Periodic Solutions of Abel Equation and Signal Reconstruction from Integral Measurements

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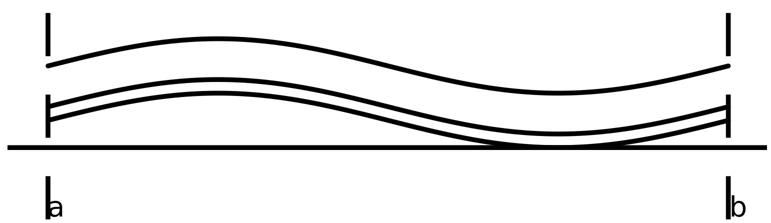
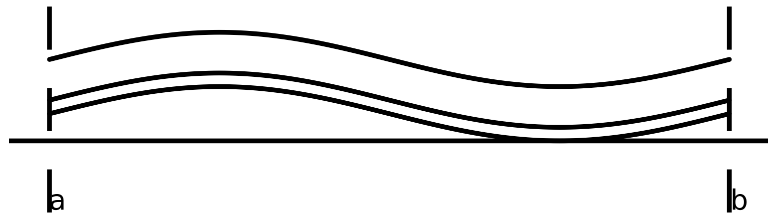
1 Motivation: Hilbert 16-th (Smale-Pugh) and Center-Focus problems

Consider the Abel differential equation

$$y' = p(x)y^2 + q(x)y^3 \tag{1.1}$$

Its solution $y(x)$ is called "periodic" on $[a, b]$ if $y(a) = y(b)$. Equation (1.1) has a "center" on $[a, b]$ if all its solutions (for $y(a)$ small enough) are periodic.





Smale-Pugh problem Bound the number of periodic solutions of (1.1). In particular, for p, q - polynomials is there a bound in terms of the degrees of p and q?

Center-Focus problem Give conditions on (p, q, a, b) for (1.1) to have a center.

Versions of the classical Hilbert 16-th and Poinceré Center-Focus problem (the simplest where the problems are still non-trivial???)

Status of the problems.

Smale-Pugh (counting periodic solutions of the Abel equation $y' = p(x)y^2 + q(x)y^3$): nothing new!

Center-Focus (conditions for all the solutions of the Abel equation to be periodic): very good progress in the last few years, especially in the case where the coefficients p, q are polynomials. ([F. Pakovich], [A. Cima, A. Gasull, F. Manosas], [J. Gine, M. Grau, J. Llibre], [M. Briskin, N. Roytvarf, Y. Y.]). More progress in understanding the Algebraic Geometry of the Center-Focus problem is expected.

So here the hope that the Abel equation case is indeed more tractable gets certain confirmation!

Counting periodic solutions requires new approaches. I'll present some initial steps in one possible direction: Analytic Continuation.

Consider the Poincaré "first return" mapping $G(y)$ of Abel differential equation (1.1)

$$y' = p(x)y^2 + q(x)y^3,$$

which associates to each initial value y at a the value $G(y)$ of the corresponding solution of the Abel equation at b.

Periodic solutions of (1.1) correspond to solutions of $G(y) = y$, and (1.1) has a center if and only if $G(y) \equiv y$.

In order to approach the problems above we have to understand the analytic nature of G, in particular, to bound the number of zeroes of $G(y) - y$, and to give conditions for $G(y) - y \equiv 0$.

Unfortunately, G does not allow for any apparent "close form representation" or even a good approximation in this form (Dynamics). The only known and pretty well understood way to analytically represent G is through Taylor expansion:

$G(y)$ is given by a convergent for small y power series

$$G(y) = y + \sum_{k=2}^{\infty} v_k(p, q, 1) y^k. \tag{1.2}$$

Here the Taylor coefficients $v_k(p, q, x)$ of G are determined through the following recurrence relation:

$$\begin{gathered} \frac{dv_k}{dx}(x) = (1-k)p(x)v_{k-1}(x) + (2-k)q(x)v_{k-2}(x), \\ v_0 \equiv 0, \quad v_1 \equiv 1, \; v_k(a) = 0, \; k \geq 2. \end{gathered} \tag{1.3}$$

So we have to read out the global analytic properties of G from its Taylor expansion (1.2), or from (1.3). This is a classical setting of Analytic Continuation.

