# Two weight $L^2$ inequality for the Hilbert transform

Eric T. Sawyer reporting on joint work with
Michael T. Lacey     Chun-Yen Shen     Ignacio Uriarte-Tuero

Abel Symposium
Oslo, Norway

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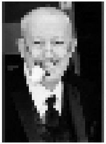
# Dedication

This talk is dedicated to the memory of

Joseph Csima

March 2, 1933 - August 17, 2012



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1. Statement of the theorem: an indicator/interval characterization of the two weight inequality for the Hilbert transform.
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3. What is left: comments on the *NTV* conjecture.

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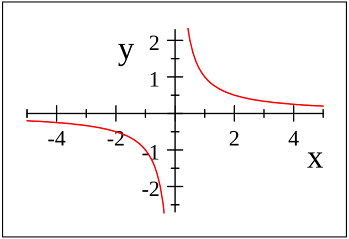
        3. The Nazarov-Treil-Volberg theorem,
        4. Our indicator/interval characterization.

# The Hilbert transform
## as singular integral

The Hilbert transform $Hf$ arose in 1905 in connection with Hilbert's twenty-first problem, and for $f \in L^2(\mathbb{R})$ is defined almost everywhere by the *principle value* singular integral

$$\begin{aligned} Hf(x) &= p.v. \int \frac{1}{y-x} f(y)\, dy \\ &\equiv \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \frac{1}{y-x} f(y)\, dy, \qquad a.e.\, x \in \mathbb{R}. \end{aligned}$$




The convolution kernel of $H$

# Helson-Szego theorem (1960)

a function theoretic characterization of the one weight inequality

- A locally finite positive Borel measure $\omega$ on $\mathbb{T}$ satisfies the property

$$\int |Hf|^2 \, d\omega \leq C \int |f|^2 \, d\omega, \qquad f \in C^\infty(\mathbb{T}),$$

if and only if

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$$\int |Hf|^2 \, d\omega \le C \int |f|^2 \, d\omega, \qquad f \in C^\infty(\mathbb{T}),$$

if and only if

- $d\omega(x) = w(x)\, dx$ and where there are bounded real-valued functions $u, v$ on the circle such that the Helson-Szegö condition holds:

$$\begin{aligned} w(x) &= e^{u(x)+Hv(x)}, \qquad a.e. x \in \mathbb{R}, \\ \|u\|_{L^\infty(\mathbb{R})} &< \infty \text{ and } \|v\|_{L^\infty(\mathbb{R})} < \frac{\pi}{2}. \end{aligned}$$

- In 1972 B. Muckenhoupt showed that the 'poor cousin' maximal function

**Definition (maximal function)**

$$Mf(x) \equiv \sup_{\text{intervals } Q:\ x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

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### Definition (maximal function)

$$Mf(x) \equiv \sup_{\text{intervals } Q:\ x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

- satisfies the $L^2$ weighted norm inequality with weight $w$,

$$\int Mf(x)^2 w(x)\, dx \leq C \int |f(x)|^2 w(x)\, dx,$$

# Toward a geometric characterization

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**Definition ($A_2$ condition)**

$$\left(\frac{1}{|Q|}\int_Q w(y)\,dy\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{w(y)}dy\right) \leq C.$$

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# The two weight Hilbert transform inequality

## a function theoretic characterization analogous to Helson-Szegö

- In 1979 Cotlar and Sadosky showed that

$$\int_{\mathbb{T}} |Hf|^2 \, d\omega_1 \leq A \int_{\mathbb{T}} |f|^2 \, d\omega_2, \qquad f \in C^\infty(\mathbb{T}),$$

if and only if

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$$\int_{\mathbb{T}} |Hf|^2 \, d\omega_1 \leq A \int_{\mathbb{T}} |f|^2 \, d\omega_2, \qquad f \in C^{\infty}(\mathbb{T}),$$

if and only if

- $$d\omega_1 \ll d\theta, d\omega_1 \leq A d\omega_2,$$

and there exists a holomorphic function $h \in \mathcal{H}^1(\mathbb{D})$, i.e.

$$\|h\|_{\mathcal{H}^1(\mathbb{D})} \equiv \sup_{0<r<1} \int_{\mathbb{T}} \left| h\left(re^{i\theta}\right) \right| d\theta < \infty,$$

such that

$$|A d\omega_2 + d\omega_1 - h d\theta| \leq |A d\omega_2 - d\omega_1| .$$

# Toward a geometric characterization

## The two weight inequality for the maximal function

- In 1981 Sawyer showed that the maximal function $Mf$ satisfies the $L^2$ two weight norm inequality with weight pair $(\omega, \sigma)$,

$$\int M(f\sigma)(x)^2 \, d\omega(x) \leq C \int |f(x)|^2 \, d\sigma(x),$$

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**Definition (maximal testing condition)**

$$\int_Q M(\chi_Q \sigma)(x)^2 \, d\omega(x) \leq C \, |Q|_\sigma .$$

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- if and only if the following two *testing* conditions hold:

$$\int_Q I_\alpha (\chi_Q \sigma)^2 \, d\omega \le C\, |Q|_\sigma \quad \text{and} \quad \int_Q I_\alpha (\chi_Q \omega)^2 \, d\sigma \le C\, |Q|_\omega .$$

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- and a similar result for the Poisson integral

$$\mathbb{P}f(x,t) = \int_{\mathbb{R}} \frac{t}{t^2+x^2} f(t)\, dt.$$

- In 1984 David and Journé showed that if $K(x,y)$ is a standard kernel on $\mathbb{R}^n$,

$$\begin{aligned} |K(x,y)| &\leq C|x-y|^{-n}, \\ |K(x',y)-K(x,y)|+\ldots &\leq C|x-y|^{-n}\left(\frac{|x'-x|}{|x-y|}\right)^{\delta}, \end{aligned}$$

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- and if $Tf(x) \equiv \int_{\mathbb{R}^n} K(x,y) f(y)\,dy$ for $x \notin supp\ f$, then $T$ is bounded on $L^2(\mathbb{R}^n)$ if and only if $T \in WBP$ and

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**Definition ($T1$ or *testing* conditions)**

