Non-separable AF-algebras

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§1. AF-algebras

Def. Λ : directed set

☆ $(A_\lambda, \varphi_{\lambda,\lambda'})$: inductive system of finite dimensional C^*-algebras

$\underset{\text{def}}{\Longleftrightarrow}$ A_λ : fin. dim. C^*-alg. $(\lambda \in \Lambda)$

$\varphi_{\lambda,\lambda'} : A_\lambda \to A_{\lambda'}$ $*$-hom. $(\lambda \lesssim \lambda')$

s.t. $\varphi_{\lambda,\lambda} = \mathrm{id}_{A_\lambda}$

$\varphi_{\lambda',\lambda''} \circ \varphi_{\lambda,\lambda'} = \varphi_{\lambda,\lambda''}$

☆ A : AF-algebra

$\underset{\text{def}}{\Longleftrightarrow}$ $A \cong \varinjlim_{\Lambda} (A_\lambda, \varphi_{\lambda,\lambda'})$ for some ind. sys. of fin. dim. C^*-algs $(A_\lambda, \varphi_{\lambda,\lambda'})$

Problem

AF-alg. $\overset{?}{=}$ locally fin. dim. C^*-alg.

②

Rem If an AF-alg. is separable, then we can take $\Lambda = \mathbb{N}$.

Thm (Elliott '76)

A, B : separable AF-algs

$$A \cong B \iff K_0(A) \cong K_0(B)$$

(as scaled ordered groups)

Ex. (Kishimoto)

X : set

$$A_X := \bigotimes_X M_2(\mathbb{C})$$

$$A_X \cong A_{X'} \iff X \cong X'$$

same cardinality

$$K_0(A_X) \cong \mathbb{Z}[\tfrac{1}{2}] \quad (\forall X)$$

Def A **Bratteli diagram** of ind. sys. of fin. dim. C*-algs $(A_\lambda, \varphi_{\lambda,\lambda'})$

is $$\left(\begin{array}{ll} V = \coprod_{\lambda\in\Lambda} V_\lambda , & P = \coprod_{\lambda \lneq \lambda'} P_{\lambda,\lambda'} \\ m : V \to \mathbb{Z}_+ , & s \times r : P_{\lambda,\lambda'} \to V_\lambda \times V_{\lambda'} \end{array} \right)$$

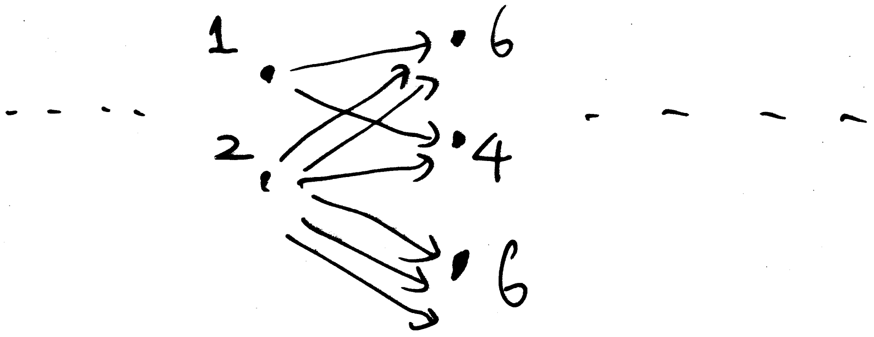
where

$$A_\lambda \cong \bigoplus_{v\in V_\lambda} M_{m(v)}(\mathbb{C}) \qquad (\lambda\in\Lambda)$$

and

$$\begin{tikzcd} \varphi_{\lambda,\lambda'} : A_\lambda \arrow[r] & A_{\lambda'} \\ M_{m(v)}(\mathbb{C}) \arrow[u, phantom, "\cup"] \arrow[r, dashed] & M_{m(v')}(\mathbb{C}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\#\{p\in P_{\lambda,\lambda'} : s(p)=v,\ r(p)=v'\}$ = multiplicity



Rem

For an AF-alg $A = \varinjlim_{\lambda \in \Lambda} (A_\lambda, \varphi_{\lambda,\lambda'})$,

$K_0(A)$ is determined by the Bratteli diagram of $(A_\lambda, \varphi_{\lambda,\lambda'})$.

Thm (Bratteli '72)

For a separable AF-alg $A = \varinjlim (A_\lambda, \varphi_{\lambda\lambda'})$, its Bratteli diagram determines A up to isomorphism.

This is not the case for non-separable AF-algebras

§2. A construction

X: uncountable set

$X_2 := \{ z \subset X : |z| = 2 \}$

Def For $z \in X_2$,

$M_z := M_2(\mathbb{C})$ w/ matrix unit $\{e^z_{i,j}\}_{i,j=1}^{2}$.