At present we can handle only very special cases of (1.3), so most of results are for other *(simpler but still non-trivial)* recurrence relations.

2 Taylor Domination

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a series with the radius of convergence $R > 0$. Let a natural N, and a positive sequence $S(k)$ of a sub-exponential growth be fixed.

Definition 2.1. *The function f possesses a (N, R, S) - Taylor domination property if for each $k \geq N+1$ we have*

$$|a_k| R^k \leq S(k) \max{}_{i=0,\dots,N} |a_i| R^i,$$

The property of Taylor domination allows us to compare the behavior of $f(z)$ with the behavior of the polynomial $P_N(z) = \sum_{k=0}^{N} a_k z^k$.

In particular, the number of zeroes of f can be easily bounded in this way.

Taylor domination property is essentially equivalent to the bound on the number of zeroes of $f - c$ for each c:

Theorem 2.1 (Biernacki, 1932)**.** *If f is p-valent in D_R, i.e. the number of solutions in D_R of $f(z) = c$ for any c does not exceed p, then for $k > p$*

$$|a_k|R^k \leq (Ak/p)^{2p} \max{}_{i=0,\dots,p}|a_i|R^i.$$

So f possesses $(p, R, (Ak/p)^{2p})$ Taylor domination property.

For $p = 1, a_0 = 0, R = 1$ the Bieberbach conjecture proved by De Branges claims that $|a_k| \leq k|a_1|$.

2.1 Uniform Taylor domination

Consider a family

$$f_\lambda(z) = \sum_{k=0}^{\infty} a_k(\lambda) z^k, \ \ \lambda \in \mathbb{C}^n$$

with the coefficients $a_k(\lambda) \in \mathbb{C}[\lambda]$.

The position of singularities (and hence the radius of convergence $R(\lambda)$) for general $f_\lambda(z)$ depend on λ.

Definition 2.2. *The family $f_\lambda(z)$ possesses a Uniform Taylor domination property if*

$$|a_k(\lambda)| R^k(\lambda) \leq S(k) \max{}_{i=0,\ldots,N} |a_i(\lambda)| R^i(\lambda)$$

with N and $S(k)$ not depending on λ.

Uniform Taylor domination implies a uniform in λ bound on zeroes in any disk $D_{\alpha R(\lambda)}$ for any fixed $\alpha < 1$.

Here are some situations where uniform Taylor domination holds:

1. Families $f_\lambda(z)$ with the Taylor coefficients $a_k(\lambda)$ possessing certain (rather restrictive) algebraic properties. Here the key ingredient is provided by the Bautin ideals and related algebraic structures. This covers some cases of (1.3).

2. Taylor coefficients obtained via certain types of linear recurrence relations. Here the key fact is the classical Turan's lemma which, essentially, provides a uniform Taylor domination for rational functions.

3. Taylor coefficients of the Stiltjes transform (i.e. the consecutive moments) of functions obeying certain Remez-type inequalities, or of D-finite functions.

We shall continue with the case (2), (and (3), if time allows).

3 Taylor domination via Turan's lemma

We consider functions whose Taylor coefficients are obtained via certain types of linear recurrence relations.

1. Taylor coefficients of a rational function $R(z) = \frac{P(z)}{Q(z)} = \sum_{k=0}^{\infty} a_k z^k$ of degree d satisfy a linear recurrence relation

$$\sum_{j=0}^{d} c_j a_{k+j} = 0, \quad k = 0, 1, \ldots,$$

where c_j are the coefficients of the denominator $Q(z) = \sum_{j=0}^{d} c_j z^j$. Let $z_1, \ldots, z_d$ be all the poles of $R(z)$, i.e. the roots of $Q(z)$, and let $R = (\min_{i=1}^{n} |z_i|)$ be the radius of convergence.

Theorem 3.1. *(Turan, 1953)*
For each $k \geq n+1$

$a_k R^k \leq C(d)\ k^d \max_{i=1,\dots,d} |a_i| R^i.$

This is a perfect example of uniform Taylor domination.

2. Taylor coefficients of solutions of Fuchsian ODE's satisfy linear recurrence relations of "Poincaré type"

$$\sum_{j=0}^{d}[c_j + \psi_j(k)]a_{k+j} = 0, \quad k = 0, 1, \ldots, \ \lim_{k\to\infty} \psi_j(k) = 0.$$

What kind of Taylor domination (Turan-like inequalities) can we get in this case?