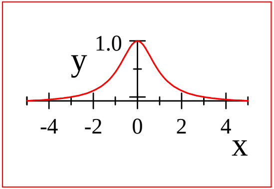
$$\begin{aligned} T1 \in BMO \quad &\left(\Leftrightarrow \int_Q |T\chi_Q|^2 \leq C|Q|\right), \\ T^*1 \in BMO \quad &\left(\Leftrightarrow \int_Q |T^*\chi_Q|^2 \leq C|Q|\right). \end{aligned}$$

# Toward a geometric characterization

- In 2004 Nazarov, Treil and Volberg showed that if a weight pair $(\omega, \sigma)$ satisfies the pivotal condition

$$\sum_{r=1}^{\infty} |I_r|_\omega \mathrm{P}(I_r, \chi_{I_0}\sigma)^2 \leq \mathcal{P}_*^2 |I_0|_\sigma ; \quad \mathrm{P}(I, \nu) = \int \frac{|I|}{|I|^2 + x^2} d\nu(x),$$

for all decompositions of an interval $I_0$ into subintervals $I_r$,



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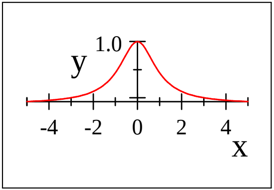


- then the Hilbert transform $H$ satisfies the two weight $L^2$ inequality

$$\int |H(f\sigma)|^2 \, d\omega \leq C \int |f|^2 \, d\sigma,$$

*uniformly* for all smooth truncations of the Hilbert transform,

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**Definition ($A_2$ condition on steroids)**

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$$\sup_I \mathrm{P}(I,\omega) \cdot \mathrm{P}(I,\sigma) \equiv \mathcal{A}_2^2 < \infty ,$$

- as well as the two *interval testing* conditions

$$\begin{aligned} \int_I |H(\chi_I \sigma)|^2 \, d\omega &\leq \mathfrak{T}^2 \, |I|_\sigma , \\ \int_I |H(\chi_I \omega)|^2 \, d\sigma &\leq (\mathfrak{T}^*)^2 \, |I|_\omega . \end{aligned}$$

# Maximal inequalities and doubling

- Nazarov, Treil and Volberg showed that the pivotal conditions are implied by the boundedness of the maximal operator and its 'dual':

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- They also showed that the pivotal conditions are implied by the testing conditions and the $\mathcal{A}_2$ condition if the measures $\sigma$ and $\omega$ are both doubling:

$$\int_{2Q} d\sigma \lesssim \int_Q d\sigma \text{ and } \int_{2Q} d\omega \lesssim \int_Q d\omega$$

for all intervals $Q$.

# The role of cancellation

- In 2002 Nazarov showed that the strengthened $A_2$ condition alone is not enough for the two weight inequality to hold. The reason for the failure lies in the fact that this condition is a consequence *solely* of the size and smoothness of the kernel.

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- It is the pair of *testing conditions* that encode the cancellation required for the $L^2$ norm inequality.

# Energy and hybrid conditions

- Two years ago, Lacey Sawyer and Uriarte-Tuero showed that the pivotal conditions are not necessary, that the following *energy condition* is,

$$\mathsf{E}(I,\omega) \equiv \left( \mathbb{E}_I^{\omega(dx)} \mathbb{E}_I^{\omega(dx')} \left( \frac{|x-x'|}{|I|} \right)^2 \right)^{1/2},$$

$$\sum_{r=1}^{\infty} \omega(I_r) [\mathsf{E}(I_r,\omega) \mathsf{P}(I_r, \chi_{I_0}\sigma)]^2 \leq \mathfrak{E}^2 \sigma(I_0),$$

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- and that the following *hybrid condition* is 'sufficient' for $0 \leq \gamma < 1$ (but still not necessary):

$$\sum_{r=1}^{\infty} \omega(I_r) [\mathsf{E}(I_r,\omega)^{\gamma} \mathsf{P}(I_r, \chi_{I_0}\sigma)]^2 \leq \mathfrak{E}_\gamma^2 \sigma(I_0),$$

for all intervals $I_0$, and decompositions $\{I_r : r \geq 1\}$ of $I_0$ into disjoint intervals $I_r \subsetneq I_0$. Note that for $\gamma = 0$ this is the pivotal condition, while for $\gamma = 1$ it is the energy condition.

# Bounded fluctuation characterization

- Last year Lacey Sawyer Shen and Uriarte-Tuero showed the Hilbert transform two weight inequality is equivalent to the $\mathcal{A}_2$ condition and the bounded fluctuation conditions taken over all dyadic grids $\mathcal{D}$:

$$\begin{aligned} \int_I H(\mathbf{1}_I f\sigma)^2 \, d\omega &\leq C\left\{ |I|_\sigma + \int_I |f|^2 \, d\sigma \right\}, \\ \int_I H(\mathbf{1}_I g\omega)^2 \, d\sigma &\leq C\left\{ |I|_\omega + \int_I |g|^2 \, d\omega \right\}, \end{aligned} \tag{1}$$

for all $I \in \mathcal{D}$ and all functions $f, g$ of unit $\mathcal{D}$-fluctuation on $I$.

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  for all $I \in \mathcal{D}$ and all functions $f, g$ of unit $\mathcal{D}$-fluctuation on $I$.

- A function $f \in L^2(\sigma)$ is of *unit $\mathcal{D}$-fluctuation on $I$*, written $f \in \mathcal{BF}_\sigma(I)$, if it is supported in $I$ and $\frac{1}{|K|_\sigma} \int_K |f| \, d\sigma \leq 1$ for all dyadic subintervals $K$ of $I$ on which $f$ is *not* constant.

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$$\begin{aligned} \int_I H\left(\mathbf{1}_I f\sigma\right)^2 d\omega &\leq C\left\{|I|_\sigma + \int_I |f|^2 d\sigma\right\}, \qquad (1) \\ \int_I H\left(\mathbf{1}_I g\omega\right)^2 d\sigma &\leq C\left\{|I|_\omega + \int_I |g|^2 d\omega\right\}, \end{aligned}$$

  for all $I \in \mathcal{D}$ and all functions $f, g$ of unit $\mathcal{D}$-fluctuation on $I$.
- A function $f \in L^2(\sigma)$ is of *unit $\mathcal{D}$-fluctuation on $I$*, written $f \in \mathcal{BF}_\sigma(I)$, if it is supported in $I$ and $\frac{1}{|K|_\sigma}\int_K |f|\, d\sigma \leq 1$ for all dyadic subintervals $K$ of $I$ on which $f$ is *not* constant.
- Such functions are special cases of dyadic $BMO_{\mathcal{D}}(\sigma)$ functions of norm 1, and include functions bounded by 1 in modulus. They arise as the *good* functions in a Calderón-Zygmund decomposition.

