Definition. Let A, B locally convex algebras, $p \in [1, \infty]$

$$kk_i^{(p)}(A, B) \underset{\mathrm{Def}}{=} \varinjlim_{n} \left[J^{n-i} A, \mathcal{C}_p \otimes_\pi S^n B \right]$$

where

$[\cdot, \cdot]$ = { $\mathcal{C}^\infty$-homotopy classes of homoms $\cdot \to \cdot$ }

$\mathcal{C}_p$ Schatten ideal ($\{x \mid \mathrm{Tr}\, |x|^p < \infty\}$)

$\otimes_\pi$ projective tensor product

$SA = \mathcal{C}_0^\infty(0,1) \otimes_\pi A$ (suspension)

$TA = \overline{(A \oplus A^{\otimes 2} \oplus \ldots\ )}^{\text{locconv}}$, $A \to TA$ cont. linear

$$0 \longrightarrow JA \longrightarrow TA \longrightarrow A \longrightarrow 0$$

$$x_1 \otimes \ldots \otimes x_n \longmapsto x_1 x_2 \ldots x_n$$

Properties (same formal properties as Kasparov's KK)

- $kk_0^{(p)}$ additive (triangulated) category
  objects: loc. conv. alg. morphism sets: $kk_0^{(p)}(A,B)$
- every extension $0 \to I \to A \to B \to 0$ with continuous linear splitting gives long exact sequences in $kk_0^{(p)}(D, \cdot\,)$ and $kk_0^{(p)}(\,\cdot\,, D)$
- $kk_n^{(p)}(SA,B) = kk_{n-1}^{(p)}(A,B)$, $kk_n^{(p)}(A,SB) = kk_{n+1}^{(p)}(A,B)$
  and $S^2B \underset{kk}{\cong} B$ (Bott periodicity)
- Pimsner - Voiculescu sequence for crossed products of loc. conv. algebras.

etc. etc ...

Remark. Analogous construction on cat. of C^*-algebras ($\mathcal{C}_p \rightsquigarrow \mathcal{K}$, $\otimes_\pi \rightsquigarrow \otimes_{min}$, $SA \to SA$, $TA \to T_{C^*}A$, $\mathcal{C}^\infty \rightsquigarrow \mathcal{C}$)

gives $$KK_i(A,B) = \varinjlim_n \left[J_{C^*}^{n-i} A, \mathcal{K} \otimes S^n B \right]$$

The natural map

$$kk_n^{(1)}(A, B) \longrightarrow kk_n^{(p)}(A, B)$$

is an isomorphism for all $1 \leq p < \infty$.

Thus only two theories

$$kk_*^{(1)}(\cdot, \cdot) \quad \text{and} \quad kk_*^{(\infty)}(\cdot, \cdot)$$

The "coefficient ring" $R^{(p)} = kk_0^{(p)}(\mathbb{C},\mathbb{C})$

is a commutative unital ring (for each $p \in [1,\infty]$).

Product in $kk_*^{(p)}$ and all natural constructions in $kk_*^{(p)}$ are $R^{(p)}$-linear.

Problem: $R^{(p)} = ?$

(I could show that $R^{(p)} = \mathbb{Z}$ for an analogous construction of kk_* on category of "m-algebras").

<u>Example</u>. The Weyl algebra

$$W = \{ \text{n.c. polynomials in } x, y \,,\; xy - yx = 1 \}$$

is a locally convex algebra with "fine" topology.

Can show that $W \underset{kk_0^{(p)}}{\cong} \mathbb{C}$.

