Modular Invariants, Subfactors and Twisted K-theory

David E. Evans (Cardiff)

- examples

  Ising

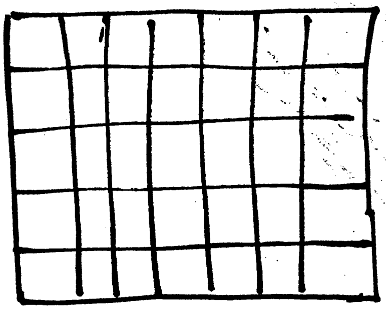
  SU(n)

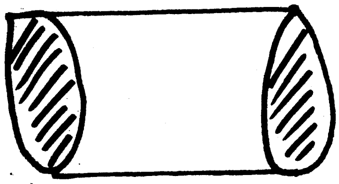
  q doubles
  including Haagerup

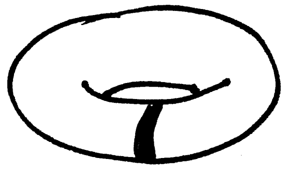
- twisted equivariant K-theory

Böckenhauer
Kawahigashi
Pinto
Behrend

Nest

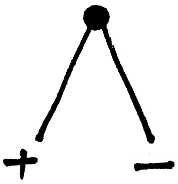


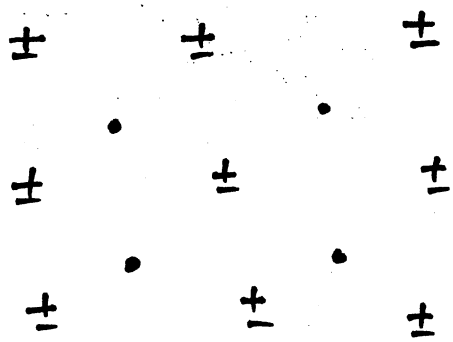


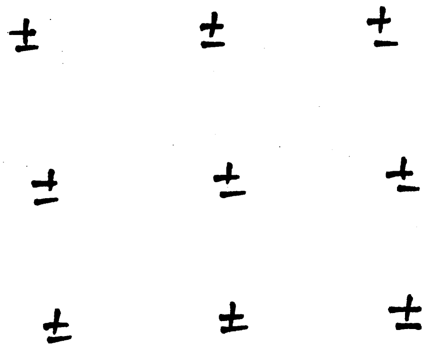


Stat Mech

Conformal field theory

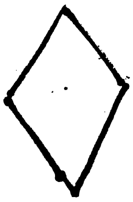






Ising

$SU(2)_2$

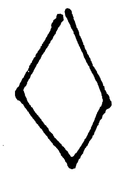
 $\in \left(\otimes M_n\right)^{SU_q(n)} =$ Hecke

Hecke	loop group
state / spin	+ve energy rep
II_1	III_1
bimodules	endo / sectors
6j Ocneanu	Q Longo

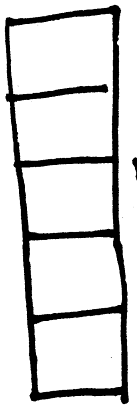
Ising $\quad C\{\pm\}^{\mathbb{Z}^2} = \otimes_{\mathbb{Z}^2} \mathbb{C}^2 \rightsquigarrow \otimes M_2$

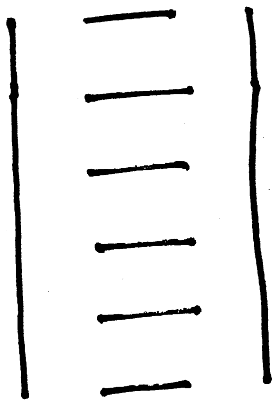
$$\mu(F) = \varphi_\mu(F_\beta)$$

$$Z = \sum_\sigma \exp - H(\sigma) = \sum_\sigma \prod \text{weights}$$



$$T = V^{1/2} W V^{1/2}$$

ξ  η

 $\in M_{2^n}$

$$V = \exp K_2 \sum \sigma_j^x \sigma_{j+1}^x$$

$$W = \exp K_1^* \sum \sigma_j^z$$

$$\sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$T = 0$ $\qquad\qquad$ $T = \infty$

