RENORMALIZATION AND MOTIVIC GALOIS THEORY

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RENORMALIZATION GROUP
$\cong$
AMBIGUÏTY OF ϕ THEORY

1st PART : RECALL Jt WORK WITH KREIMER

2nd PART : ANSWER TO RELATION : RENORMALIZATION AND THEORY OF AMBIGUÏTY (ABEL + GALOIS)

QUANTUM FIELD THEORY
(COMPUTATIONAL ASPECT)

PROB. AMPLITUDE : $e^{i\frac{S(A)}{\hbar}}$

$S(A)=\int \mathcal{L}(A)\,d^4x$,

$\mathcal{L}(A)=\frac{(\partial A)^2}{2}-\frac{m^2}{2}A^2+\mathcal{L}_{int}(A)$

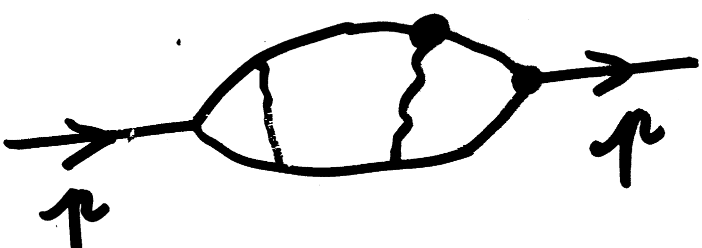
$$G_N(x_1,\dots,x_N)=N\int \underbrace{e^{i\frac{S(A)}{\hbar}}A(x_1)..A(x_N)[DA]}_{\text{FEYNMAN INT.}}$$

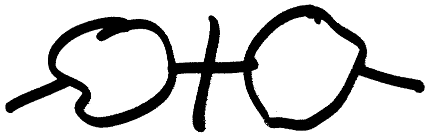
SOURCE J $\rightarrow$ $W(J)=\operatorname{Log}\int$

LEGENDRE TRANSFO $\longrightarrow$

$$S_{eff}(A)=S(A)+\sum_{\substack{\text{1PI}\\ \text{GRAPHS}}}\frac{U(\Gamma)(A)}{S(\Gamma)}$$

A^3

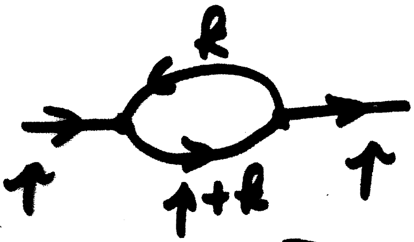


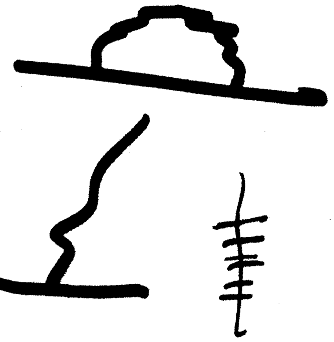


FEYNMAN GRAPHS

GRAPH $\longrightarrow$ INTEGRAL

EXAMPLE





$$U(p) = \int \frac{1}{k^2+m^2}\,\frac{1}{(p+k)^2+m^2}\,d^D k$$

DIVERGENT

PHYSICS RESOLUTION $\int_{|k|<\Lambda}$

Λ = CUTOFF $\quad \delta Z(\Lambda) \quad \delta m^2(\Lambda)$

$$\frac{1}{2}(\partial_\mu\varphi)^2(1-\delta Z) - \left(\frac{m^2+\delta m^2}{2}\right)\varphi^2 + g\,\delta g\,\varphi^3$$

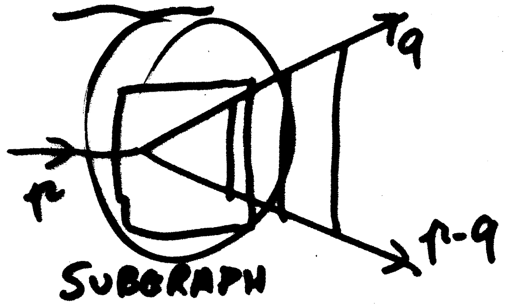
COUNTERTERMS $\qquad (\delta g)(\Lambda)$

ADJUST COUNTERTERMS ORDER BY ORDER IN PERT. THEORY TO CANCELL THE ∞

δZ, δm^2, δg ALL GO CRAZY !!

