ satisfies $c \geq 0$, $\|c\| = 1$ and $ca = ac = 0$ for $a \in A$. Thus $\mathrm{Ann}(A) \neq \{0\}$.

If $a \in A_+$ is a strictly positive contraction in A, then the positive contraction $e := \pi_\omega(a, a^{1/2}, a^{1/3}, \ldots)$ satisfies $ea = ae = a$. Thus $e - e^2 \in \mathrm{Ann}(A)$ and $b - be, b - eb \in \mathrm{Ann}(A)$ for all $b \in A^c$. Thus $e + \mathrm{Ann}(A)$ is a unit element of $A^c/\mathrm{Ann}(A)$.

Let $d \in A_+$ a positive contraction that is full in A. $N(b) := \sup \|bd^{1/n}\|$ is a semi-norm on A^c with $N(b) \leq \|b\|$, $N(b^*) = N(b)$ and $N(b) = 0$ if and only if $bd = db = 0$. $bd = 0$ holds if and only if $bg = gb = 0$ for every $g \in A$ because d is full in A, i.e. every $g \in A$ can be approximated by finite sums $\sum_j e_j d f_j$ with $e_j, f_j \in A$. Thus $N(b) = 0$ if and only if $b \in \mathrm{Ann}(A)$. $N(bc) \leq N(b)N(c)$ because $\|bcd^{1/n}\| \leq \|bd^{1/(2n)}\|\|cd^{1/(2n)}\|$. $\|bd^{1/n}\|^2 = \|b^*bd^{2/n}\| \leq \|b^*bd^{1/n}\|$ because b^*b and d commute. Thus $N(b)^2 \leq N(b^*b)$, and N is a C*–norm on A^c with $N(b) = 0$ if and only if $b \in \mathrm{Ann}(A)$, i.e. $N(b) := \|b + \mathrm{Ann}(A)\|$.

It follows that the natural C*-algebra homomorphism from $A^c \otimes^{max} A$ into $D_A := \overline{aA_\omega a} \subset A_\omega$ given by $b \otimes x \mapsto bx$ factorizes over

$$(A^c/\mathrm{Ann}(A)) \otimes^{\max} A \cong (A^c \otimes^{max} A)/(\mathrm{Ann}(A) \otimes^{max} A)$$

and defines a *-epimorphism ρ from $(A^c/\mathrm{Ann}(A))\otimes^{\max} A$ onto the C*–algebra generated by $A^c \cdot A$. We get that ρ is well-defined, satisfies $\rho((b + \mathrm{Ann}(A)) \otimes x) = bx$ for $b \in A^c$, $x \in A$, and $c = 0$ if $\rho(c \otimes d) = 0$ and $\mathrm{span}(AdA)$ is dense in A.

Let $D \subset A$ a full hereditary C*–subalgebra of A. Since D is separable, D contains a strictly positive contraction $d \in D_+$. Let $f := \pi_\omega(d^{1/2}, d^{1/3}, \ldots) \in D_\omega \subset A_\omega$ and let $T(b) := fbf$ for T is a completely positive contraction from A^c into D^c such that $bg = T(b)g = gT(b)$ for all $g \in D$, $T(\mathrm{Ann}(A)) \subset \mathrm{Ann}(D) := \mathrm{Ann}(D, D_\omega)$ and $T(b^*b) - T(b)^*T(b) \in \mathrm{Ann}(D)$. Thus $\iota(b + \mathrm{Ann}(A)) := T(b) + \mathrm{Ann}(D)$ (for $b \in A^c$) is a well-defined *-homomorphism from $A^c/\mathrm{Ann}(A)$ into $D^c/\mathrm{Ann}(D)$ with

$$\rho_A((b + \mathrm{Ann}(A)) \otimes g) = bg = T(b)g = \rho_D((T(b) + \mathrm{Ann}(D)) \otimes g)$$

for $g \in D$ ι is a unital *-monomorphism, because $fefd = d$ and $0 = T(b)d = bd$ implies $b \in \mathrm{Ann}(A)$ for $b \in A^c$. ι is uniquely determined by the values $\rho_D(\iota(b+\mathrm{Ann}(A)) \otimes d) = bd$, because $\rho_D((\iota(b+\mathrm{Ann}(A))-x)\otimes d) = 0$ implies $x = \iota(b+\mathrm{Ann}(A))$ if $x \in D^c/\mathrm{Ann}(D)$. (Here d can be any *full* element of D.)