# The Nazarov Treil Volberg conjecture

- A question raised in Volberg's 2003 CBMS book, which we refer to as the *NTV conjecture*, is whether or not

$$\int_{\mathbb{R}} |H(f\sigma)|^2 \omega \leq \mathfrak{N} \int_{\mathbb{R}} |f|^2 \sigma, \qquad f \in L^2(\sigma), \tag{2}$$

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  is equivalent to the $\mathcal{A}_2$ condition and the two interval testing conditions.

- A weaker conjecture, that we refer to as the *indicator/interval* NTV conjecture, is that (2) is equivalent to the $\mathcal{A}_2$ condition and the two *indicator/interval* testing conditions,

$$\int_I |H(\mathbf{1}_E\sigma)|^2 \omega \le \mathfrak{A}\, |I|_\sigma, \quad \int_I |H(\mathbf{1}_E\omega)|^2 \sigma \le \mathfrak{A}^* \, |I|_\omega, \tag{3}$$

  for all intervals $I$ and closed subsets $E$ of $I$. Note that $E$ does *not* appear on the right side of these inequalities, and that if $H$ were a positive operator we could take $E = I$.

# Our characterization of the two weight Hilbert transform inequality

## The indicator/interval NTV conjecture

### Theorem

*The best constant* $\mathfrak{N}$ *in the two weight inequality (2) for the Hilbert transform satisfies*

$$\mathfrak{N} \approx \sqrt{\mathcal{A}_2} + \mathfrak{A} + \mathfrak{A}^*,$$

*i.e.* $H_\sigma$ *is bounded from* $L^2(\sigma)$ *to* $L^2(\omega)$ *if and only if the strong* $A_2$ *and indicator/interval testing conditions hold.*

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## Corollary

*The Hilbert transform $H_\sigma$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if both it and its dual $H_\omega$ are weak type $(2,2)$, i.e.*

$$\lambda^2 \left|\{|H_\sigma f| > \lambda\}\right|_\omega \lesssim \int |f|^2 d\sigma \text{ and } \lambda^2 \left|\{|H_\omega g| > \lambda\}\right|_\sigma \lesssim \int |g|^2 d\omega.$$

# Outline of Part II: the proof of the theorem

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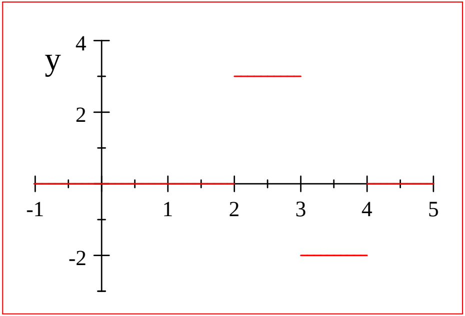
   2. The two weight norm inequality for the Poisson operator

# Haar functions adapted to a measure

- The Haar function $h_I^\sigma$ adapted to a positive measure $\sigma$ and a dyadic interval $I \in \mathcal{D}$ is a positive (negative) constant on the left (right) child, has vanishing mean $\int h_I^\sigma d\sigma = 0$, and is normalized $\|h_I^\sigma\|_{L^2(\sigma)} = 1$. For example if $|[2,3]|_\sigma = \frac{1}{15}$ and $|[3,4]|_\sigma = \frac{1}{10}$, then




The Haar function $h_{[2,4]}^\sigma$

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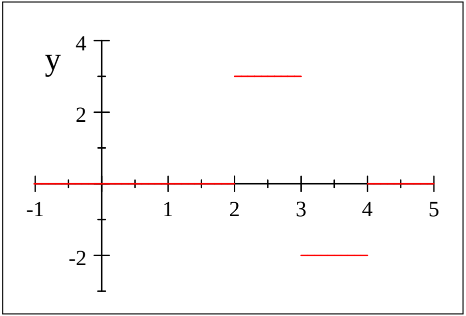


The Haar function $h^\sigma_{[2,4]}$

- The supremum norm of $h_I^\sigma$ is quite large if $\sigma$ is very unbalanced (not doubling).

# The good dyadic grids of NTV

- For any $\beta = \{\beta_l\} \in \{0,1\}^{\mathbb{Z}}$, define the dyadic grid $\mathbb{D}_\beta$ to be the collection of intervals
$$\mathbb{D}_\beta = \left\{ 2^n \left( [0,1) + k + \sum_{i<n} 2^{i-n} \beta_i \right) \right\}_{n\in\mathbb{Z},\, k\in\mathbb{Z}}$$
and place the usual uniform probability measure $\mathbb{P}$ on the space $\{0,1\}^{\mathbb{Z}}$.

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  and place the usual uniform probability measure $\mathbb{P}$ on the space $\{0,1\}^{\mathbb{Z}}$.
- For weights $\omega$ and $\sigma$, consider random choices of dyadic grids $\mathcal{D}^\omega$ and $\mathcal{D}^\sigma$. Fix $\varepsilon > 0$ and for a positive integer $r$, an interval $J \in \mathcal{D}^\omega$ is said to be *r-bad* if there is an interval $I \in \mathcal{D}^\sigma$ with $|I| \geq 2^r |J|$, and

$$\mathrm{dist}(e(I), J) \leq \tfrac{1}{2} |J|^\varepsilon |I|^{1-\varepsilon}.$$

  where $e(I)$ is the set of the three discontinuities of $h_I^\sigma$. Otherwise, $J$ is said to be *r-good*.