(THOMSON, LORENTZ, KRAMERS BETHE, LAMB, SCHWINGER, TOMONAGA, FEYNMAN, DYSON .....) $\qquad$ 1947 JUNE

Problems $\underline{D-\varepsilon}$



$$\frac{1}{\varepsilon^3}\, 0 + \frac{1}{\varepsilon^2} \cdots$$

① COEFF OF $\frac{1}{\varepsilon}$ NON LOCAL ($\log|p|$) ....

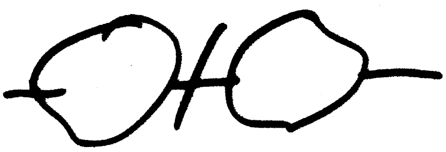
② HIERARCHY OF SUBGRAPHS

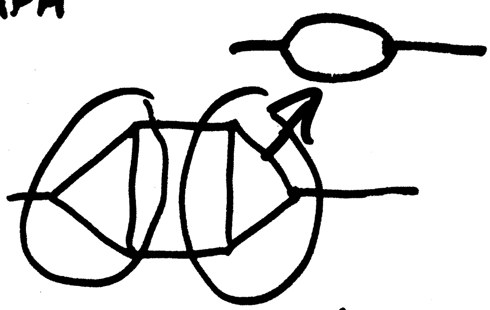
THM (BPHZ) "IT'S OK

COMBINATORIAL RECIPEE

$$\text{COUNTERTERMS} = \sum_{\Gamma} \frac{C(\Gamma)}{S(\Gamma)}$$

1PI GRAPH SYM. FACTOR





B.P. PREPARATION

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma)\, U(\Gamma/\gamma)$$

THEN

$$C(\Gamma) = -T(\bar{R}(\Gamma)) \quad \text{IS LOCAL}$$

POLE PART

RENORMALIZED GRAPH

$$R(\Gamma) = \underbrace{\bar{R}(\Gamma)}_{\text{BP PREP.}} + \underbrace{C(\Gamma)}_{\text{COUNTERTERM}}$$

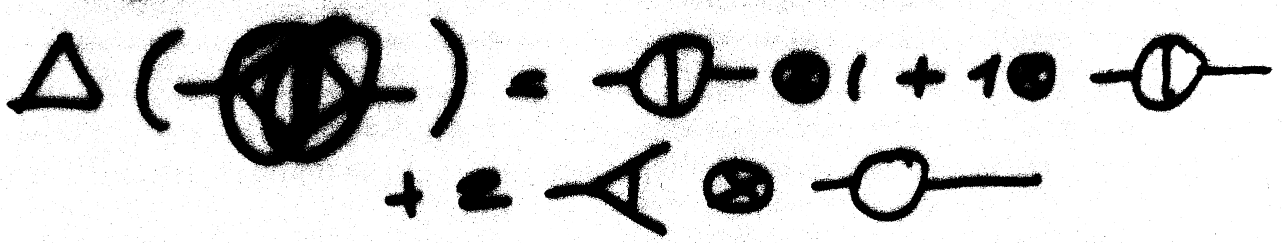
$R(\Gamma)$ IS FINITE. 1997.

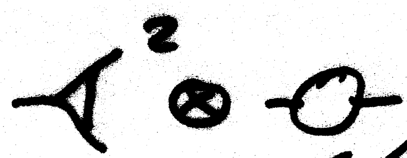
DIRK KREIMER: ROOTED TREES + ac

$$\Delta\Gamma = \underbrace{\Gamma}_{\text{1PI}} \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma$$

IS A HOPF ALGEBRA COPRODUCT!

EXAMPLE



 APPEARS IN

$(\mathcal{H}, \Delta)$ = HOPF ALGEBRA

LOOP # = GRADING.

