Abel Symposium

Voksenåsen, Oslo, 2004/09/04

Def. Let $A \in \mathcal{B}(\mathcal{H})$. A (closed) subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ is A-invariant if $A(\mathcal{H}_0) \subseteq \mathcal{H}_0$

(i.e. $Ap = pAp$ where $p : \mathcal{H} \longrightarrow \mathcal{H}_0$ is the projection)

Def. We say $\mathcal{H}_0$ is affiliated to $vN(A)$ if $p \in vN(A)$

Def. $\mathcal{H}_0$ is A-hyperinvariant if $\mathcal{H}_0$ is B-invariant whenever $B \in \mathcal{B}(\mathcal{H})$ and $AB = BA$

The hyperinvariant subspace problem

Does every $A \in \mathcal{B}(\mathcal{H}) \setminus \mathbb{C}I$ have a nontrivial hyperinvariant subspace ?

<u>Prop.</u> If $\mathcal{H}_0$ is A-hyperinvariant, then $\mathcal{H}_0$ is affiliated to $vN(A)$.

The converse is not true. (finite dimensional examples)

<u>Prop.</u> $\exists\ A \in \mathcal{B}(\mathcal{H})$ s.t. $vN(A)$ is a II_1-factor and s.t. $\exists$ an A-invariant subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ that is affiliated to $vN(A)$ but that is not A-hyperinvariant.

<u>Prop.</u> [Tucci] We can arrange that $vN(A)$ is the hyperfinite II_1-factor.

Note:

$$\{A\text{-invariant subspaces affiliated to } vN(A)\}$$
$$\simeq \{p \in \mathcal{P}roj(vN(A)) \mid Ap = pAp\}$$

Prop. Given a von Neumann algebra $\mathcal{M}$ generated by $A \in \mathcal{M}$,

(*) $\{p \in \mathcal{P}roj(\mathcal{M}) \mid$ the range of p is an A-hyperinvariant subspace $\}$

does not depend on the representation $\mathcal{M} \hookrightarrow \mathcal{B}(\mathcal{H})$.

Qn. is $\exists$ a representation-independent description of the set (*) ?

LCV

<u>Qn.</u> Suppose $p \in \mathrm{Proj}(vN(A))$, satisfies

$$B p = p B p$$

for all $B \in vN(A)$ s.t.

$$A B = B A.$$

Is p necessarily A-hyperinvariant?

<u>ans.</u> No!

(finite dimensional counter-example).

Suppose additionally $vN(A)$ is a factor.

??

V

The Brown measure of $A \in \mathcal{M}$

II_1-factor

is $\mu_A = \frac{1}{2\pi} \text{Laplace}_\lambda \left(\log \Delta(A - \lambda 1) \right)$.

Kadison-Fuglede determinant

It is a "spectral distribution" measure for A.

$$\operatorname{supp}(\mu_A) \subseteq \sigma(A).$$

<u>Prop.</u> [Haagerup] $\operatorname{supp}(\mu_A) = \{0\}$ iff $((A^*)^n A^n)^{1/n} \to 0$ strongly as $n \to \infty$.

<u>Thm.</u> [Haagerup] If F is closed in $\mathbb{C}$, then $\exists$ a maximal $p(F) = p \in \mathcal{P}roj(\mathcal{M})$ s.t. $Ap = pAp$ and

$$\operatorname{supp}(\mu_{pAp}) \subseteq F.$$

$$\underset{p\mathcal{M}p}{\uparrow}$$

In fact, p is A-hyperinvariant

<u>Thm.</u> [Haagerup + Schultz]

$$\tau(p(F)) = \mu_A(F)$$

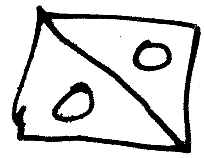
<u>Cor.</u> If $A \in \mathcal{M}$, a II_1-factor and if μ_A has support of more than one point, then A has a nontrivial, hyperinvariant subspace.

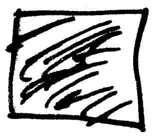
a construction of the quasi-nilpotent DT-operator T.

[D + Haagerup].

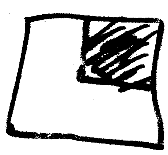
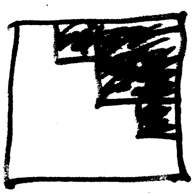
$L(F_2)$ – generated by a copy $\mathcal{D}$ of $L^\infty[0,1]$ and a semicircular element X that are free.

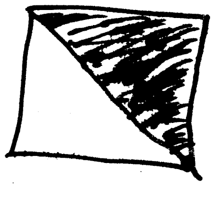
From the random matrix model [Voiculescu]:

$\mathcal{D} \ni f =$ 

, $X =$ 

Use projections from $\mathcal{D}$ to cut X:

$T_1 =$  , $T_2 =$ 

$T_n \longrightarrow T =$ 

Thm. [Śniady].

$$\tau\left(\left((T^*)^k T^k\right)^n\right) = \frac{n^{nk}}{(nk+1)!}$$

(Conjectured by [D + Haagerup]).

Consequence.

$k\left((T^*)^k T^k\right)^{1/k}$ and T^*T have the same moments.

Thm. [D + Haagerup]. For some F $F\left(k\left((T^*)^k T^k\right)^{1/k}\right)$ converges in s.o.-topology to $f \in D$, with $f(t) = t$.

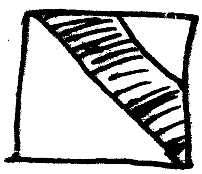
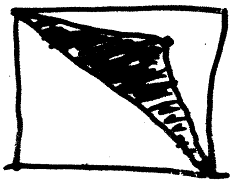
Consequence: The T-invariant projections $\chi_{[0,t]} \in D$ are affiliated to $vN(T)$.