$$\varphi_0^+ = \bigotimes_{\mathbb{N}} \omega_{\begin{pmatrix}1\\0\end{pmatrix}}$$

$$\varphi_0^- = \bigotimes_{\mathbb{N}} \omega_{\begin{pmatrix}0\\1\end{pmatrix}}$$

$$\varphi_\infty = \bigotimes_{\mathbb{N}} \omega_{\begin{pmatrix}1/2\\1/2\end{pmatrix}}$$

$$\varphi_0^+ = \varphi_\infty \circ \nu$$

$$\nu \quad \sigma_j^z = \sigma_j^x \sigma_{j+1}^x \qquad \nu\, \sigma_j^x \sigma_{j+1}^x = \sigma_{j+1}^z$$

$$\nu\, \sigma_j^x = \prod_{i=1}^{j} \sigma_i^z$$

$\nu^2 \neq$ shift $\qquad\qquad$ $\nu^2 \sigma_j^x = \sigma_1^x \sigma_{j+1}^x$

$\nu^2|_{\text{even}} = \text{shift}|_{\text{even}}$

$$\bigotimes_{\mathbb{N}} M_2 \subset \mathcal{O}_2 = C^*(S_+, S_-)$$

$$S_+ S_+^* + S_- S_-^* = 1$$

$$\nu \quad \frac{1}{\sqrt{2}}(S_+ + \sigma S_-) = S_+ S_\sigma S_\sigma^* + S_- S_{-\sigma} S_{-\sigma}^*$$

$$\sigma = \pm$$

$$\nu^2 S_\sigma = S_+ S_\sigma S_+^* + S_- S_{-\sigma} S_-^*$$

$$\nu^2 x = S_+ x S_+^* + S_- \alpha(x) S_-^*$$

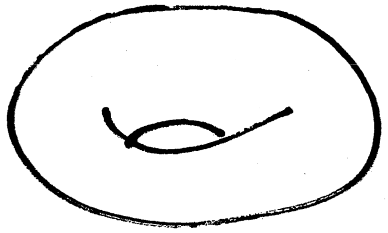
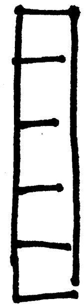
$\alpha : S_+ \leftrightarrow S_-$

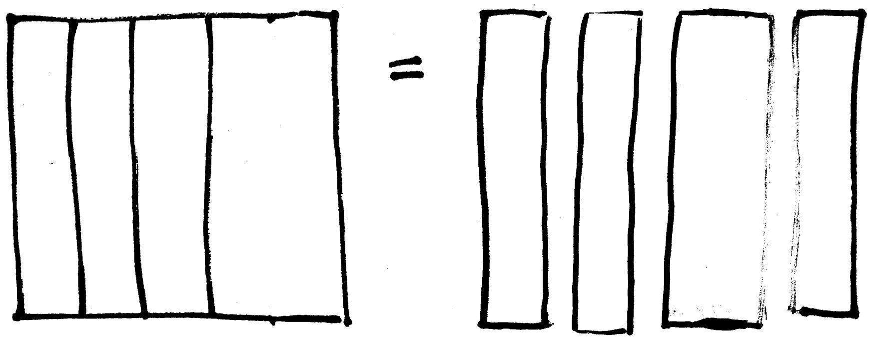
$$\nu^2 = id \oplus \alpha$$

$$\alpha \nu = \nu$$

$$\begin{tikzcd} & \nu \arrow[dl, dash] \arrow[dr, dash] & \\ id & & \alpha \end{tikzcd}$$

- states / spins
- +ve energy reps
- endos
- boundary conditions
- twisted K-theory

 $T =$  $= e^{-\mathcal{H}}$

$Z =$  $=$ trace T^N

$\rightsquigarrow \sum' Z_{\lambda\mu}\, \chi_\lambda\, \bar{\chi}_\mu \qquad \chi_\lambda = \mathrm{tr}\, q^{L_0 - c/2}$

$q = e^{2\pi i \tau}$

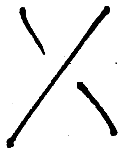
$= Z(\tau) \quad SL(2, \mathbb{Z})$ invariant