Now suppose that D is a full corner of A, $P \in \mathcal{M}(A)$ is the projection with $PAP = D$, and that $d \in D_+$ a strictly positive contraction of D.

There exists a partial isometry V in $\mathcal{M}(A \otimes \mathcal{K})$ with $V^*V = 1 - (P \otimes e_{1,1})$ and $VV^* = (P \otimes 1) - (P \otimes e_{1,1})$, because $P \otimes (1 - e_{1,1})$ and $1 = 1 \otimes 1$ are Murray–von-Neumann equivalent in $\mathcal{M}(A \otimes \mathcal{K})$ by [4, lem. 2.5],

$$1 \geq 1 - (P \otimes e_{1,1}) \geq 1 \otimes (1 - e_{1,1}) \geq P \otimes (1 - e_{1,1})$$

are properly infinite projections and $K_0(\mathcal{M}(A \otimes \mathcal{K})) = 0$ (*cf.* [5]).

Let $c \in D^c_+$ and $(c_1, c_2, \ldots) \in \ell_\infty(D)_+$ a representing sequence for c, i.e. $c := \pi_\omega(c_1, c_2, \ldots)$. We define $h_n \in A \otimes \mathcal{K}$ by

$$h_n := c_n \otimes e_{1,1} + V^*(c_n \otimes (e_{2,2} + \cdots + e_{n,n}))V$$

and $b_n \in A_+$ by $b_n \otimes e_{1,1} := (1 \otimes e_{1,1}) h_n (1 \otimes e_{1,1})$. (Here $e_{j,k}$ denote the matrix units of $\mathcal{K}$.) $P \otimes (1 - e_{1,1})$ It is easy to check that $b := \pi_\omega(b_1, b_2, \ldots)$ is in A^c and $V(b)d = bd = cd$. Thus $\rho_D((\iota(b + \mathrm{Ann}(A)) - (c + \mathrm{Ann}(D))) \otimes d) = 0$, i.e. $\iota(b + \mathrm{Ann}(A)) = c + \mathrm{Ann}(D)$, and ι is surjective.

The general case of a full hereditary C*–subalgebra $D \subset A$ reduces to the case of a full corner of A:
We may identify A with $A \otimes e_{1,1} \subset A \otimes M_2$ and D with $D \otimes e_{1,1} \subset E := D \otimes M_2 \subset A \otimes M_2$. Let B denote the hereditary C*–subalgebra of $A \otimes M_2$ generated by $(A \otimes e_{1,1}) + (D \otimes e_{2,2})$. Then A and $F := D \otimes e_{2,2}$ are full corners of B, and of $E \subset B$. Consider the unital *-monomorphisms $\iota_1 \colon B^c/\mathrm{Ann}(B) \to A^c/\mathrm{Ann}(A)$, $\iota_2 \colon B^c/\mathrm{Ann}(B) \to D^c/\mathrm{Ann}(D)$, $\iota_3 \colon B^c/\mathrm{Ann}(B) \to E^c/\mathrm{Ann}(E)$, $\iota_4 \colon B^c/\mathrm{Ann}(B) \to F^c/\mathrm{Ann}(F)$, $\iota_5 \colon E^c/\mathrm{Ann}(E) \to D^c/\mathrm{Ann}(D)$, and $\iota_6 \colon E^c/\mathrm{Ann}(E) \to F^c/\mathrm{Ann}(F)$.

Then $\iota_2 = \iota \circ \iota_1$, $\iota_2 = \iota_5 \circ \iota_3$ and $\iota_4 = \iota_6 \circ \iota_3$ (by uniqueness with respect to ρ). ι_1, ι_4, ι_5 and ι_6 are isomorphisms, because $A \subset B$, $F \subset B$, $D \subset E$ and $F \subset E$ are full corners. It follows that ι_3, ι_2 and ι must be isomorphisms (i.e. must be surjective). $\square$

(A.2) If J is a non-trivial closed ideal of A, then J_ω is a closed ideal of A_ω. The ideal $A^c \cap J_\omega$ is not contained in $\mathrm{Ann}(A)$ and $\mathrm{Ann}(A) + (A^c \cap J_\omega)$ does not contain A^c. (I.e. $(A^c \cap J_\omega)/(\mathrm{Ann}(A) \cap J_\omega)$ is a non-trivial closed ideal of $A^c/\mathrm{Ann}(A)$.)