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where $e(I)$ is the set of the three discontinuities of $h_I^\sigma$. Otherwise, $J$ is said to be *$r$-good*.
- We have
$$\mathbb{P}\left(J \text{ is } r\text{-bad}\right) \leq C2^{-\varepsilon r}.$$

# Reduction to good projections

- Let $\mathcal{D}^\sigma$ be randomly selected with parameter $\beta$, and $\mathcal{D}^\omega$ with parameter $\beta'$. Define a projection

$$\mathsf{P}^\sigma_{good} f \equiv \sum_{I \text{ is } r\text{-good } \in \mathcal{D}^\sigma} \Delta^\sigma_I f \,,$$

and likewise for $\mathsf{P}^\omega_{good} g$.

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- Define $\mathsf{P}^\sigma_{bad} f \equiv f - \mathsf{P}^\sigma_{good} f$. Then

$$\mathbb{E}_{\beta'} \left\| \mathsf{P}^\sigma_{bad} f \right\|_{L^2(\sigma)} \leq C 2^{-\frac{\varepsilon r}{2}} \left\| f \right\|_{L^2(\sigma)} .$$

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  and likewise for $\mathsf{P}^\omega_{bad} g$.

- There is an absolute choice of $r$ so that if $T : L^2(\sigma) \to L^2(\omega)$ is a bounded linear operator, then

$$\left\| T \right\|_{L^2(\sigma) \to L^2(\omega)} \leq 2 \sup_{\|f\|_{L^2(\sigma)}=1} \sup_{\|g\|_{L^2(\omega)}=1} \mathbb{E}_\beta \mathbb{E}_{\beta'} \left| \left\langle T \mathsf{P}^\sigma_{good} f, \mathsf{P}^\omega_{good} g \right\rangle_\omega \right| .$$

# The Haar expansion

- Let $\mathcal{D}^\sigma$ and $\mathcal{D}^\omega$ be an $r$-good pair of grids, and let $\{h_I^\sigma\}_{I\in\mathcal{D}^\sigma}$ and $\{h_J^\omega\}_{J\in\mathcal{D}^\omega}$ be the corresponding Haar bases, so that

$$\begin{aligned} f &= \sum_{I\in\mathcal{D}^\sigma} \triangle_I^\sigma f = \sum_{I\in\mathcal{D}^\sigma} \langle f, h_I^\sigma\rangle \ h_I^\sigma = \sum_{I\in\mathcal{D}^\sigma} \widehat{f}(I) \ h_I^\sigma, \\ g &= \sum_{J\in\mathcal{D}^\omega} \triangle_J^\omega g = \sum_{J\in\mathcal{D}^\omega} \langle g, h_J^\omega\rangle \ h_J^\omega = \sum_{J\in\mathcal{D}^\omega} \widehat{g}(J) \ h_J^\omega, \end{aligned}$$

  where the appropriate grid is understood in the notation $\widehat{f}(I)$ and $\widehat{g}(J)$.

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  where the appropriate grid is understood in the notation $\widehat{f}(I)$ and $\widehat{g}(J)$.
- Inequality (2) is equivalent to boundedness of the bilinear form

$$\mathcal{H}(f,g) \equiv \langle H(f\sigma), g\rangle_\omega = \sum_{I\in\mathcal{D}^\sigma \text{ and } J\in\mathcal{D}^\omega} \langle H(\sigma \triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega$$

  on $L^2(\sigma)\times L^2(\omega)$, i.e.

$$|\mathcal{H}(f,g)| \leq \mathfrak{N} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

# Splitting of the form by interval size

Virtually all attacks on the two weight inequality (2) to date have proceeded by first splitting the bilinear form $\mathcal{H}$ into three natural forms determined by the relative size of the intervals $I$ and $J$ in the inner product:

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_{lower} + \mathcal{H}_{diagonal} + \mathcal{H}_{upper}; \qquad (4)\\
\mathcal{H}_{lower}(f,g) &\equiv \sum_{\substack{I\in\mathcal{D}^\sigma \text{ and } J\in\mathcal{D}^\omega \\ |J|<2^{-r}|I|}} \left\langle H\left(\sigma \triangle_I^\sigma f\right), \triangle_J^\omega g\right\rangle_\omega ,\\
\mathcal{H}_{diagonal}(f,g) &\equiv \sum_{\substack{I\in\mathcal{D}^\sigma \text{ and } J\in\mathcal{D}^\omega \\ 2^{-r}|I|\leq|J|\leq 2^r|I|}} \left\langle H\left(\sigma \triangle_I^\sigma f\right), \triangle_J^\omega g\right\rangle_\omega ,\\
\mathcal{H}_{upper}(f,g) &\equiv \sum_{\substack{I\in\mathcal{D}^\sigma \text{ and } J\in\mathcal{D}^\omega \\ |J|>2^r|I|}} \left\langle H\left(\sigma \triangle_I^\sigma f\right), \triangle_J^\omega g\right\rangle_\omega ,
\end{aligned}$$

and then continuing to establish boundedness of each of these three forms.

# Boundedness of the split forms

- Now the boundedness of the diagonal form $\mathcal{H}_{diagonal}$ is an automatic consequence of that of $\mathcal{H}$ since it is shown by NTV that

$$\begin{aligned} \left|\mathcal{H}_{diagonal}\left(f,g\right)\right| &\lesssim \left(\sqrt{\mathcal{A}_2}+\mathfrak{T}+\mathfrak{T}^*\right)\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)} \\ &\lesssim \mathfrak{N}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}. \end{aligned}$$

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- However, it is *not* known if the boundedness of $\mathcal{H}_{lower}$ and $\mathcal{H}_{upper}$ follow from that of $\mathcal{H}$, which places in jeopardy the entire method of attack based on the splitting (4) of the form $\mathcal{H}$.

- The triple corona decomposition consists of a series of three reductions performed with two Calderón-Zygmund corona decompositions, followed by an energy corona decomposition, in order to identify the extremal functions that fail to yield to the standard analyses.

# Circumventing the obstacles

## The triple coronas

- The triple corona decomposition consists of a series of three reductions performed with two Calderón-Zygmund corona decompositions, followed by an energy corona decomposition, in order to identify the extremal functions that fail to yield to the standard analyses.
- These extremals are certain *bounded* functions, and functions of *minimal bounded fluctuation*, occurring in a corona with *energy* control.

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- These extremals are certain *bounded* functions, and functions of *minimal bounded fluctuation*, occurring in a corona with *energy* control.
- In the end, the standard NTV methodology is, to some extent, decisive when used on these extremal functions with very special structure.