This question is very close to Poincaré-Perron type results on asymptotic behavior of the solutions. Closely related to Linear Non-autonomous Dynamics, Lyapunov Exponents, Difference Equations.

Application: bounding zeroes of solutions of Fuchsian equations. This is a very active field, also closely related to Hilbert 16-th problem (recent results of G. Binyamini, D. Novikov, and S. Yakovenko).

Weak Turan inequality

We are given a Poincaré type recurrence relation

$$\sum_{j=0}^{d}[c_j+\psi_j(k)]a_{k+j}=0, \quad k=0,1,\ldots, \ \lim_{k\to\infty}\psi_j(k)=0.$$

Let $z_1,\ldots,z_d$ be all the roots of $Q(z)=\sum_{j=0}^{d}c_jz^j$, and let $R=(\min_{i=1}^{n}|z_i|)$ be the radius of convergence of the corresponding series, $\rho=\frac{1}{R}$. Let us now define $\hat{N}$ as the first index such that for $k\geq\hat{N}+1$ we have $|\psi_j(k)|\leq 2^d\rho^j$, and let us put $N=\hat{N}+d$.

Theorem 3.2. *Let $a_0,a_1,\ldots$ satisfy (5.9). Then for each $k\geq N+1$ we have*

$$|a_k|R^k\leq 2^{(d+2)k}\max_{j=0}^{N}|a_j|R^j.$$

The problem is that in this result the constant grows exponentially with k.

The inequality of Theorem 3.2 implies Taylor domination for $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in a disk of a smaller radius $R' = 2^{-(d+2)} R$:

Corollary 3.1. *Under conditions of Theorem 3.2 we have*

$$|a_k| R'^k \leq \ \max{}_{j=0}^{N} |a_j| R'^j,$$

and the corresponding bound on the number of zeroes of $f(z)$ in any concentric disk strictly inside the disk of radius R'.

4 Signal Reconstruction from Integral Measurements

The second topic of this talk is related to a certain approach in Signal Processing, which is under active development today, with different names: “Algebraic Sampling”, “Algebraic Signal Reconstruction”, “Signals with finite rate of innovation”, “Moments Inversion” (K.S. Eckhoff, G. Kvernadze, A. Gelb and E. Tadmor, M. Vetterli, Th. Peter and G. Plonka, D.B. and Y.Y., ...).

Very shortly, the approach is as follows: assume that a parametric form of the signal is a priori known, but not the specific values of the parameters. Substitute this expression *symbolically* into the symbolic expression for the measurements (like moments or Fourier integrals). We get an algebraic system of equations. Solve this system for the specific measurements values and get the unknown signal parameters.

The names above give a very small sample - much more groups work in this direction, but a general framework for this kind of techniques apparently does not exist. However, very recently some general lines have appeared, and some promising connections with "Compressed Sensing" have emerged ([E. Candes and C. Fernandez-Granda], [Th. Peter and G. Plonka], [D.B and Y.Y]).

In particular, the following result partly settles conjecture of Eckhoff (1995):

Theorem 4.1. *([D.B and Y.Y, 2012]) A piecewise C^k function can be reconstructed from its first N Fourier coefficients with an error of order $N^{-\frac{k}{2}}$.*

The conjecture is: N^{-k}, Fourier truncation gives: N^{-1}.

Not less inspiring are the connections (some very recently discovered) with other mathematical fields.

An example

Assume that the signal $F(x)$ is a priori known to be a linear combination of δ-functions:

$$F(x) = \sum_{i=1}^{d} \alpha_i \delta(x - x_i), \tag{4.1}$$

with the unknown parameters α_i, x_i. Our measurements are the moments $m_k(F) = \int x^k F(x) dx, \ k = 0, 1, ...$

Symbolic substitution gives immediately

$$m_k(F) = \int x^k \sum_{i=1}^{d} \alpha_i \delta(x - x_i) = \sum_{i=1}^{d} \alpha_i x_i^k.$$

So for any set of specific measurements $m_k(F) = \mu_k$ we get the following system of equations ("Prony system") with respect to the unknown parameters α_i, x_i:

$$\sum_{i=1}^{n} \alpha_i x_i^k = \mu_k, \; k = 0, 1, \ldots, 2n.$$

Turan lemma appear as follows: consider a rational function

$$R(z) = \sum_{i=1}^{n} \frac{\alpha_i}{1 - x_i z} = \sum_{k=0}^{\infty} m_k z^k,$$

with $m_k = \sum_{i=1}^{d} \alpha_i x_i^k$ as above (a sum of geometric progressions).