In particular, A *is simple if* $A^c/\mathrm{Ann}(A)$ *is simple.* A *is simple and unital if* A^c *is simple.*

Proof. It is clear that J_ω is a closed ideal of A_ω, that $J_\omega \cap A = J$ and that $A^c \cap J_\omega$ is a closed ideal of A^c. If a is a strictly positive contraction in A_+ and $b \in C_+$ a strictly positive contraction for C, then there are $b_1, b_2, \ldots \in C^*(b)_+$ with $\|b_n\| = 1$, $b_n b_{n+1} = b_n$, $\|b_n - b_n b\| < 1/n$ and $\lim_{n\to\infty} \|b_n d - d b_n\| = 0$ for all $d \in A$. Thus $c := \pi_\omega(b_1, b_2, \ldots)$ is in $A^c \cap J_\omega$ and $cb = b \neq 0$. Thus $c \notin \mathrm{Ann}(A)$. A^c is not contained in $\mathrm{Ann}(A) + (A^c \cap J_\omega)$, because otherwise $\rho(1 \otimes a) = a$ is in J_ω, i.e. $a \in J$, which contradicts the non-triviality of J. □

(A.3) If B is a separable C*–subalgebra of A_ω and μ a pure state on B, then there exists a sequence of pure states $\mu_1, \mu_2, \ldots$ on A such that μ is the restriction of $\mu_\omega \colon A_\omega \to \mathbb{C} \cong \mathbb{C}_\omega$ to B. Further there are positive contractions $g_n \in A_+$ with $\mu_n(g_n) = 1$ and $gbg = \mu(b)g^2$ for $b \in B$, where $g := \pi_\omega(g_1, g_2, \ldots)$. g commutes with B if (and only if) μ is a character of B.

Proof. By an old observation of J. Glimm there exists $b \in B_+$ with $\mu(b) = \|b\| = 1$ such that $\nu(b) = 1$ and $\|\nu\| = 1$ implies $\nu = \mu$. It follows $\lim_{n\to\infty} \|b^n a b^n - \mu(a) b^{2n}\| = 0$ for every $a \in B$.

Further there exist a sequence $b_1, b_2, \ldots \in B_+$ with $\|b_n\| = 1$ and $\pi_\omega(b_1, b_2, \ldots) = b$.

Let $\mu_1, \mu_2, \ldots$ pure states on B with $\mu_n(b_n) = 1$. Then $\mu_\omega(b) = 1$. Thus $\mu_\omega | B = \mu$.

If $f_n(t) = \max(0, 1 - n(1-t))$ and $g_n := f_n(b_n)$, then $g := \pi_\omega(g_1, g_2, \ldots)$ is as desired. □

(A.4) Suppose that $P_1, P_2, \ldots$ is a sequence of (non-commutative) polynomials in in non-commuting variables x, x^* *with coefficients in* A_ω.

If, for every $n \in \mathbb{N}$ and $\varepsilon > 0$, there is a contraction $a \in A^c$ with $\|P_k(a, a^*)\| < \varepsilon$ for $k = 1, \ldots, n$, then there is a contraction $x_0 \in A^c$ with $P_n(x_0, x_0^*) = 0$ for all $n \in \mathbb{N}$.

Proof. The result is true for A_ω in place of A^c by [12, lem. 2.5], *cf.* also [9, sec. 2]. One gets the corresponding result for A^c if one adds to the sequence $P_1, P_2, \ldots$ the sequence of polynomials $Q_1, Q_2, \ldots$ given by $Q_n(x, x^*) := d_n x - x d_n$ for a dense sequence $d_1, d_2, \ldots$ in the selfadjoint contractions in A. □

(A.5) Suppose that there exists a positive element $b \in A_\omega$ with $\|b\| = 1$,such that $bA \neq \{0\}$, $Ab + bA$ is contained in the hereditary C*–subalgebra $E := \overline{bA_\omega b}$ of A_ω, and $A \cap E \neq A$. Then there are $a \in A_+$ and $d \in A^c_+$ with $\|d\| = 1$, $da \neq a$ and $da \neq 0$.

In particular, $d + \mathrm{Ann}(A)$ is a non-scalar element of $A^c/\mathrm{Ann}(A)$.