- We use *parallel corona* splittings of the bilinear form, followed by an analysis of the extremal functions that fail both the energy and Calderón-Zygmund stopping time methodology.

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- The parallel corona splitting involves defining upper and lower and diagonal forms relative to the tree of triple corona stopping time intervals, rather than the full tree of dyadic intervals.
- The enemy of Calderón-Zygmund stopping times is degeneracy of the doubling property, while the enemy of energy stopping times is degeneracy of the energy functional (since nondegenerate doubling implies nondegenerate energy, it is really the failure of doubling in both weights that is the common enemy).

# CZ stopping trees

- In order to improve on the splitting in (4), we introduce stopping trees $\mathcal{F}$ and $\mathcal{G}$ for the functions $f \in L^2(\sigma)$ and $g \in L^2(\omega)$. Let $\mathcal{F}$ be a collection of Calderón-Zygmund stopping intervals for $f$, and let $\mathcal{D}^\sigma = \bigcup_{F \in \mathcal{F}} \mathcal{C}_F$ be the associated corona decomposition of the dyadic grid $\mathcal{D}^\sigma$.

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- For $I \in \mathcal{D}^\sigma$ let $\pi_{\mathcal{D}^\sigma} I$ be the $\mathcal{D}^\sigma$-parent of $I$ in the grid $\mathcal{D}^\sigma$, and let $\pi_{\mathcal{F}} I$ be the smallest member of $\mathcal{F}$ that contains $I$. For $F, F' \in \mathcal{F}$, we say that $F'$ is an $\mathcal{F}$-child of $F$ if $\pi_{\mathcal{F}}(\pi_{\mathcal{D}^\sigma} F') = F$, and we denote by $\mathfrak{C}(F)$ the set of $\mathcal{F}$-children of $F$.

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- For $F \in \mathcal{F}$, define the projection $\mathsf{P}^\sigma_{\mathcal{C}_F}$ onto the linear span of the Haar functions $\{h^\sigma_I\}_{I \in \mathcal{C}_F}$ by

$$\mathsf{P}^\sigma_{\mathcal{C}_F} f = \sum_{I \in \mathcal{C}_F} \triangle^\sigma_I f = \sum_{I \in \mathcal{C}_F} \langle f, h^\sigma_I \rangle_\sigma h^\sigma_I; \quad f = \sum_{F \in \mathcal{F}} \mathsf{P}^\sigma_{\mathcal{C}_F} f,$$

$$\int \left(\mathsf{P}^\sigma_{\mathcal{C}_F} f\right) \sigma = 0, \quad \|f\|^2_{L^2(\sigma)} = \sum_{F \in \mathcal{F}} \left\|\mathsf{P}^\sigma_{\mathcal{C}_F} f\right\|^2_{L^2(\sigma)}.$$

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  3. $\mathsf{P}^\sigma_{\mathcal{C}_S}\left(\mathsf{P}^\sigma_{\mathcal{C}_K}\left(\mathsf{P}^\sigma_{\mathcal{C}_F} f\right)\right)$ is as in step 2 but with additional energy control after the third energy decomposition, analogous to the pivotal stopping time corona of $NTV$, but using the necessary energy condition instead.

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- This is called the triple corona decomposition for $f$, and there is an analogous decomposition for $g$.

# The parallel corona splitting

- Consider the following *parallel corona splitting* of the inner product $\langle H(f\sigma), g\rangle_\omega$ that involves the projections $\mathsf{P}^\sigma_{\mathcal{C}_F}$ acting on $f$ and the projections $\mathsf{P}^\omega_{\mathcal{C}_G}$ acting on $g$. We have

$$\langle H(f\sigma), g\rangle_\omega = \sum_{(F,G)\in\mathcal{F}\times\mathcal{G}} \left\langle H\left(\sigma \mathsf{P}^\sigma_{\mathcal{C}_F} f\right), \left(\mathsf{P}^\omega_{\mathcal{C}_G} g\right)\right\rangle_\omega \tag{5}$$

$$\begin{aligned} &= \left\{ \sum_{(F,G)\in\mathrm{Near}(\mathcal{F}\times\mathcal{G})} + \sum_{(F,G)\in\mathrm{Disjoint}(\mathcal{F}\times\mathcal{G})} + \sum_{(F,G)\in\mathrm{Far}(\mathcal{F}\times\mathcal{G})} \right\} \\ &\qquad \times \left\langle H\left(\sigma \mathsf{P}^\sigma_{\mathcal{C}_F} f\right), \left(\mathsf{P}^\omega_{\mathcal{C}_G} g\right)\right\rangle_\omega \\ &\equiv \mathsf{H}_{near}(f,g) + \mathsf{H}_{disjoint}(f,g) + \mathsf{H}_{far}(f,g). \end{aligned}$$

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- Consider the following *parallel corona splitting* of the inner product $\langle H(f\sigma), g\rangle_\omega$ that involves the projections $\mathsf{P}^\sigma_{\mathcal{C}_F}$ acting on $f$ and the projections $\mathsf{P}^\omega_{\mathcal{C}_G}$ acting on $g$. We have

$$\langle H(f\sigma), g\rangle_\omega = \sum_{(F,G)\in\mathcal{F}\times\mathcal{G}} \left\langle H\left(\sigma \mathsf{P}^\sigma_{\mathcal{C}_F} f\right), \left(\mathsf{P}^\omega_{\mathcal{C}_G} g\right)\right\rangle_\omega \tag{5}$$

$$\begin{aligned} &= \left\{ \sum_{(F,G)\in\mathrm{Near}(\mathcal{F}\times\mathcal{G})} + \sum_{(F,G)\in\mathrm{Disjoint}(\mathcal{F}\times\mathcal{G})} + \sum_{(F,G)\in\mathrm{Far}(\mathcal{F}\times\mathcal{G})} \right\} \\ &\qquad \times \left\langle H\left(\sigma \mathsf{P}^\sigma_{\mathcal{C}_F} f\right), \left(\mathsf{P}^\omega_{\mathcal{C}_G} g\right)\right\rangle_\omega \\ &\equiv \mathsf{H}_{near}(f,g) + \mathsf{H}_{disjoint}(f,g) + \mathsf{H}_{far}(f,g) . \end{aligned}$$

- These forms are no longer linear in $f$ and $g$ as the 'cut' is determined by the coronas $\mathcal{C}_F$ and $\mathcal{C}_G$, which depend on $f$ and $g$.