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma)\, U(\Gamma/\gamma)$$

$$C(\Gamma) = -T(\bar{R}(\Gamma)) \qquad U \leftrightarrow \phi$$

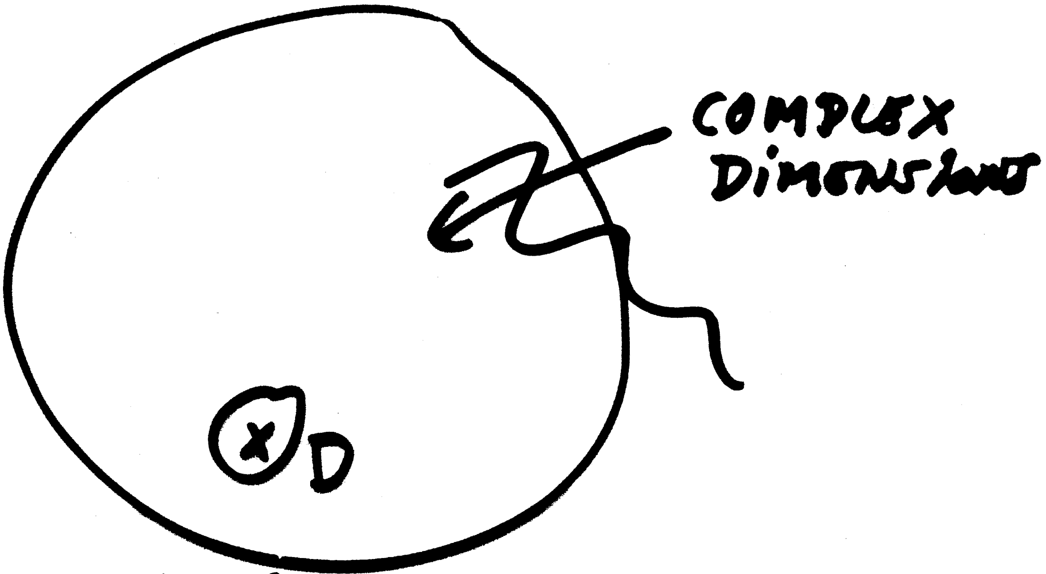
$$R(\Gamma) = \bar{R}(\Gamma) + C(\Gamma) \qquad C \leftrightarrow \phi_- \qquad R \leftrightarrow \phi_+$$

THM (ac+dk)

1) UNRENORMALIZED THEORY GIVES LOOP $\gamma(\varepsilon) \in G$

2) RENORMALIZED THEORY $= \gamma_+(0)$

$$\left(\gamma(z) = \gamma_-(z)^{-1}\, \gamma_+(z)\right)$$



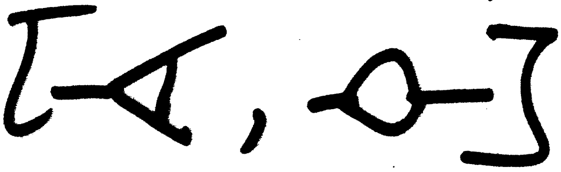
GENERAL PRINCIPLE TO GET FINITE VALUE

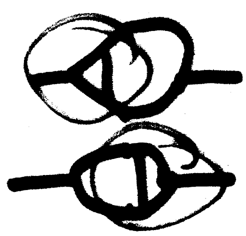
(HOPF-SAMELSON-LERAY-BOREL-CARTIER
MILNOR MOORE) THEOREM
QUILLEN / P. MAY

GRADED HOPF (COMMUTATIVE AS ALGEBRA) $= \mathcal{U}(\text{Lie})^*$

$\uparrow$ GRADED LIE

$$[\Gamma_1, \Gamma_2] = \sum \Gamma_1 \circ \Gamma_2 - \sum \Gamma_2 \circ \Gamma_1$$

1PI GRAPHS 

GRADED LIE ALGEBRA OF FEYNMAN GRAPHS 

PROBLEM WHAT IS THE MEANING OF RENORMALIZATION ???

$$\mathcal{H} \xrightarrow{\Phi} \mathbb{C}$$

ALGEBRAIC	GEOMETRIC
$\mathcal{H} \xrightarrow{\phi} \mathcal{A}$	$\varepsilon \to \gamma(\varepsilon) \in G$
ALGEBRA OF LAURENT SERIES	LOOP OF ELTS OF G
$\mathcal{H} \xrightarrow{\phi_+} \mathcal{A}_+$ REGULAR AT $\varepsilon = 0$	LOOP REGULAR AT $\varepsilon = 0$
$\mathcal{H} \xrightarrow{\phi_-} \mathcal{A}_-$ (GENERATED BY $\frac{1}{\varepsilon}$)	γ REGULAR FOR $\varepsilon \neq 0$, $\gamma_-(\infty) = 1$
ϕ $\quad \phi_-, \phi_+$	

THM (ackr) $\quad \gamma(\varepsilon) = \gamma_-(\varepsilon)^{-1}\gamma_+(\varepsilon)$

BIRKHOFF DEC. OF ϕ IS GIVEN BY $\Delta X = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

$$\phi_-(X) = -T\left(\phi(X) + \sum \phi_-(X')\phi(X'')\right)$$

$$\phi_+(X) = \phi_-(X) + \underbrace{\phi(X) + \sum \phi_-(X')\phi(X'')}$$

IT AGREES WITH

BPHZ !!!!!