Modular invariant:

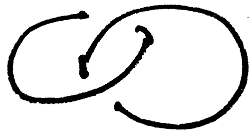
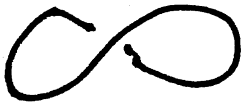
- $[Z_{\lambda\mu}] \in SL(2, \mathbb{Z})'$
- $Z_{\lambda\mu} \in 0, 1, 2, \ldots$
- $Z_{00} = 1$

subfactor / inclusion setting

$\mathcal{A} \subset \mathrm{End}(N)$

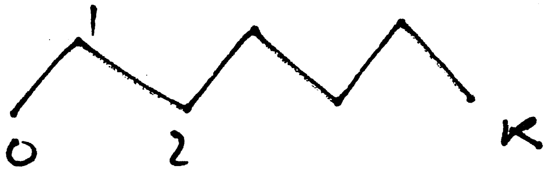
braided $\qquad \varepsilon =$ 

$\lambda\mu = \mathrm{Ad}\,\varepsilon\,\mu\lambda$

S, T  

e.g.:

- loop groups $SU(n)$ — Wassermann, Laredo-Toledano

$SU(2)_k$ 

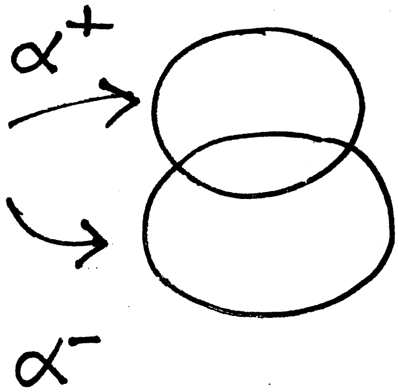
- q double of subfactor — Ocneanu, Popa, Longo-Rehren

$N \subset M \qquad \tau L \in \Sigma \mathcal{A}$

$Z_{\lambda\mu} := \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ is mod invt

BEK

$\mathcal{A} = {}_N\mathcal{X}_N$



G : action of ${}_N\mathcal{X}_N$ on ${}_N\mathcal{X}_M$

$G_\lambda = [G_{\lambda ab}]_{ab}$

$\sigma(G_\lambda) = \left\{ \dfrac{S_{\lambda\rho}}{S_{0\lambda}} \ : \ \text{multiplicity } Z_{\rho\rho} \right\}$

$\# \, N\text{-}M \quad = \quad \operatorname{tr} Z$

$\# \, M\text{-}M \quad = \quad \operatorname{tr} Z^* Z$

$Z \qquad N \subset M$

$Z^*Z \qquad N \subset \tilde{M}$

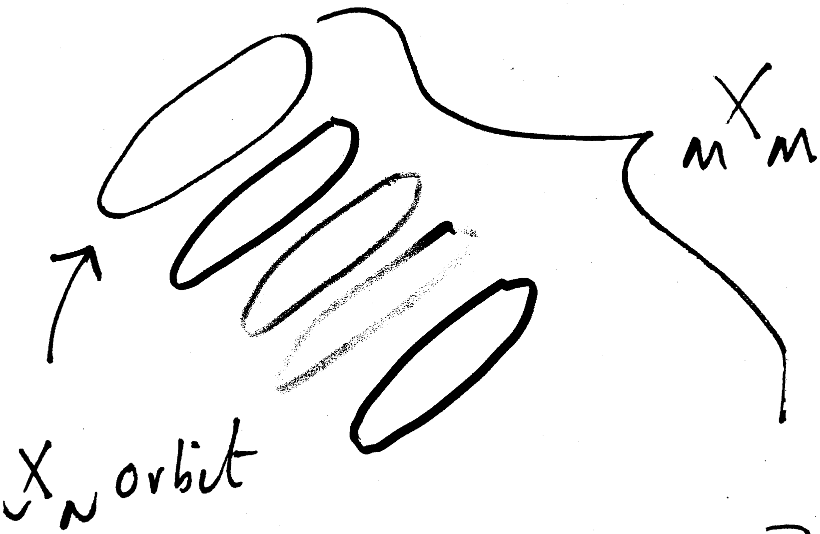
st $\quad {}_M X_M \simeq {}_N X_{\tilde{M}}$