For $x \in X$, define $p_x \in \prod_{z \in X_2} M_z$ by

$$p_x := \sum_{x \in z} e^z_{1,1} \quad \text{(converges strongly)}.$$

$$A := C^*\Big(\bigoplus_{z \in X_2} M_z , \{p_x\}_{x \in X} \Big)$$

Def For $z \in X_2$,

$M'_z := M_2(\mathbb{C})$ w/ matrix unit $\{e^z_{x,y}\}_{x,y \in z}$.

For $x \in X$, define $q_x \in \prod_{z \in X_2} M'_z$ by

$$q_x := \sum_{x \in z} e^z_{x,x} .$$

$$B := C^*\Big(\bigoplus_{z \in X_2} M'_z , \{q_x\}_{x \in X} \Big)$$

Lem $x, y \in X$ with $x \neq y$

$$p_x \cdot p_y = e^{\{x,y\}}_{1,1} \neq 0$$

$$q_x \cdot q_y = 0$$

Def For $\lambda \subset X$ with $|\lambda| < \infty$,

$$A_\lambda := \mathrm{span}\Big\{ \bigoplus_{z \subset \lambda} M_z ,\ \{p_x\}_{x \in \lambda} \Big\} \subset A$$

$$B_\lambda := \mathrm{span}\Big\{ \bigoplus_{z \subset \lambda} M'_z ,\ \{q_x\}_{x \in \lambda} \Big\} \subset B.$$

Lem

★ For $\lambda \subset X$ with $|\lambda| < \infty$,

$$A_\lambda \cong \bigoplus_{z \subset \lambda} M_2(\mathbb{C}) \oplus \bigoplus_{x \in \lambda} \mathbb{C} \cong B_\lambda .$$

★ For $\lambda \subset \lambda' \subset X$ with $|\lambda'| < \infty$,

$$\begin{tikzcd}
A_\lambda \arrow[r, phantom, "\cong"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_{z \subset \lambda} M_2(\mathbb{C}) \oplus \bigoplus_{x \in \lambda} \mathbb{C} \arrow[r, phantom, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B_\lambda \arrow[d, hook] \\
A_{\lambda'} \arrow[r, phantom, "\cong"] & \bigoplus_{z \subset \lambda'} M_2(\mathbb{C}) \oplus \bigoplus_{x \in \lambda'} \mathbb{C} \arrow[r, phantom, "\cong"] & B_{\lambda'}
\end{tikzcd}$$

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Lem

☆ Ideal gen. by $[A, A] = \bigoplus_{z\in X_2} M_z$

Ideal gen. by $[B, B] = \bigoplus_{z\in X_2} M'_z$

☆ $0 \to \bigoplus_{z\in X_2} M_z \to A \xrightarrow{\pi_A} \bigoplus_{x\in X} \mathbb{C} \to 0$

$0 \to \bigoplus_{z\in X_2} M'_z \to B \xrightarrow{\pi_B} \bigoplus_{x\in X} \mathbb{C} \to 0$

(exact)

Lem

π_A does not split, while π_B splits.

(proof)

- $\delta_x \mapsto q_x$ gives a splitting map for π_B.
- To the contrary, suppose
  $\exists\, \delta_x \mapsto P'_x$ a splitting map for π_A.

$\Rightarrow {}^{\exists}\{P'_x\}_{x\in X}$: mutually orthogonal projections in A

s.t. $P_x - P'_x \in \bigoplus_{z\in X_2} M_z$

Choose $Y \subset X$ countable infinite subset.

$\forall y \in Y$

$$\mathcal{F}_y := \{x \in X : \|(P_y - P_y')|_{M_{\{x,y\}}}\| \geqq \tfrac{1}{2}\}$$

is a finite set. Take $x_0 \notin Y \cup \bigcup_{y \in Y} \mathcal{F}_y$.

$$\mathcal{F}_{x_0} := \{y \in X : \|(P_{x_0} - P_{x_0}')|_{M_{\{x_0,y\}}}\| \geqq \tfrac{1}{2}\}$$

is a finite set. Take $y_0 \in Y \setminus \mathcal{F}_{x_0}$.

Then for $z = \{x_0, y_0\} \in X_2$, in M_z,

$$P_{x_0} = P_{y_0} = e^{z}_{1,1}, \quad P'_{x_0} \perp P'_{y_0},$$

$$\|P_{x_0} - P'_{x_0}\| < \tfrac{1}{2}, \quad \|P_{y_0} - P'_{y_0}\| < \tfrac{1}{2}.$$

It is a contradiction. ▨

Thm 1 (TK)

$A = \varinjlim_{\lambda \subset X} A_\lambda$ and $B = \varinjlim_{\lambda \subset X} B_\lambda$ are non-isomorphic AF-algebras whose Bratteli diagrams are the same.