WHAT is THE GROUP G ?

C.M INDEX THEORY FOR TRANSV. HYPOELLIPTIC OP. ON FOL.

$$\Downarrow$$

HOPF ALGEBRA $\mathcal{H}_{cm}$

$\mathcal{H}_{cm} \supset$ DUAL OF $\mathcal{U}$(LIE ALG. OF FORMAL VECTOR FIELDS)

$$(\operatorname{Log} \varphi'(x))^{(n)} = \delta_n$$

PB: RELATE G WITH DIFF

EFF. COUPLING CONSTANT

$$g_{eff} = g Z_1 Z_3^{-3/2}$$

$$= \Big(\sum g^{2l+1} \frac{\Gamma}{S(\Gamma)}\Big)\Big(\sum g^{2k} \frac{\Gamma}{S(\Gamma)}\Big)^{-3/2}$$

THM (ac+dk)

1) THE ABOVE GIVES A HOMOMORPHISM

$\mathcal{H}_{CM} \to \mathcal{H}_K$, $G \to$ FORMAL DIFF

2) LET THE UNRENORMALIZED EFF. $g_{eff}(\varepsilon)$ (FORMAL POWER SERIES IN g) BE CONSIDERED AS A LOOP OF F. DIFF. THEN

Birk $\to$ $g_{eff}^{-}(\varepsilon)$ IS THE BARE COUPLING CST

RENORMALIZATION GROUP

INTEGRATION IN D-ε DIM

$$\int f(k) \frac{dk^{D-\varepsilon}}{\mu^{D-\varepsilon}} \Rightarrow \text{CHOICE OF SCALE } \mu$$

LOOP $\gamma_\mu(\varepsilon)$

μ (UNIT OF MASS)

$\theta_t \in \mathrm{AUT}(G)$ (LOOP #)

$$\gamma_{e^t\mu}(\varepsilon) = \theta_{\varepsilon t}\, \gamma_\mu(\varepsilon)$$

<u>FACT</u> : γ_μ^- IS INDEPENDENT OF μ!

<u>THM</u> $\gamma^-(\varepsilon)\,\theta_{\varepsilon t}(\gamma^{-1}(\varepsilon))$ IS FINITE AT $\varepsilon = 0$ AND GIVES A ONE PAR. SUBGROUP $F_t \in G$.

$$F_t = \underline{\exp t\beta}, \quad \beta = Y \,\mathrm{Res}\, \gamma$$

GRADING RESIDUE

<u>THM</u> LET $\tilde{G} = G \rtimes_\theta \mathbb{R}$, THEN

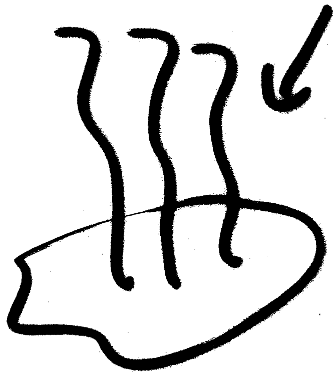
$$\gamma^-(\varepsilon) = \lim_{t \to \infty} e^{-t\left(\frac{\beta}{\varepsilon} + Z_0\right)} e^{tZ_0}$$

UNIQUELY DET. BY. RESIDUE

JT WORK WITH M. MARCOLLI

① RIEMANN-HILBERT CORRESPONDENCE

DIFFERENTIAL SYSTEM



MONODROMY

=

REPRESENTATION

$\longleftarrow \cdots \longrightarrow$

(CORR. DU BON DIEU)

GEOMETRY

GROUP THEORY

SINGULAR CASE . RAMIS

② CONCRETE PT OF DEPARTURE

WRITE SCATTERING FORMULA IN TERMS OF SOLUTIONS OF A DIFFERENTIAL SYSTEM (ex. C.M.;

TIME ORDERED EXPONENTIAL

EXPANSIONAL (ARAKI)