$\qquad\; \| \qquad\qquad\quad \|$

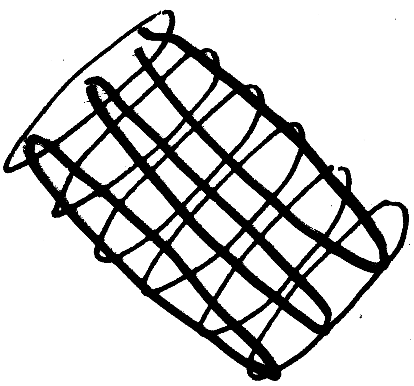
$\quad \operatorname{tr} Z^*Z \qquad \operatorname{tr} Z^*Z$

$N \subset \tilde{M}_p$

$Z^*Z = \sum' Z_p$



$Z_a Z_b = \sum_c n_{ab}^c Z_c$



Z^*Z

ZZ^*

Pinto, DEt

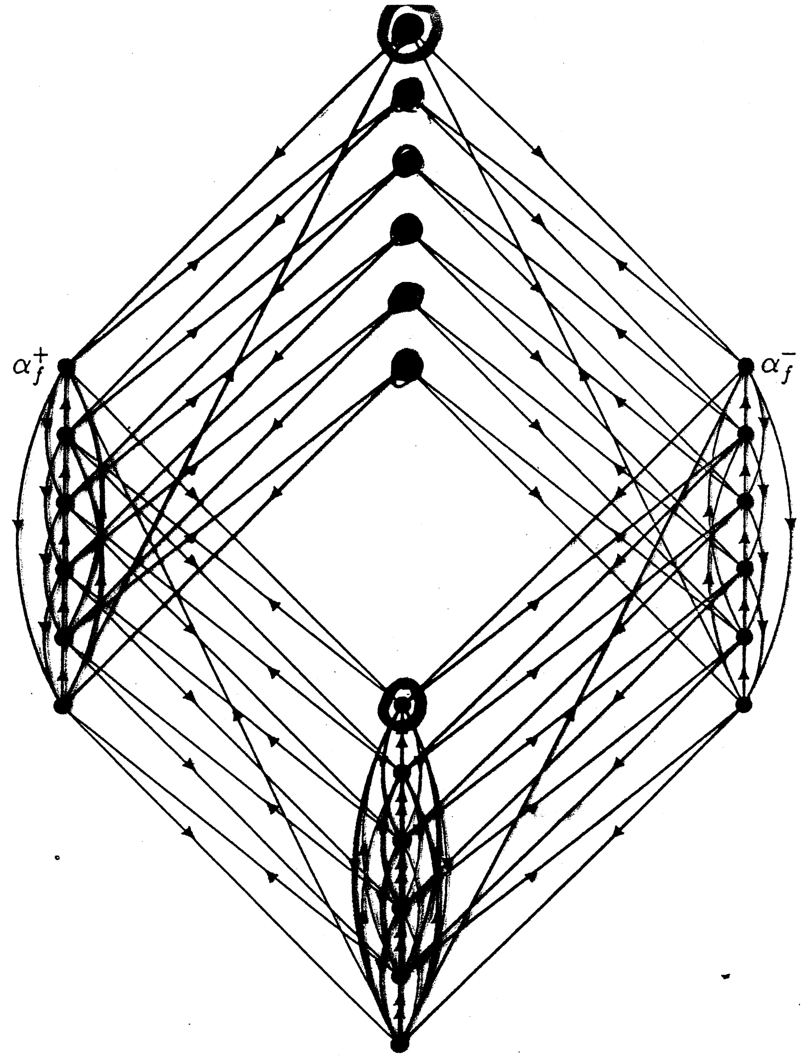


Figure 31: $SU(3)_5 \subset SU(6)_1$, $\mathcal{E}^{(8)}$: Fusion graph of the two generators