Therefore $kk_0^{(p)}(\mathbb{C}, W) = R^{(p)}$

Analysis of $R^{(p)}$. (with A. Thom)

Lemma. E functor : {loc. conv. alg.} $\to$ {ab. groups} such that

- E $\mathcal{K}_{-\infty}$-stable ($\mathcal{K}_{-\infty}$ = {rap. decr. matrices})
- E $\mathcal{C}^\infty$-homotopy invariant
- E half-exact: $0 \to I \to A \to B \to 0$ induces $E(I) \to E(A) \to E(B)$

Then (a) E has long exact sequences

$$\dots \to E(SA) \to E(SB) \to E(I) \to E(A) \to E(B)$$

(b) $E(JA) \cong E(SA)$

(c) E is Bott periodic $\quad (E(S^2A) = E(A))$

Proof (a) ✓

(b) Consider $0 \to JA \to TA \to A \to 0$, $E(TA) = 0$

$$\to \underset{0}{\underset{\shortparallel}{E(STA)}} \to E(SA) \xrightarrow{\cong} E(JA) \to \underset{0}{\underset{\shortparallel}{E(TA)}} \to$$

(c) Consider $0 \to \mathcal{K}_{-\infty} \hat{\otimes}_\pi A \to J_0 \hat{\otimes}_\pi A \to SA \to 0$

Toeplitz extension. Show $E(J_0 \hat{\otimes}_\pi A) = 0$

$$\to \underset{0}{\underset{\shortparallel}{E(SJ_0\hat{\otimes}_\pi A)}} \to E(S^2A) \to \underset{E(A)}{\underset{\shortparallel}{E(\mathcal{K}_{-\infty}\hat{\otimes}_\pi A)}} \to \underset{0}{\underset{\shortparallel}{E(J_0\hat{\otimes}_\pi A)}} \to$$

Assume moreover that $E : \{\text{loc. conv. alg.}\} \to \{\text{ab. groups}\}$ is weakly $\mathcal{C}_p$ - stable :

$$\begin{tikzcd} E(A) \arrow[r, "E(j)"] \arrow[rr, bend right, "id"'] & E(\mathcal{C}_p \otimes_\pi A) \arrow[r, "\exists"] & E(A) \end{tikzcd}$$

Then $\alpha \in kk_0^{(p)}(A, B)$ induces $E(\alpha) : E(A) \to E(B)$ as follows : Let α be represented by $[\varphi] \in [J^{2n}A, \mathcal{C}_p \otimes_\pi S^{2n}B]$

$$\begin{tikzcd} E(J^{2n}A) \arrow[r, "E(\varphi)"] \arrow[d, equal] & E(\mathcal{C}_p \otimes_\pi S^{2n}B) \arrow[r, "\exists"] & E(S^{2n}B) \arrow[d, equal] \\ E(A) \arrow[rr, "E(\alpha)"] & & E(B) \end{tikzcd}$$

By construction of product in $kk_0^{(p)}$, have

$$E(\alpha \cdot \beta) = E(\alpha) E(\beta)$$

Example. Chern-Connes character into cyclic homology.

Put $E(\cdot) = HP_0(A, \cdot)$, functor with properties above for $p < \infty$.

Get
$$kk_0^{(p)}(A,B) \longrightarrow HP_0(A,B)$$
$$\alpha \longmapsto E(\alpha)\cdot 1_A$$
$$1_A \in HP_0(A,A)$$

In particular
$$R^{(p)} \longrightarrow HP_0(\mathbb{C},\mathbb{C}) = \mathbb{C} \quad (p<\infty!)$$

unital ring homomorphism, whence $R^{(p)} \neq 0$.

//

Consider now case $p = \infty$, $\mathcal{C}_p = \mathcal{K}$ C^*-algebra

<u>Thm</u> (Kasparov - Higson). Let $F : \{C^*\text{-alg.}\} \to \{\text{ab. groups}\}$ be a (cov.) functor such that

- F split exact: $0 \to I \to A \leftrightarrows B \to 0$ induces
$$0 \to E(I) \to E(A) \leftrightarrows E(B) \to 0$$
- F $\mathcal{K}$-stable : $F(A) \xrightarrow{\cong} F(\mathcal{K} \otimes A)$