$\alpha(s) \in \mathrm{Lie}(G)$ $\qquad T e^{\int_a^b \alpha(t)\,dt}$

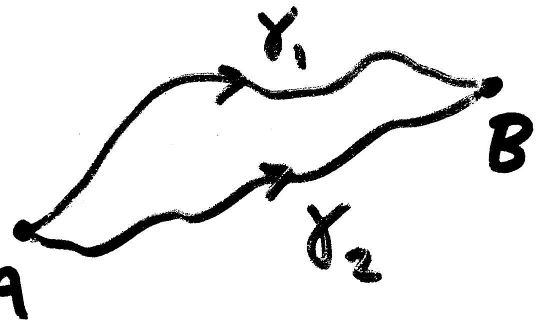
$$T e^{\int_a^b \alpha(t)\,dt} = 1 + \sum_1^\infty \int_{a \le s_1 \le \cdots \le s_n \le b} \alpha(s_1)\cdots\alpha(s_n) \prod ds_j\,.$$

$$\frac{d}{ds} g(s) = g(s)\,\alpha(s)$$

$$T e^{\int_a^c \alpha(t)\,dt} = T e^{\int_a^b \alpha(t)\,dt}\, T e^{\int_b^c \alpha(t)\,dt}$$

FLAT CONNECTION

$T e^{\int_0^1 \gamma^* \omega}$



EXPLICIT FORM OF γ_N

(2.3) $$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)) \qquad \forall t \in \mathbb{R},$$

(2.4) $$\frac{\partial}{\partial\mu}\gamma_{\mu^-}(z) = 0\,.$$

$$\gamma_N = (\gamma_N^-)^{-1}\gamma_N^+$$

Theorem *Let $\gamma_\mu(z)$ be a family of G-valued loops fulfilling* (2.3) *and* (2.4). *Then there exists uniquely $\beta \in \mathfrak{g}$ and a loop $\gamma_{\text{reg}}(z)$ regular at $z = 0$ such that*

$$\gamma_\mu(z) = \mathrm{T}e^{-\frac{1}{z}\int_\infty^{-z\log\mu}\theta_{-t}(\beta)\,\mathrm{d}t}\,\theta_{z\log\mu}(\gamma_{\text{reg}}(z))\,.$$

For any β and regular loop $\gamma_{\text{reg}}(z)$ the Birkhoff decomposition of the loop $\gamma_\mu(z)$ is given by

$$\gamma_\mu^+(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^{-z\log\mu}\theta_{-t}(\beta)\,\mathrm{d}t}\,\theta_{z\log\mu}(\gamma_{\text{reg}}(z))\,,$$

$$\gamma_\mu^-(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^\infty\theta_{-t}(\beta)\,\mathrm{d}t}\,.$$

$$\mathrm{T}e^{\underbrace{-\frac{1}{z}\int_0^{-z\log N}\theta_{-t}(\beta)\,dt}_{\to\,\log N\,\beta}}$$

FOR $z \to 0$.