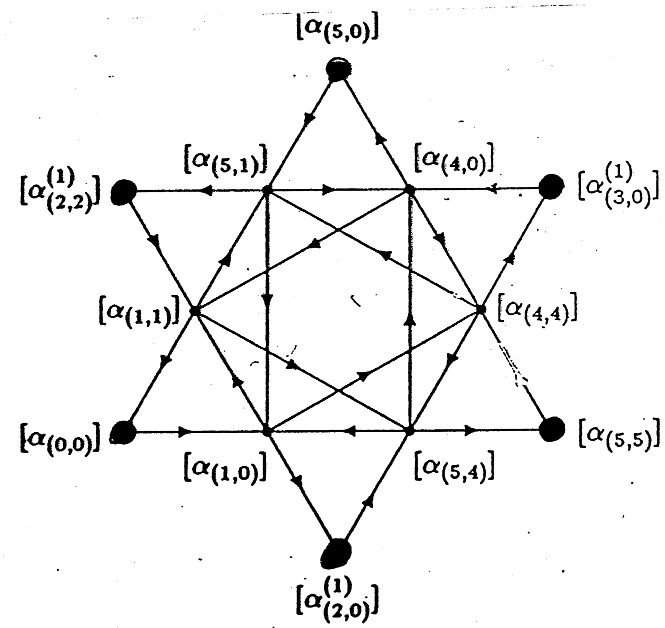


Figure 6: $SU(3)_5 \subset SU(6)_1$: Fusion graph of $[\alpha_{(1,0)}]$: $\mathcal{E}^{(8)}$

$$
\begin{aligned}
Z_{\mathcal{E}^{(8)}} &= |\chi_{(0,0)} + \chi_{(4,2)}|^2 + |\chi_{(2,0)} + \chi_{(5,3)}|^2 + |\chi_{(2,2)} + \chi_{(5,2)}|^2 \\
&\quad + |\chi_{(3,0)} + \chi_{(3,3)}|^2 + |\chi_{(3,1)} + \chi_{(5,5)}|^2 + |\chi_{(3,2)} + \chi_{(5,0)}|^2
\end{aligned}
$$

$N \subset M$ $\quad \sum_{t \subset B} x_t \overline{x_t}$

$\cancel{A} \quad \mathcal{B}$ or $\sum x_t \overline{x_{\sigma(t)}}$

$$Z = \sum \left(\sum b_{t\lambda} \chi_\lambda \right) \left(\sum b_{t\mu} \chi_\mu \right)^-$$

$$Z_{\lambda\mu} = \sum_t b_{t\lambda} b_{t\mu}$$ type I $\Rightarrow Z_{\lambda\mu} = Z_{\mu\lambda}$

$$Z_{\lambda\mu} = \sum_t b_{t\lambda} b_{\sigma(t)\mu}$$ type II $\Rightarrow Z_{0\mu} = Z_{\mu 0}$