Proof. Let $B := C^*(b, A)$, then $J := \overline{bBb}$ is a closed ideal of B such that $A + J = B$, $A \not\subset J$ and $J \not\subset \mathrm{Ann}(A)$. There is a strictly positive contraction $a \in A_+$ of A with $\|a + J\| = 1$. (Indeed, there is a strictly positive contraction $f \in A/(A \cap J)$ with $\|f\| = 1$

and a positive and contractive lift $f_1 \in A$ of f. Then $C^*(f_1, J \cap A)/(J \cap A) = C^*(f)$. Let χ the (unique) character on $C^*(f_1, J \cap A)$ with $\chi(J \cap A) = 0$ and $\chi(f_1) = 1$. A strictly positive contraction $a \in C^*(f_1, J \cap A)_+$ with $\chi(a) = 1 = \|a\|$ exists by the argument in the beginning of the proof of (A.3).)

It holds $ba \neq 0$, because $bA \neq \{0\}$. We find pure states μ, ν on B with $\mu(a) = 1$, $\mu(J) = 0$, $\nu(b) = 1$. By (A.3) there are positive contractions $g, h \in A_\omega$ with $\|g\| = 1 = \|h\|$, $gdg = \mu(d)g^2$ and $hdh = \nu(d)h^2$ for $d \in B$. This implies $bg = 0$, $bh = h$, and $ag = g$.

We find in $C^*(b) \subset J$ a sequence of positive contractions $b_1, b_2, \ldots$ with $b_n b_{n+1} = b_n$, $\|b - b_n b\| < 1/n$ and $\lim_{n\to\infty} \|b_n c - c b_n\| = 0$ for all $c \in A$, *cf.* [15, thm. 3.12.14]. Note that $b_n g = 0$ and $b_n h = h$ for all $n \in \mathbb{N}$.

If $a_1, a_2, \ldots$ is a dense sequence in the positive part of the unit ball of A, then the sequence of polynomials $P_1(x, x^*) := b - x^*xb$, $P_2(x, x^*) := x^*xg$, $P_3(x, x^*) := h - x^*xh$, $P_{n+3}(x, x^*) := x^*xa_n - a_n x^*x$ have approximate zeros given by contractions $x := (b_n)^{1/2}$.

Thus there is a contraction $x_0 \in A_\omega$ with $P_n(x_0, x_0^*) = 0$ for all $n \in \mathbb{N}$, *cf.* [12, lem. 2.5].

It follows that $d := x_0^* x_0$ is a contraction in A^c with $db = b$, $dg = 0$ and $dh = h$. $da \neq 0$ because $bda = ba \neq 0$. $da \neq a$ because $dag = dg = 0$ and $ag = g$ and $g \neq 0$. □

(A.6) If the closed hereditary C*-subalgebra $D_A := \overline{AA_\omega A}$ of A_ω contains a non-zero hereditary C*–subalgebra D with $AD + DA \subset D$ and $D \neq D_A$, then $A^c/\mathrm{Ann}(A)$ contains a non-scalar element.

In particular, $A^c/\mathrm{Ann}(A)$ contains a non-scalar element if the closed ideal J_A of A_ω generated by A is not simple.

Proof. Let $c \in D_+$ a non-zero positive element. Since $c \in D \subset D_A$, we have $Ac \neq \{0\}$. Consider the separable C*–subalgebra C of D generated by $Ac \cup cA \cup \{c\}$. Then $AC + CA \subset C \subset J$. A strictly positive element $b \in C_+ \subset A_\omega$ with $\|b\| = 1$ satisfies the assumptions of (A.5), because $b \in D \subset D_A$. Thus, there exist a non-scalar element in $A^c/\mathrm{Ann}(A)$ by (A.5).

The closed hereditary C*–subalgebra $D_A := \overline{AA_\omega A}$ is full in J_A. Thus, if J_A is not simple and J is a non-trivial closed ideal of J_A, then $D := D_A \cap J$ is a non-trivial ideal of D_A with $A \not\subset D$. □

(A.7) For positive contractions $a \in A^c$ and $b \in A_\omega$ with $ab = 0$ there exist positive contractions $c, d \in A^c$ with $cd = 0$ and $ca = a$, $db = b$.

In particular, A^c is "sub-Stonean". This property passes to quotients. Thus $A^c/\mathrm{Ann}(A)$ is also "sub-Stonean".

Proof. It suffices to find $d \geq 0$ in A^c with $\|d\| \leq 1$, $da = 0$ and $db = b$ (because then one can repeat with (b, d) in place of (a, b)).

Let $(b_1, b_2, \ldots)$ a sequence of positive contractions in A with $b = \pi_\omega(b_1, b_2, \ldots)$ and let $f \in A_+$ a strictly positive contraction. There are $k_n \in \mathbb{N}$ with $\|f^{1/k_n} b_n - b_n\| < 1/n$.