# Near and far definitions

- Here Near $(\mathcal{F} \times \mathcal{G})$ is the set of pairs $(F, G) \in \mathcal{F} \times \mathcal{G}$ such that $F$ is maximal in $G$, or $G$ is maximal in $F$, more precisely: either

$$F \subset G \text{ and there is no } G_1 \in \mathcal{G} \setminus \{G\} \text{ with } F \subset G_1 \subset G,$$

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$$G \subset F \text{ and there is no } F_1 \in \mathcal{F} \setminus \{F\} \text{ with } G \subset F_1 \subset F.$$

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- The set $\mathrm{Far}\,(\mathcal{F}\times\mathcal{G})$ is the complement of $\mathrm{Near}\,(\mathcal{F}\times\mathcal{G})\cup\mathrm{Disjoint}\,(\mathcal{F}\times\mathcal{G})$ in $\mathcal{F}\times\mathcal{G}$:

$$\mathrm{Far}\,(\mathcal{F}\times\mathcal{G})=\mathcal{F}\times\mathcal{G}\setminus\{\mathrm{Near}\,(\mathcal{F}\times\mathcal{G})\cup\mathrm{Disjoint}\,(\mathcal{F}\times\mathcal{G})\}.$$

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- The set Far $(\mathcal{F} \times \mathcal{G})$ is the complement of Near $(\mathcal{F} \times \mathcal{G}) \cup$ Disjoint $(\mathcal{F} \times \mathcal{G})$ in $\mathcal{F} \times \mathcal{G}$:

$$\text{Far}\,(\mathcal{F} \times \mathcal{G}) = \mathcal{F} \times \mathcal{G} \setminus \{\text{Near}\,(\mathcal{F} \times \mathcal{G}) \cup \text{Disjoint}\,(\mathcal{F} \times \mathcal{G})\}.$$

- The parallel corona splitting (5) is somewhat analogous to the splitting (4) except that corona blocks are used in place of individual intervals to determine the 'cut'.

# The form estimates

- The disjoint form $\mathrm{H}_{disjoint}(f,g)$ is easily controlled by the strong $\mathcal{A}_2$ condition and the interval testing conditions:

$$\left|\mathrm{H}_{disjoint}(f,g)\right| \lesssim \left(\sqrt{\mathcal{A}_2}+\mathfrak{T}+\mathfrak{T}^*\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

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- We show that the far form satisfies

$$\left|\mathsf{H}_{far}(f,g)\right| \lesssim \left(\sqrt{\mathcal{A}_2} + \mathfrak{T} + \mathfrak{T}^*\right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} ,$$

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  using our functional energy inequality.

- Finally we show that the near form $\mathsf{H}_{near}(f,g)$ is controlled by the strong $\mathcal{A}_2$ condition and the indicator testing conditions:

$$\left|\mathsf{H}_{near}(f,g)\right| \lesssim \left(\sqrt{\mathcal{A}_2}+\mathfrak{A}+\mathfrak{A}^*\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

# The near form

## Bounded fluctuation

- Recall that $f \in \mathcal{BF}_\sigma(K)$ if there is a pairwise disjoint collection $\mathcal{K}_f$ of $\mathcal{D}^\sigma$-subintervals of $K$ such that

$$\int_K f\sigma = 0 \text{ and } \frac{1}{|I|_\sigma}\int_I |f|\,\sigma \leq 1, \qquad I \in \widehat{\mathcal{K}_f},$$
$$f = a_{K'} \in \mathbb{R} \text{ on } K' \text{ and } |a_{K'}| > 2, \qquad K' \in \mathcal{K}_f,$$

where $\widehat{\mathcal{K}_f}$ is the corona determined by $K$ and $\mathcal{K}_f$:

$$\widehat{\mathcal{K}_f} = \left\{ I \in \mathcal{D}^\sigma : I \subset K \text{ and } I \supsetneqq K' \text{ for some } K' \in \mathcal{K}_f \right\}.$$

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- Using the facts that $\frac{1}{|I|_\sigma}\int_I |f|\,\sigma \leq 1$ for $I \in \widehat{\mathcal{K}}$ and $\frac{1}{|I|_\sigma}\int_I |f|\,\sigma > 2$ for $I \in \mathcal{K}$, the collection $\mathcal{K}$ is uniquely determined by the simple function $f$ of bounded fluctuation, and we write $\mathcal{K}_f$ for this collection.

- Define the collection $\mathcal{MBF}_\sigma(K)$ of functions of *minimal* bounded fluctuation by

$$\mathcal{MBF}_\sigma(K) = \left\{ f \in \mathcal{BF}_\sigma(K) : supp\ \widehat{f} \subset \pi\mathcal{K}_f \right\},$$

where $\widehat{f} : \mathcal{D} \to \mathbb{C}$ by $\widehat{f}(I) \equiv \langle f, h_I^\sigma \rangle_\sigma$ is the Haar coefficient map (with underlying measure $\sigma$ being understood), and

$$\pi\mathcal{K}_f \equiv \left\{ \pi_{\mathcal{D}} K' : K' \in \mathcal{K}_f \right\}.$$

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Thus the functions $f \in \mathcal{MBF}_\sigma(K)$ have their Haar support $supp\ \widehat{f}$ as small as possible given that they satisfy the conditions for belonging to $\mathcal{BF}_\sigma(K)$.
- Note that while $\mathcal{K}_f$ consists of pairwise disjoint intervals for $f \in \mathcal{MBF}_\sigma(K)$, the collection of parents $\pi\mathcal{K}_f$ may have considerable overlap, and this represents the main difficulty for further investigation.

# An essential property of minimal bounded fluctuation

- If $f \in \mathcal{MBF}_{\sigma}(I)$ is of minimal bounded fluctuation, then there is a collection $\mathcal{K}_f$ of pairwise disjoint subintervals of $I$ such that

$$f = \sum_{I \in \pi\mathcal{K}_f} \widehat{f}(I)\ h_I^{\sigma} = \sum_{I \in \pi\mathcal{K}_f} \triangle_I^{\sigma} f,$$

where if $I = \pi K$, then $K = I_-$, the child of $I$ with smallest $\sigma$-measure.