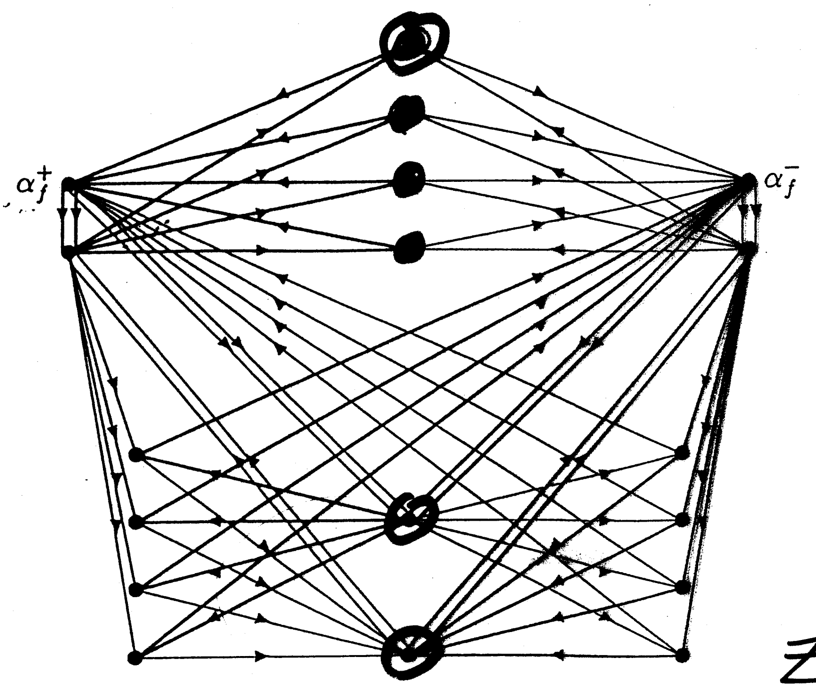
$$N \subset M_\pm \subset M$$

$$\mathcal{B}_\pm \subset \mathrm{End}(M_\pm)$$

Böckenhauer
E

$\exists$ examples $M_+ \neq M_-$
even for $SU(n)$

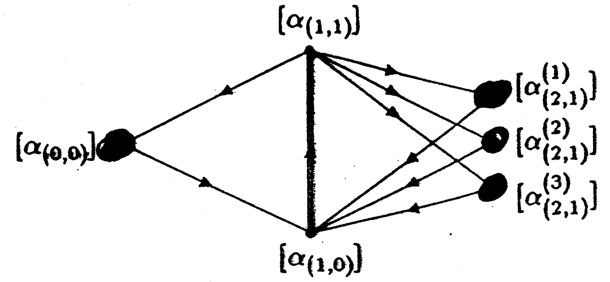
math OA 0204278



$\mathrm{tr}\, ZZ^* = 18$

$ZZ^* = 3Z$

Figure 30: $SU(3)_3 \subset SO(8)_1$, $\mathcal{D}^{(6)}$: Fusion graph of the two generators



$n\chi_M$

$\mathrm{tr}\, Z = 6$

Figure 5: $SU(3)_3 \subset SO(8)_1$: Fusion graph of $[\alpha_{(1,0)}]$: $\mathcal{D}^{(6)}$

$$Z_{\mathcal{D}^{(6)}} = |\chi_{(0,0)} + \chi_{(3,0)} + \chi_{(3,3)}|^2 + 3\,|\chi_{(2,1)}|^2$$

$$N \underset{\iota}{\subset} M \longrightarrow Z$$

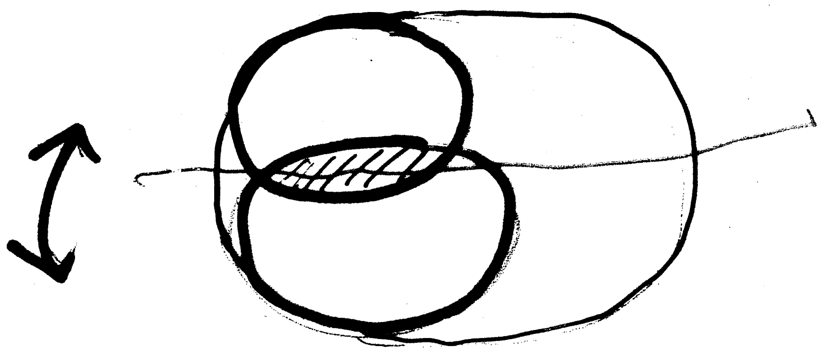
? $\exists$ canonical subfactor

for SU(2)

$$[\bar{\iota}\iota] = \sum_{\lambda \text{ even}} Z_{\lambda\lambda} [\lambda]$$

$$[\iota\bar{\iota}] = \sum_{\substack{\beta \text{ even} \\ \text{"real"}}} [\beta]$$

$$\beta \in {}_M X_M = X^+ \vee X^-$$



X^+

$\updownarrow$

X^-

$SU(n)_k \qquad n > 2$

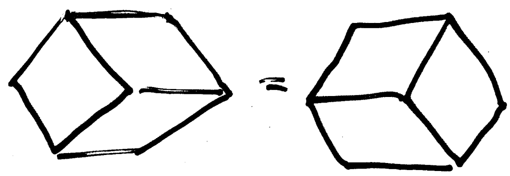
$$Z = C = [\delta_{\lambda \bar{\mu}}]$$

nimreps

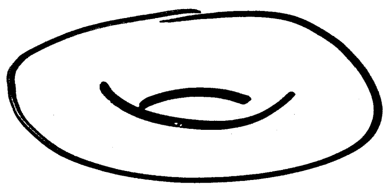
- $n = 3$
  - Behrend, Pearce, Petkova, Zuber
  - Böckenhauer, DEE
  - Ocneanu
- $n = 4$ Ocneanu
- $n \geq 3$
  - Birke, Fuchs, Schweigert
  - Gannon Gaberdiel
  - Petkova Zuber