Consider $P_1(x, x^*) := x^*xb$ and $P_2(x, x^*) := a - x^*xa$ and apply (A.4). $d := x_0^* x_0$ for a contractive solution x of $P_1 = P_2 = 0$. The approximate solutions are given by $x = (1 - a^{1/n})f \in A^c$, where $e \in A^c$ is given by $e := \pi_\omega(f^{1/k_1}, f^{1/k_2}, \ldots)$. □

(A.8) For every non-zero positive contraction $c \in A^c$ and positive contraction $g \in A_\omega$ with $cg = g$ there is a positive contraction $d \in A^c$ with $\|d\| = 1$ and $dc = cd = \|c\|d$ and $dg = g$.

In particular, for every non-zero positive contraction $c \in A^c$ there is a positive contraction $d \in A^c$ with $\|d\| = 1$ and $dc = cd = \|c\|d$.

Proof. It follows that $\|c\| = 1$. By (A.4) one gets $d := x_0^* x_0$ as contractive solution of $P_1 = 0 = P_2$ for $P_1(x, x^*) = g - x^*xg$ and $P_2(x, x^*) = x^*xc - x^*x$. The approximate solutions are given by $x := c^n$.

If only $c \in A^c$ is given, we can suppose $\|c\| = 1$. By (A.3) there is $g \in (A_\omega)_+$ with $gc = cg = g$. Thus there is a positive contraction $d \in A^c$ with $dg = g$ and $cd = d$. □

(A.9) If A is separable and B is a separable C*–subalgebra of A^c such that the image of B in $A^c/\mathrm{Ann}(A)$ contains 1, then for every separable C*–subalgebra C of A_ω there is a *-homomorphism h from B into $(A+B+C)' \cap A_\omega$ such that $h(B \cap \mathrm{Ann}(A)) \subset \mathrm{Ann}(A)$ and the image of $h(B)$ in $A^c/\mathrm{Ann}(A)$ contains 1.

Proof. Let H_∞ denote the free involutive semi-group on countably many generators, and let $C^*(H_\infty) := C^*(\ell_1(H_\infty))$. Since $C^*(H_\infty)$ is projective, there are *-homomorphisms $h_n \colon C^*(H_\infty) \to A$ such that

$$h_\omega = (h_1, h_2, \ldots)_\omega \colon\ f \in C^*(H_\infty) \mapsto \pi_\omega(h_1(f), h_2(f), \ldots) \in A_\omega$$

is an epimorphism from $C^*(H_\infty)$ onto B. Let e_1 a strictly positive contraction of the kernel of h_ω, e_2 a strictly positive contraction in $h_\omega^{-1}(B \cap \mathrm{Ann}(A))$ and e_3 a positive contraction in $C^*(H_\infty)$ with $h_\omega(e_3) + \mathrm{Ann}(A) = 1$ in $A^c/\mathrm{Ann}(A)$, and let $a \in A_+$ a strictly positive contraction in A. A suitable subsequence $(h_{k_n})_{n\in\mathbb{N}}$ induces the desired homomorphism from $C^*(H_\infty)$ into $(A+B+C)' \cap A_\omega$ with $(h_{k_1}, h_{k_2}, \ldots)_\omega(e_1) = 0$. More precisely, given a separable C*–algebra D of $\ell_\infty(A)$ with $\pi_\omega(D) \supset A + B + C$, one can find the subsequence $k_1, k_2, \ldots$, such that $\lim_{n\to\infty} \|h_{k_n}(e_1)\| = 0$, $\lim_{n\to\infty} \|h_{k_n}(e_2)a\| +$

$\|a - h_{k_n}(e_3)a\| = 0$ and $(h_{k_1}(f)d_1 - d_1h_{k_1}(f), h_{k_2}(f)d_2 - d_2h_{k_2}(f), \ldots)$ is in $c_0(A)$ for every $f \in C^*(H_\infty)$ and $d \in D$. $\square$

(A.10) If A is antiliminary (=NGCR) then for every positive $b \in A_\omega$ with $\|b\| = 1$ there exists a *-monomorphism ψ from $C_0((0,1], \mathcal{K})$ into A_ω with $b\psi(c) = \psi(c)$ for every $c \in C_0((0,1], \mathcal{K})$.