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  where if $I = \pi K$, then $K = I_-$, the child of $I$ with smallest $\sigma$-measure.

- The key additional property, besides that of bounded fluctuation, of such an $f$ is

$$\mathbb{E}_{I_+}^\sigma \triangle_I^\sigma f \geq 0, \qquad \text{for all } I \in \mathcal{K}_f.$$

# Analysis of the near form

- There is the decomposition

$$\begin{aligned} \mathsf{P}^{\sigma}_{\mathcal{C}_F} f &= \left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_1 + \left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_2 ; \\ \left\| \left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_1 \right\|_\infty &\lesssim \mathbb{E}^{\sigma}_F |f| , \\ \frac{1}{3\mathbb{E}^{\sigma}_F |f|} \left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_2 &\in \mathcal{BF}_\sigma (F) , \end{aligned} \tag{6}$$

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- A second more complicated CZ decomposition produces blocks $\mathsf{P}^{\sigma}_{\mathcal{C}_{KF}} \left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)$ satisfying

$$\frac{1}{C\mathbb{E}^{\sigma}_K |f|} \mathsf{P}^{\sigma}_{\mathcal{C}_{KF}} \left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right) \in \left(L^{\infty}\right)_1 (K) + \mathcal{MBF}_{\sigma}(K) .$$

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- This decomposition leads to control of the near form by the $\mathcal{A}_2$ and indicator/interval testing conditions. Indeed, the I/I testing conditions apply to $\left(L^\infty\right)_1 (K)$, while the special properties of $\mathcal{MBF}_\sigma (K)$ permit control by $\mathcal{A}_2$ and interval testing.

# Analysis of the far form

- Now we decompose the far form $\mathsf{H}_{far}(f,g)$ into lower and upper forms in analogy with $\mathcal{H}_{lower}$ and $\mathcal{H}_{upper}$ in (4):

$$\begin{aligned}\mathsf{H}_{far}(f,g) &= \left\{ \sum_{\substack{(F,G)\in \mathrm{Far}(\mathcal{F}\times\mathcal{G})\\ G\subset F}} + \sum_{\substack{(F,G)\in \mathrm{Far}(\mathcal{F}\times\mathcal{G})\\ F\subset G}} \right\} \left\langle H\left(\sigma \mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right), \mathsf{P}^{\omega}_{\mathcal{C}_G} g \right\rangle \\ &\equiv \mathsf{H}_{far\ lower}(f,g) + \mathsf{H}_{far\ upper}(f,g).\end{aligned}$$

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- We will use a *functional energy inequality* to control $\mathsf{H}_{far\ \ lower}(f,g)$, which is defined in terms of *F-adapted* collections of intervals.

# F-adapted collections of intervals

## Definition

Let $\mathcal{F}$ be a collection of dyadic intervals satisfying a Carleson condition

$$\sum_{F\in\mathcal{F}:\ F\subset S} |F|_\sigma \leq C_{\mathcal{F}} |S|_\sigma , \qquad S\in\mathcal{F},$$

where $C_{\mathcal{F}}$ is referred to as the Carleson norm of $\mathcal{F}$. A collection of functions $\{g_F\}_{F\in\mathcal{F}}$ in $L^2(w)$ is said to be $\mathcal{F}$*-adapted* if there are collections of intervals $\mathcal{J}(F)\subset\{J\in\mathcal{D}^\sigma : J\Subset F\}$, with $\mathcal{J}^*(F)$ consisting of the *maximal* dyadic intervals in $\mathcal{J}(F)$, such that the following three conditions hold:

# F-adapted conditions

## Definition

1. for each $F \in \mathcal{F}$, the Haar coefficients $\widehat{g_F}(J) = \langle g_F, h_J^\omega \rangle_\omega$ of $g_F$ are nonnegative and supported in $\mathcal{J}(F)$, i.e.

$$\left\{ \begin{array}{lll} \widehat{g_F}(J) \geq 0 & \text{for all} & J \in \mathcal{J}(F) \\ \widehat{g_F}(J) = 0 & \text{for all} & J \notin \mathcal{J}(F) \end{array} \right. , \qquad F \in \mathcal{F},$$

2. the collection $\{g_F\}_{F \in \mathcal{F}}$ is pairwise orthogonal in $L^2(\omega)$,
3. and there is a positive constant $C$ such that for every interval $I$ in $\mathcal{D}^\sigma$, the collection of intervals

$$\mathcal{B}_I \equiv \{J^* \subset I : J^* \in \mathcal{J}^*(F) \text{ for some } F \supset I\}$$

has overlap bounded by $C$, i.e. $\sum_{J^* \in \mathcal{B}_I} \mathbf{1}_{J^*} \leq C$, for all $I \in \mathcal{D}^\sigma$.

# The functional energy condition

- The *functional energy condition* is:

## Definition

Let $\mathfrak{F}$ be the smallest constant in the inequality below, holding for all non-negative $h \in L^2(\sigma)$, all $\sigma$-Carleson collections $\mathcal{F}$, and all $\mathcal{F}$-adapted collections $\{g_F\}_{F\in\mathcal{F}}$:

$$\sum_{F\in\mathcal{F}} \sum_{J^*\in\mathcal{J}^*(F)} \mathrm{P}(J^*, h\sigma) \left| \left\langle \frac{x}{|J^*|}, g_F \mathbf{1}_{J^*} \right\rangle_\omega \right| \leq \mathfrak{F} \|h\|_{L^2(\sigma)} \left[ \sum_{F\in\mathcal{F}} \|g_F\|^2_{L^2(\omega)} \right]^{1/2}. \tag{7}$$

Here $\mathcal{J}^*(F)$ consists of the *maximal* intervals $J$ in the collection $\mathcal{J}(F)$.

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Here $\mathcal{J}^*(F)$ consists of the *maximal* intervals $J$ in the collection $\mathcal{J}(F)$.

- The dual version of this condition has constant $\mathfrak{F}^*$.
- The functional energy condition (7) controls the lower far form $\mathsf{H}_{far\ lower}(f, g)$ using a monotonicity property of the Hilbert transform.