So m_k are the Taylor coefficients of the rational function of degree d. By Turan lemma we have

$$m_k R^k \leq C(d)\ k^d \max{}_{i=1,\dots,d} |m_i| R^i,\ \ k = d+1, d+2, \dots$$

This statement certainly provides information on robustness of solutions of the Prony system. Indeed, its solutions *may "blow up"*: as the points x_i collide, the coefficients x_i may tend to infinity. Turan's lemma shows that this happens in such a way that all the moments remain bounded.

However, there is a much more accurate result, which implies, in particular, Turan's lemma.

As the points $x_1, ..., x_d$ are fixed, we can consider divided finite difference $\Delta_j = \Delta(x_1, ...' x_j)$. In a natural way Δ_j can be interpreted as linear combinations of δ-function, which form a basis for such combinations.

Represent F as $F = \sum_{l=1}^{d} \beta_l \Delta_l$.

Theorem 4.2. *There are constants C_1, C_2 depending only on d such that*

$$C_1(d) C_1 \max{}_{i=1}^{d} |\beta_j| \leq \max{}_{i=1}^{d} |m_i(F)| \leq C_2 \max{}_{j=1}^{d} |\beta_j|.$$

Turan's lemma easily follows from Theorem 4.1.

More connections: Turan - Nazarov inequality and its discrete version by [O. Friedland and Y.Y.]

Abel equation $y' = py^2 + qy^3$ is to be reconstructed. Measurements - the Taylor coefficients of the Poincaré mapping $G(y)$. Center-Focus problem - the question non-uniqueness of the reconstruction. Turan-type inequality = Taylor domination - to be found.

1. Basis of divided differences. Let us recall a definition of the divided finite differences (see, for example, [**?**]). Let $X = \{x_1, \dots, x_n\}$ be a set of points in $\mathbb{C}$, and let $Y = Y(x)$ be a complex function on $\mathbb{C}$, $Y(x_i) = y_i,\ i = 1, \dots, n$. Initially we assume that all the points x_j in X are pairwise different, but later we shall drop this assumption.

Definition 4.1. *The $n-1$-st divided finite difference $\Delta[X, Y] = \Delta_{n-1}[X, Y]$ is defined as the sum*

$$\Delta_n[X, Y] = \sum_{i=1}^{n} \frac{y_i}{(x_i - x_1) \dots (x_i - x_n)} = \sum_{i=1}^{n} \alpha_i^j y_i.$$

In particular, for $X = \{x_1\}$ we have $\Delta_0[X, Y] = y_1$, for $X = \{x_1, x_2\}$ we have $\Delta_1[X, Y] = \frac{y_2 - y_1}{x_2 - x_1}$.

For our purposes it is convenient to interpret the divided differences as linear combinations of δ-functions. For $X = \{x_1, \dots, x_n\} \subset$

$\mathbb{C}$ let us denote by δ_X the function $\delta_X = \sum_{i=1}^{n} \alpha_i^n \delta(x - x_i)$. Then for each "probe function" f we have $\Delta_n[X, f] = \int f(x)\delta_X(x)dx$.

Our construction of the basis of finite differences in the space of linear combinations of δ-functions at the points of $X = \{x_1, \dots, x_n\}$ will depend on the choice of a chain C of subsets $X_1 \subset X_2 \subset \dots \subset X_{n-1} \subset X_n = X$, with X_j containing exactly j points for $j = 1, \dots, n$. As the chain C has been fixed, we have a natural order of the points $x_1, \dots, x_n$ for which $X_j = \{x_1, \dots, x_j\}$, $j = 1, \dots, n$. This order will be used below.