§3. A prime but not primitive AF-algebra

Def. A : C^*-algebra

- ★ A : prime $\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ $I_1 \cap I_2 \neq 0$ for non-zero ideals I_1, I_2 of A
- ★ A : primitive $\underset{\text{def}}{\Longleftrightarrow}$ A has a faithful irreducible representation

Rem. A : primitive $\Rightarrow$ A : prime

Thm (Dixmier '60) A : separable C^*-alg.

A : primitive $\Longleftrightarrow$ A : prime

Thm (Weaver '03)

$^\exists$A : prime but not primitive C^*-alg.

Thm 2 (TK)

$^\exists$prime but not primitive AF algebra

X : uncountable set

$X_n := \{\lambda \subset X : |\lambda| = n\}$

$\Lambda := \coprod_{n=1}^{\infty} X_n$: directed set

Def $\lambda \in X_n \subset \Lambda$

$\ell(\lambda) := \{t : \{1, \dots, n\} \to \lambda : \text{bijection}\}$

$M_\lambda := M_{n!}(\mathbb{C})$

with matrix unit $\{e^{\lambda}_{s,t}\}_{s,t \in \ell(\lambda)}$

Def $\lambda \in X_n \subset \Lambda$

For $s, t \in \ell(\lambda)$, define $f^{\lambda}_{s,t} \in \prod_{\lambda \in \Lambda} M_\lambda$ by

$$f^{\lambda}_{s,t} := \sum_{\substack{\lambda \subset \mu \\ u : \{n+1, \dots, k\} \to \mu \setminus \lambda}} e^{\mu}_{su,tu}$$

$A_\lambda := \mathrm{span}\{f^{\lambda}_{s,t} : s, t \in \ell(\lambda)\} \cong M_{n!}(\mathbb{C})$

Lem $\lambda, \mu \in \Lambda$

$$A_\lambda \cdot A_\mu = \begin{cases} A_\lambda & \text{if } \lambda \supset \mu \\ A_\mu & \text{if } \lambda \subset \mu \\ 0 & \text{else.} \end{cases}$$

Def $A := C^*(A_\lambda : \lambda \in \Lambda) = \overline{\sum_{\lambda \in \Lambda} A_\lambda}$

Prop A is an AF-algebra.

Lem

$$\begin{array}{ccc} \{\mathcal{F} \subset \Lambda : \lambda \in \mathcal{F}, \lambda < \mu \Rightarrow \mu \in \mathcal{F}\} & \overset{1:1}{\longleftrightarrow} & \overset{\text{ideals}}{\{I \triangleleft A\}} \\ \rotatebox{90}{$\in$} & & \\ \mathcal{F} & \longmapsto & I_{\mathcal{F}} := \overline{\sum_{\lambda \in \mathcal{F}} A_\lambda} \\ \mathcal{F}_I = \{\lambda \in \Lambda : A_\lambda \subset I\} & \longleftarrow\!\shortmid & I \end{array}$$

A : prime

$I_1, I_2 \triangleleft A$: non-zero ideals

$$\Rightarrow \underset{\lambda_1}{\underset{\rotatebox{90}{$\in$}}{\mathcal{F}_{I_1}}}, \underset{\lambda_2}{\underset{\rotatebox{90}{$\in$}}{\mathcal{F}_{I_2}}} \subset \Lambda : \text{non-empty}$$

$$\Rightarrow \underset{\lambda_1 \vee \lambda_2}{\underset{\rotatebox{90}{$\in$}}{\mathcal{F}_{I_1} \cap \mathcal{F}_{I_2}}} \subset \Lambda : \text{non-empty}$$

$$\Rightarrow I_1 \cap I_2 \neq 0.$$

A : not primitive

Suffice to see

$\forall \varphi \in S(A), {}^{\exists} I \triangleleft A$ non-zero $\varphi(I) = 0$.

Take φ : state of A.

Let $n \in \mathbb{Z}_+$. $\{f^{\lambda}_{\star,\star}\}_{\lambda \in X_n, \star \in \ell(\lambda)}$: mutually ortho.

$\Omega_n = \{\lambda \in X_n : {}^{\exists}\star \in \ell(\lambda), \varphi(f^{\lambda}_{\star,\star}) \neq 0\}$

is countable.

${}^{\exists} x_0 \in X$, s.t. $x_0 \notin \lambda \quad {}^{\forall}\lambda \in \Omega_n, {}^{\forall} n$.

$\mathcal{F}_{x_0} := \{\lambda \in \Lambda : x_0 \in \lambda\}$

$I := I_{\mathcal{F}_{x_0}} := \overline{\sum_{\lambda \in \mathcal{F}_{x_0}} A_\lambda}$: non-zero ideal

$x_0 \in \lambda \Rightarrow \lambda \notin \bigcup_n \Omega_n \Rightarrow \varphi(A_\lambda) = 0$

Hence $\varphi(I) = 0$. We are done. ▨

Question

A : primitive $\overset{?}{\Longleftrightarrow}$ A : prime and has a faithful cyclic representation.