Integrable weights on A^c

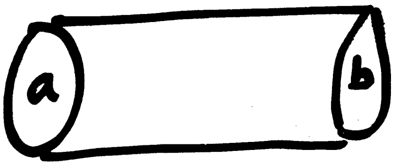
- YBE



Behrend E

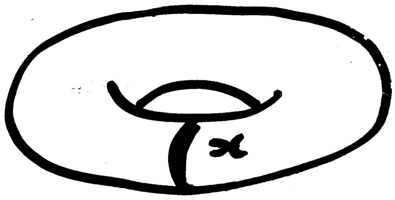


torus partition function



boundary CFT

$a, b \in {}_N X_M$



twists or defect lines

$x \in {}_M X_M$

eg $H = \bigoplus Z_{\lambda\mu} H_\lambda \otimes \overline{H}_\mu$

$Z_x = \text{trace } x \, q^{L_0 - c/24} \, \bar{q}^{L_0 - c/24}$

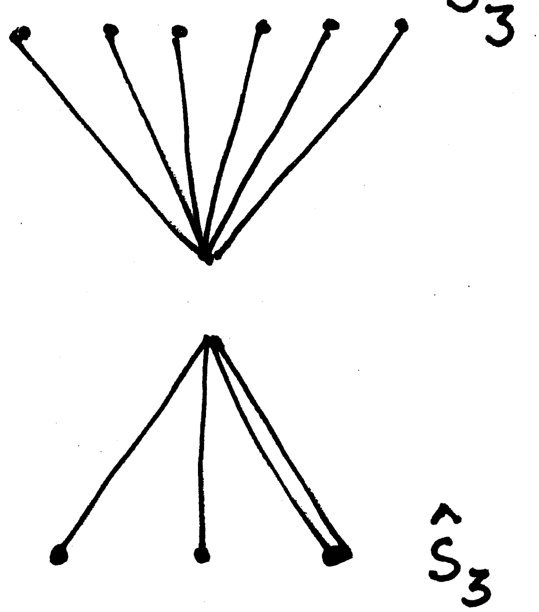
degree of freedom

$= \sum Z_{\lambda\mu}^2 = \# {}_M X_M$

q doubles

Pinto, E

$q S_3$

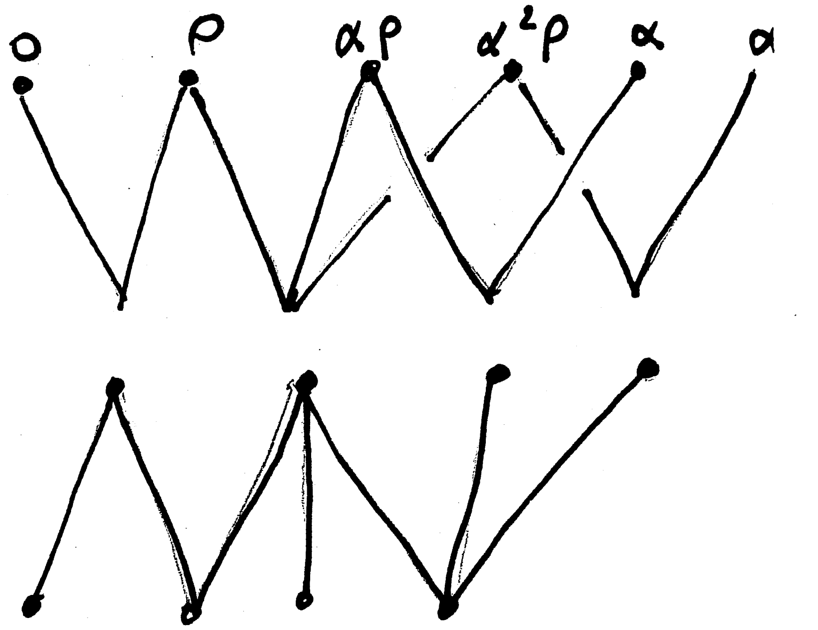


$$\alpha^3 = 1$$
$$\alpha\rho = \rho\alpha^2$$
$$\rho^2 = 1$$

48 MI

14	6	28
nimless	nimble not realised	realised

q Haagerup



$$\alpha^3 = 1$$
$$\alpha\rho = \rho\alpha^2$$
$$\rho^2 = 1 +$$

28 MI

20	1	7
not realised	?	realised

K-theory ?

Verlinde algebra $\simeq$ ${}^{\tau}K_{*}^{SU(n)}\, SU(n$

$SU(n)_k$ — Freed, Hopkins, Telem