Proof. Let $(b_1, b_2, \ldots) \in \ell_\infty(A)_+$ a representing sequence for b with $\|b_n\| = 1$, and let $D_n := \overline{(b_n - (n-1)/n)_+ A (b_n - (n-1)/n)_+}$. Then $bc = c$ for all elements c in $\prod_\omega \{D_n\,;\, n \in \mathbb{N}\} \subset A_\omega$.

Since $C_0((0,1], \mathcal{K}) \subset \prod_\omega \{C_0((0,1], M_n)\,;\, n \in \mathbb{N}\}$, is suffices to find faithful *-homomorphisms $\psi_n\colon C_0((0,1], M_n) \to D_n$. By the Glimm halving lemma (*cf.* [15, lem. 6.7.1]) there is a non-zero *-homomorphism $h_n\colon C_0((0,1], M_n) \to D_n$. Let E_n the hereditary C*-subalgebra of $D_n \subset A$ generated by $h_n(f_0 \otimes e_{1,1})$. If M is a maximal Abelian C*-subalgebra of E_n with $h_n(f_0 \otimes e_{1,1}) \in M$, then M can not contain a minimal idempotent, because A is antiliminary. It follows that h_n can be replaced by a *-*mono*morphism $\psi\colon C_0((0,1], M_n) \to D_n$. $\square$

Remarks 3.1. The below listed additional properties of A^c and $\mathrm{Ann}(A)$ are not needed for the proofs of our main results. A is not necessarily separable.

(i) Suppose that A is a σ-unital C*–algebra. The double annihilator $\mathrm{Ann}(\mathrm{Ann}(A))$ of $\mathrm{Ann}(A)$ in A_ω is nothing else the hereditary C*–subalgebra D_A of A_ω generated by A.

(ii) If A is a simple C*-algebra, then for every $g, h \in (A_\omega)_+$ with $\|g\| = \|h\| = 1$ there is $z \in A_\omega$ with $\|z\| = 1$ and $zz^*g = zz^*$, $z^*zh = z^*z$. In particular, $\mathrm{Ann}(A)$ does not contain a non-zero closed ideal of A_ω if A is simple.

(iii) Suppose that A is σ-unital. A^c contains an approximate unit of A_ω. More precisely: For every countable subset $X \subset \mathcal{M}(A_\omega)$ there is $b \in (A^c)_+$ with $ba = a = ab$ for $a \in A$, $\|b\| = 1$, $cb = bc$, $A(c - cb) = \{0\} = (c - cb)A$ and $\|bc\| = \|b\|$ for all $c \in X$ (*cf.* [9]).
Thus, the inclusion map $A^c \hookrightarrow A_\omega$ is non-degenerate and the induced natural *-monomorphism from $\mathcal{M}(A^c)$ into $\mathcal{M}(A_\omega)$ is a *-isomorphism from $\mathcal{M}(A^c)$ onto $A' \cap \mathcal{M}(A_\omega)$. The isomorphism maps $\mathcal{M}(A^c, \mathrm{Ann}(A)) := \{t \in \mathcal{M}(A^c)\,;\, tA^c \subset \mathrm{Ann}(A)\}$ onto $\mathrm{Ann}(A, \mathcal{M}(A_\omega))$. It follows

$$(A' \cap \mathcal{M}(A)_\omega)/\mathrm{Ann}(A, \mathcal{M}(A)_\omega) \cong \mathcal{M}(A^c)/\mathcal{M}(A^c, \mathrm{Ann}(A)) \cong A^c/\mathrm{Ann}(A)\,,$$

because $A^c \subset A' \cap \mathcal{M}(A)_\omega \subset A' \cap \mathcal{M}(A_\omega)$.

(iv) Suppose that A is a σ-unital C*–algebra. $D_A := \overline{AA_\omega A}$. The non-degenerate *-homomorphism ρ from $(A^c/\mathrm{Ann}(A)) \otimes^{\max} A$ into D_A defines a natural unital *-monomorphism from

$$A^c/\mathrm{Ann}(A) \cong (A^c/\mathrm{Ann}(A)) \otimes 1_{\mathcal{M}(A)} \subset \mathcal{M}((A^c/\mathrm{Ann}(A)) \otimes^{\max} A)$$

into $A' \cap \mathcal{M}(D_A) = \mathcal{M}(A)' \cap \mathcal{M}(D_A)$. It is an isomorphism from $A^c/\mathrm{Ann}(A)$ onto $A' \cap \mathcal{M}(D_A)$, because A is σ-unital.

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