# The monotonicity property of the Hilbert transform

## Lemma (Monotonicity Property)

*Suppose that $\nu$ is a signed measure, and $\mu$ is a positive measure with $\mu \geq |\nu|$, both supported outside an interal $I$. Then for $J \Subset I$ we have*

$$\left|\langle H\nu, h_J^\omega \rangle_\omega\right| \leq \langle H\mu, h_J^\omega \rangle_\omega \approx \left\langle \frac{x}{|J|}, h_J^\omega \right\rangle_\omega \mathrm{P}\left(J, \mu\right).$$

The proof uses that

$$\langle H\nu, h_J^\omega \rangle_\omega = \int_J \left\{ \int_{\mathbb{R}\setminus I} \left( \frac{1}{y-x} - \frac{1}{y-x_J} \right) d\nu\left(y\right) \right\} h_J^\omega\left(x\right) d\omega\left(x\right),$$

and then that the following expression is positive for all $y$ not in $I$:

$$\left( \frac{1}{y-x} - \frac{1}{y-x_J} \right) h_J^\omega\left(x\right) = \frac{\left(x-x_J\right) h_J^\omega\left(x\right)}{\left(y-x\right)\left(y-x_J\right)}.$$

- It remains to prove that the functional energy conditions are implied by the strong $\mathcal{A}_2$ and interval testing conditions.

**Lemma**

$\mathfrak{F} \lesssim \mathcal{A}_2 + \mathfrak{T}$ *and* $\mathfrak{F}^* \lesssim \mathcal{A}_2 + \mathfrak{T}^*$.

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**Lemma**

$\mathfrak{F} \lesssim \mathcal{A}_2 + \mathfrak{T}$ *and* $\mathfrak{F}^* \lesssim \mathcal{A}_2 + \mathfrak{T}^*$.

- To prove this lemma we fix $\mathcal{F}$ as in (7) and set

$$\mu \equiv \sum_{F\in\mathcal{F}} \sum_{J^*\in\mathcal{J}^*(F)} \left\| \mathsf{P}^{\omega}_{F,J^*} \frac{x}{|J^*|} \right\|^2_{L^2(\omega)} \cdot \delta_{(c(J^*),|J^*|)} , \tag{8}$$

where the projections $\mathsf{P}^{\omega}_{F,J^*}$ onto Haar functions are defined by

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- Here $\delta_q$ denotes a Dirac unit mass at a point $q$ in the upper half plane $\mathbb{R}^2_+$. Note that we can replace $x$ by $x-c$ for any choice of $c$ we wish.

# Two weight Poisson inequality

- We prove the two-weight inequality

$$\left\|\mathbb{P}(f\sigma)\right\|_{L^2(\mathbb{R}^2_+,\mu)} \lesssim \|f\|_{L^2(\sigma)}\,, \tag{9}$$

for all nonnegative $f$ in $L^2(\sigma)$, noting that $\mathcal{F}$ and $f$ are *not* related here.

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for all nonnegative $f$ in $L^2(\sigma)$, noting that $\mathcal{F}$ and $f$ are *not* related here.

- Above, $\mathbb{P}(\cdot)$ denotes the Poisson extension to the upper half-plane, so that in particular

$$\|\mathbb{P}(f\sigma)\|^2_{L^2(\mathbb{R}^2_+,\mu)} = \sum_{F\in\mathcal{F}} \sum_{J^*\in\mathcal{J}^*(F)} \mathbb{P}(f\sigma)(c(J^*),|J^*|)^2 \left\| \mathsf{P}^{\omega}_{F,J^*} \frac{x}{|J^*|} \right\|^2_{L^2(\omega)}$$

and so (9) implies (7) by the Cauchy-Schwarz inequality.

# Reduction to Poisson tent testing

By the two-weight inequality for the Poisson operator, inequality (9) requires checking these two inequalities

$$\int_{\mathbb{R}^2_+} \mathbb{P}\left(\mathbf{1}_I \sigma\right)(x,t)^2 \, d\mu(x,t) \equiv \left\| \mathbb{P}\left(\mathbf{1}_I \sigma\right) \right\|^2_{L^2(\widehat{I},\mu)} \lesssim \left(A_2 + \mathfrak{T}^2\right) \sigma(I), \quad (10)$$

$$\int_{\mathbb{R}} [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2 \sigma(dx) \lesssim \mathcal{A}_2 \int_{\widehat{I}} t^2 \mu(dx,dt), \quad (11)$$

for all *dyadic* intervals $I \in \mathcal{D}$, where $\widehat{I} = I \times [0,|I|]$ is the box over $I$ in the upper half-plane, and

$$\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu) = \int_{\widehat{I}} \frac{t^2}{t^2 + |x-y|^2} \mu(dy,dt).$$

# Outline of Part III: what is left?

1. What could prove the $NTV$ conjecture?

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2. What can we prove from the $NTV$ hypotheses?

# What could prove the NTV conjecture?

## The bounded over square integrable stopping form

- What is needed is to show that the indicator/interval condition is controlled by the *NTV* hypotheses:

$$\int_I |H_\sigma \mathbf{1}_E|^2 \, d\omega \lesssim (\mathfrak{NTV}) \, |I|_\sigma \,, \qquad \text{for all intervals } I.$$

- What is needed is to show that the indicator/interval condition is controlled by the *NTV* hypotheses:

$$\int_I |H_\sigma \mathbf{1}_E|^2 \, d\omega \lesssim (\mathfrak{NTV}) \, |I|_\sigma \,, \qquad \text{for all intervals } I.$$

- Our proof reduces this to bounding the $L^\infty/L^2$ stopping form by *NTV*:

$$|\mathrm{B}_{stop}(\mathbf{1}_E, g)| \leq (\mathfrak{NTV}) \sqrt{|I|_\sigma} \, \|g\|_{L^2(\omega)} \,,$$

for all compact $E \subset I$ and $g \in L^2(\omega)$ with support in $I$, an interval.

# What we can prove from the NTV hypotheses

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$$\left|\mathrm{B}_{stop}\left(\mathbf{1}_E, \mathbf{1}_F\right)\right| \leq \left(\mathfrak{NTV}\right) \sqrt{|I|_\sigma |I|_\omega},$$

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- Thanks.