Definition 4.2. *For a chain C as above the basis B_C of finite differences in the space of linear combinations of δ-functions at the points of $X = \{x_1, \dots, x_n\}$ is given by the divided finite differences $\delta_1 = \delta_{X_1}$, $\delta_2 = \delta_{X_2}, \dots, \delta_n = \delta_{X_n}$.*

$B_C = \{\delta_1, \dots, \delta_n\}$ is indeed a basis, since its transformation matrix to the standard basis is triangular, with non-zero coefficients on the diagonal. For $F(x) = \sum_{s=1}^{n} \alpha_s \delta(x - x_s)$ we have $F(x) = \sum_{r=1}^{n} \beta_r \delta_r$. The explicit transformation matrices cam be easily written down (see, for example, [**?**]).

2. Two norms of F. We put $\rho = \max_{i=1}^{n} |x_i|$, and, as above, $R = \rho^{-1}$. For $X = \{x_1, \ldots, x_n\}$ consider the space L_X of linear combinations of δ-functions $\delta(x - x_i)$. Now for $F \in L_X$ let us define $\|F\|$ as $\max_{l=0}^{n-1} m_l(F) R^l$ and let $\|F\|_1 = \sum_{r=1}^{n} |\beta_r| R^r$. We show equivalence of the norms $\|F\|$ and $\|F\|_1$ with the bounds depending only on n. which is defined as $\max_{k=0,\ldots,n-1} |m_k(F)|$. To simplify the presentation we consider here only the real case. So we assume that $x_1, \ldots, x_n, \ \alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and the moments are given by $m_k(F) = \int_{\mathbb{R}} x^k F(x) dx$.

We shall need the following property of the divided differences, which in the real case follows easily from the Rolle lemma:

Proposition 4.1. *Assume that $f(x)$ is a C^n-function. Then* *the divided finite differences of f satisfy $\Delta_j[X,f] = \frac{1}{j!}f^{(j)}(\eta)$ for some $\eta \in [x_1, x_j]$.*

There are certain analogies of this fact in the complex setting which we discuss in [**?**].

Let us return to the basis of finite differences B_C. The functions $\delta_j(x)$ forming this bases are linear combinations of δ-functions with the coefficients tending to infinity as some of the points $x_1, \dots, x_j$ approach one another. Still, their moments remain uniformly bounded:

Proposition 4.2. *For each $x_1, \dots, x_n$ in $[0,1]$ and for each k we have*

$$0 < m_k(\delta_j) \le \tbinom{k}{j}\rho^{k-j}.$$

Proof: Indeed,

$$m_k(\delta_j) = \int x^k \delta_j(x)dx = \Delta_j[X, x^k] = \frac{1}{j!}(x^k)^{(j)}(\eta_j) = \tbinom{k}{j}\eta_j^{k-j}$$

with $\eta_j \in [x_1, x_{j+1}] \subset [0, \rho]$. $\square$

3. Equivalence of two norms

The following theorem shows that the divided differences δ_j and their bounded linear combinations are, essentially, *the only* linear combinations of δ-functions with uniformly bounded moments.

Theorem 4.3. *The norms $\|F\|$ and $\|F\|_1$ on L_X are equivalent, i.e. there are constants C_1, C_2 depending only on n such that*

$$C_1\|F\| \leq \|F\|_1 \leq C_2\|F\|. \tag{4.2}$$

Turan lemma can be interpreted

Prony system, and its various modifications appears in numerous application

Represent our rational function $R(z)$ as a sum of elementary fractions:

$R(z) = \sum_{i=1}^{n} \frac{\alpha_i}{1 - x_i z} = \sum_{k=0}^{\infty} a_k z^k,$

with $a_k = \sum_{i=1}^{n} \alpha_i x_i^k$.

So Turan's lemma can be considered as a result on exponential polynomials. One of the inherent difficulties is that while a_k remain bounded, α_i may "blow up". Finite differences naturally appear in this context.

Deep relations with Harmonic Analysis, Uncertainty Principle, Analytic continuation, Number Theory, Signal Processing.

In particular the following “Prony system” appears in numerous applications:

$$\sum_{i=1}^{n} \alpha_i x_i^k = \mu_k, \ k = 0, 1, \ldots, 2n.$$

Here $\alpha_i, \ x_i$ are unknowns, while the right hand side μ_k are “measurements”. Turan lemma is a result on the robustness of this system.

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