qdb. G — Lusztig

as rings

R. Nest + DEE : work in progress

$A \subset A \rtimes G \qquad G \subset \mathrm{Aut}(A)$

$$\alpha_g \alpha_h = \mathrm{Ad}\, u(g,h)\, \alpha_{gh}$$

$$\partial u = \omega \in H^3(G, \mathbb{T})$$

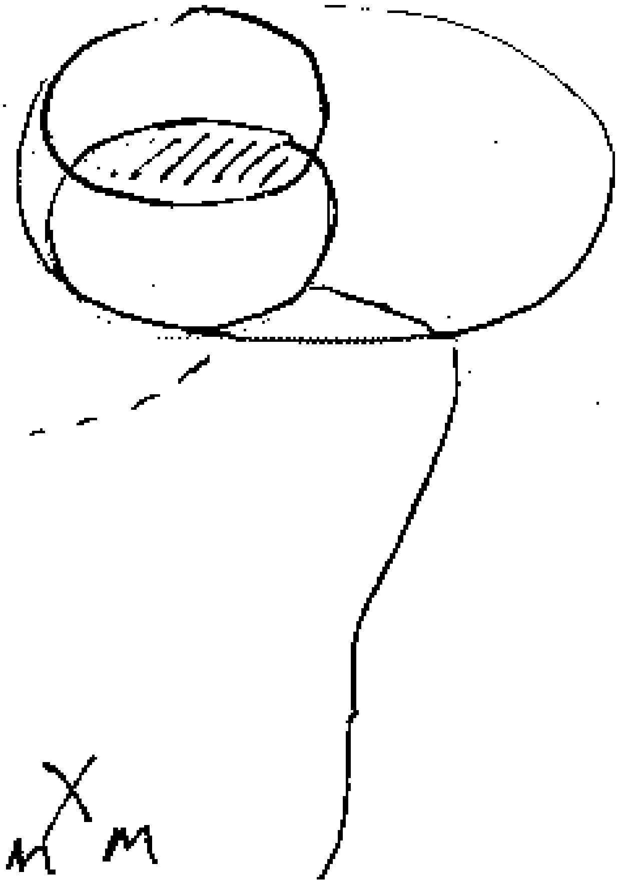
$${}^{\tau}K_0^{\Delta - \Delta^{opp}}(G \times G)$$

$\Delta \qquad \Delta^{opp}$

$\Delta(G) = \Delta \subset G \times G$
diagonal

$H \subset G \times G \qquad \psi \subset H^2(H, \mathbb{T})$

module categories (Ostrik)



$$
\begin{array}{c}
{}_N X_N \\
\circlearrowright\ K_0^{\Delta - \Delta^{opp}}(G \times G) \\
\alpha^+ \Big\downarrow \Big\downarrow \alpha^- \\
K_0^{H - H^{opp}}(G \times G) \simeq {}_M X_M
\end{array}
$$

$${}_N X_M \simeq K_0^{\Delta - H^{opp}}(G \times G)$$

$(\psi = 0,\ \sigma(H) = H$
$\sigma = \text{flip}$

$$\boxed{X^0} \simeq K_0^{L - L^{opp}}(L \times L)$$

- braiding
- α - induction

- Z, b as KK?