Then F is homotopy invariant :

$$F(A[0,1]) \xrightarrow{F(ev_t)} F(A) \quad \text{constant } t \in [0,1]$$

Consequence :
Let $E : \{\text{loc. conv. alg.}\} \to \{\text{ab. groups}\}$ be defined by
$E(A) = K_0(\mathcal{K} \otimes_\pi A)$. Then E is $\mathcal{C}^\infty$-homotopy invariant :

$$E(\mathcal{C}^\infty[0,1] \otimes_\pi A) \xrightarrow{E(ev_t)} E(A) \quad \text{constant}$$

Proof Put $F(B) = K_0(\mathcal{K} \otimes_{min} B \otimes_\pi A)$, B C^*-alg.

Then F is stable (needs proof) and split exact.

Therefore $F : \{C^*\text{-alg}\} \to \{\text{ab. gr.}\}$ homotopy inv.

Now

$$\begin{tikzcd}
K_0(\mathcal{K} \otimes_\pi \mathcal{C}^\infty[0,1] \otimes_\pi A) \arrow[rr, "K_0(ev_t)"] \arrow[dr] & & K_0(\mathcal{K} \otimes_\pi A) \\
& K_0(\mathcal{K} \otimes_{min} \mathcal{C}[0,1] \otimes_\pi A) \arrow[ur, "K_0(ev_t)"'] &
\end{tikzcd}$$

constant

— — — — — —

Therefore we get unital ring homomorphism

$$R^{(\infty)} = kk_0^{(\infty)}(\mathbb{C}, \mathbb{C}) \longrightarrow \mathrm{Hom}(\overset{\overset{\mathbb{Z}}{\shortparallel}}{E(\mathbb{C})}, \overset{\overset{\mathbb{Z}}{\shortparallel}}{E(\mathbb{C})}) = \mathbb{Z}$$

and map

$$kk_0^{(\infty)}(\mathbb{C}, A) \longrightarrow \mathrm{Hom}(\overset{\overset{\mathbb{Z}}{\shortparallel}}{E(\mathbb{C})}, E(A)) = K_0(\mathcal{K} \otimes_\pi A)$$

We constructed map

$$kk_0^{(\infty)}(\mathbb{C}, A) \longrightarrow K_0(\mathcal{K} \otimes_\pi A)$$

Show that this is isomorphism (in particular $R^{(\infty)} = \mathbb{Z}$).

<u>Lemma</u> $K_0(\mathcal{K} \otimes_\pi A) = [q\mathbb{C}, \mathcal{K} \otimes_\pi A]$

(where $0 \to q\mathbb{C} \to \mathbb{C} * \mathbb{C} \to \mathbb{C} \to 0$ defines $q\mathbb{C}$).

Proof follows from homotopy invariance of left hand side.

Since $K_0(\mathcal{K} \otimes_\pi J^{2n}\mathbb{C}) = \mathbb{Z}$ by lemma above, get map

$$\beta : q\mathbb{C} \longrightarrow \mathcal{K} \otimes_\pi J^{2n}\mathbb{C}$$

On the other hand get map $\alpha : J^{2n}\mathbb{C} \longrightarrow \mathcal{K} \otimes_\pi q\mathbb{C}$ from Toeplitz extension.

Cum grano salis these two maps are homotopy inverse.

Finally:

The Kasparov - Higson argument can be made to work for stabilization by $\mathcal{C}_p$ $(p>1)$ instead of $\mathcal{K}$ (using fact that $K_1^{alg}(\mathcal{C}_p) = 0$).

Repeating the same arguments as for $p=\infty$, one obtains for $p>1$

$$\boxed{\begin{aligned} kk_0^{(p)}(\mathbb{C}, A) &= K_0(\mathcal{C}_p \otimes_\pi A) \\ kk_0^{(p)}(\mathbb{C}, \mathbb{C}) &= \mathbb{Z} \end{aligned}}$$

Example of corollary: $kk_0^{(p)}(\mathbb{C}, W) = \mathbb{Z}$.