EQUISINGULAR FLAT CONNECTIONS

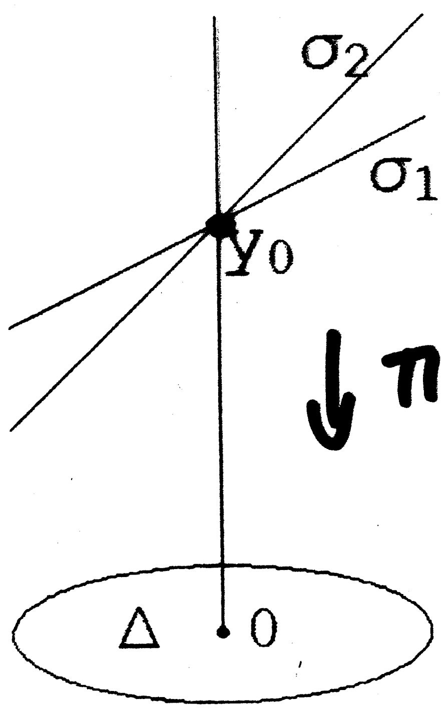
B = $\mathbb{C}^*$-PRINCIPAL BUNDLE

BASE Δ

$$Dg = g^{-1}dg$$

$$\omega_1 \sim \omega_2 \,, \quad \omega_2 = Dh + h^{-1}\omega_1 h$$

h REGULAR ON Δ

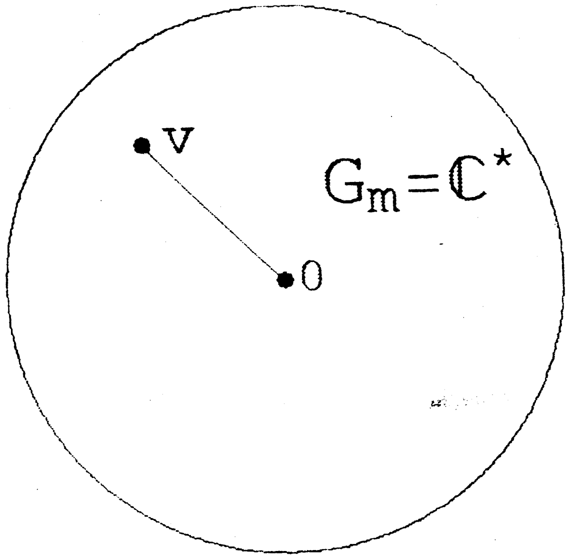


ω SINGULAR ON $V = \pi^{-1}\{0\}$

EQUISINGULAR: $\sigma^*(\omega)$ INDEPENDANT OF σ

+ PHYSICS CASE: Δ = COMPLEX DIM
FIBER = NORMAL.

CLASSIFICATION OF EQUISINGULAR FLAT CONNECTIONS



THM (ac + mm)

ω EQUISINGULAR FLAT CONNECTION

$\exists!\ \beta \in \mathrm{Lie}(G)$, $\omega \sim D\gamma$,

$$\gamma(z, v) = T e^{-\frac{1}{z}\int_0^v u^Y(\beta)\frac{du}{u}}$$

$$u^Y(\beta_n) = u^n \beta_n$$

$$\beta = \sum_1^\infty \beta_n$$

ABELIAN CATEGORY OF EQUISINGULAR FLAT BUNDLES
~

$\mathcal{E} = (E, \nabla)$

E: $\mathbb{Z}$-GRADED VECTOR SPACE

∇: FLAT EQUISINGULAR CONNECTION ON $B \times E = \tilde{E}$

$$W^{-n}(E) = \bigoplus_{m \geq n} E_m$$ WEIGHT FILTRATION

∇ COMPATIBLE WITH W.

DIFFICULTY: FILTERED $\Rightarrow$ PB. WITH STRICT.

SOLUTION 2x2 MATRIX TRICK:

$T \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ IFF

$$\begin{bmatrix} \nabla' & \nabla' T - T\nabla \\ 0 & \nabla \end{bmatrix} \sim \begin{bmatrix} \nabla' & 0 \\ 0 & \nabla \end{bmatrix}$$

TENSOR PRODUCT:

$$E_1 \otimes E_2, \ldots$$

MAIN RESULT (ac + mm)

THM EQUISINGULAR FLAT BUNDLES FORM A TANNAKIAN CATEGORY.
IT IS EQUIVALENT TO THE CAT. OF REPRESENTATIONS OF A UNIQUE AFFINE GROUP SCHEME U^*,

$U^* = \mathbb{G}_m \ltimes U$

Lie U IS THE FREE LIE ALGEBRA WITH ONE GENERATOR e_{-n} IN EACH DEGREE $n \geq 1$.

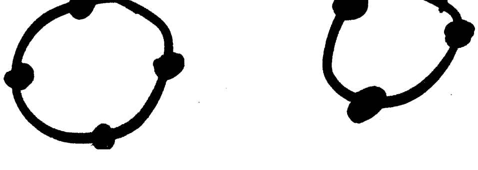
+ UNIVERSAL SINGULAR FRAME (SAME COEFFICIENTS AS IN LOCAL INDEX)

+ RENORMALIZATION GP:

$\mathbb{G}_a \to U$

+ ISOMORPHIC TO THE CATEGORY OF TATE MOTIVES

$\mathcal{M}_T(\mathcal{O}_S)$ CYCLOTOMIC



UNIVERSAL SINGULAR FRAME

$$\gamma(z,v) = T e^{-\frac{1}{z}\int_0^v u^Y(e)\,\frac{du}{u}}$$

$\sum e(-n)$

$$\gamma(z,v) = \sum_{n\geq 0}\ \sum_{k_j>0} \frac{e(-k_1)\cdots e(-k_n)}{k_1(k_1+k_2)\cdots(k_1+\cdots+k_n)}\, v^{\sum k_j} z^{-n}$$

SAME FORMULA
AS IN C.M. LOCAL
INDEX FORMULA