<u>Thm.</u> [D + Haagerup]. $\chi_{[0,t]} \in \mathcal{D}$ is the projection onto

$$\left\{ \xi \in \mathcal{H} \;\middle|\; \limsup_{n\to\infty} \frac{n}{e} \| T^n \xi \|^{2/n} \le t \right\}$$

and is, therefore, T-hyperinvariant.

What about analogues of T?

e.g.  or 

<u>Def.</u> Let $\mathcal{M}$ be a von Neumann algebra and $E: \mathcal{M} \to \mathcal{D} \subseteq \mathcal{M}$ be a conditional expectation onto a vN subalgebra $\mathcal{D}$.

Let $\alpha, \beta : \mathcal{D} \to \mathcal{D}$ be u.c.p. maps.

We say $Z \in \mathcal{M}$ is $\mathcal{D}$-circular (w.r.t. E) having covariance (α, β) if $\forall\ d \in \mathcal{D}$

$$E(Z^* d Z) = \alpha(d)$$

$$E(Z d Z^*) = \beta(d)$$

and each

$$E(Z^{\varepsilon(1)} d_1 Z^{\varepsilon(2)} d_2 \cdots Z^{\varepsilon(n-1)} d_{n-1} Z^{\varepsilon(n)}),$$

$$\varepsilon(j) \in \{*, 1\}, \quad d_j \in \mathcal{D}$$

is evaluated via "nested iteration," summing over all non-crossing pairings of

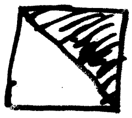
$$\varepsilon(i) = * \text{ and } \varepsilon(j) = 1.$$

(a special case of [Speicher]'s $\mathcal{D}$-Gaussian operators.)

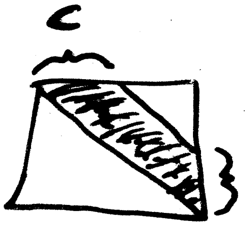
Examples:

(1) a usual circular operator is a $\mathbb{C}$-circular operator with covariance (id, id).

(2) The quasinilpotent DT-operator T is an $L^\infty[0,1]$-circular operator with covariance (α, β), where $(\alpha f)(t) = \int_t^1 f(s)\,ds$

$T =$ 

$(\beta f)(t) = \int_0^t f(s)\,ds$

(3) If $S =$ 

, then

S is $L^\infty[0,1]$-circular with covariance (α_c, β_c), where

$(\alpha_c f)(t) = \int_t^{\min(t+c,1)} f(s)\,ds$

$(\beta_c f)(t) = \int^{t} f(s)\,ds$.

Method of [Foias + Jung + Ko + Pearcy] for constructing hyperinvariant subspaces of $Q \in \mathcal{B}(\mathcal{H})$:

Use spectral resolutions of $Q^k (Q^*)^k$ applied to a particular vector $x_0 \in \mathcal{H}$.

We modify [FJKP] to treat certain classes of $\mathcal{D}$-circular operators.

Let $\mathcal{M}$ be a II_1-factor.

$E : \mathcal{M} \to \mathcal{D} \subseteq \mathcal{M}$ a τ-preserving conditional expectation.

Def. Singular numbers of $A \in \mathcal{M}$:

$$s_\theta(A) = \inf\left\{ \| A(1-p) \| \;\middle|\; \begin{array}{l} p \in \mathrm{Proj}(\mathcal{M}) \\ \tau(p) \le \theta \end{array} \right\}$$

Thm. Let $Q \in \mathcal{M}$. Suppose $\exists\ \mu_n \geq 0$
$\exists\ \theta \in (0,1)$ s.t.

$$\lim_{n\to\infty} \frac{\mu_n}{s_\theta(Q^n (Q^*)^n)} = 0$$

and $\exists\ p \in \mathbb{N} \cup \{0\}$
$\exists$ vectors $\xi_n \in L^2(\mathcal{M}, \tau)$
s.t.

$$\xi_n = 1_{[0,\mu_n]}\left(E(Q^{n+p}(Q^*)^{n+p})\right)\xi_n$$

and $\xi_n \longrightarrow \xi \neq 0$ in the norm of $L^2(\mathcal{M}, \tau)$.

Then Q has a nontrivial hyperinvariant subspace.

Given a finite Borel measure ν on $[0,1]^2$ satisfying certain conditions, let

$$\alpha_\nu, \beta_\nu : L^\infty[0,1] \longrightarrow L^\infty[0,1]$$

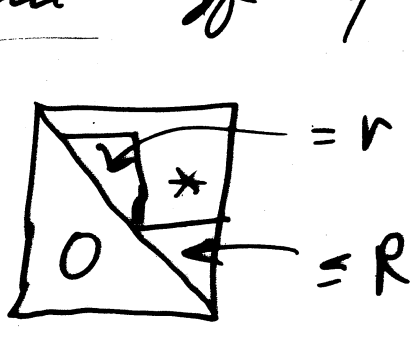
be

$$(\alpha_\nu f)(t) = \int_0^1 f(y)\, \nu(t, dy)$$

$$(\beta_\nu f)(t) = \int_0^1 f(x)\, \nu(dx, t).$$

Such (α_ν, β_ν) are precisely the covariance of $L^\infty[0,1]$ - circular operator z_ν in tracial vN algebra.

<u>Thm.</u> If ν looks like



then z_ν has a nontrivial hyperinvariant subspace.