Spectral gaps and oscillations

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Contents of the talk

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- The Gap Problem:

  Estimating the size of the gap in the Fourier spectrum of a measure.

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  Completeness of complex exponentials in L^2-spaces.

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- The Gap Problem:

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- The Type Problem:

  Completeness of complex exponentials in L^2-spaces.

- A Problem on Oscillations of Fourier Integrals:

  How often should a measure with a spectral gap change signs?

Beurling's Gap Problem

Let $X \subset \mathbb{R}$ be a closed set.

Question: Under what conditions on X does there exist a non-zero finite complex measure $\mu, \operatorname{supp}\mu \subset X$ whose Fourier transform

$$\hat{\mu}(x) = \int e^{2\pi i x t} d\mu(t)$$

vanishes on an interval? How to determine the maximal size of such an interval (spectral gap)?

The Gap Problem is a part of an area called Uncertainty Principle in Harmonic Analysis. In this context the principle says that the supports of the measure and its Fourier transform cannot both be small.

Beurling's Gap Problem

Definition

If X is a closed subset of the real line denote

$$\mathbf{G}_X = \sup\{\ a \mid \exists\ \mu \not\equiv 0,\ \operatorname{supp}\mu \subset X,\ \ \text{such that}\ \ \hat{\mu} = 0\ \ \text{on}\ \ [0, a]\}$$

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Examples:

1) If X is bounded from below or from above then $\mathbf{G}_X = 0$.

2) If μ is the counting measure of $\mathbb{Z}$ then $\hat{\mu} = \mu$ in the sense of distributions (Poisson formula). It follows that $\mathbf{G}_{\frac{1}{d}\mathbb{Z}} = d$.

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3) Since $Y \subset X \Rightarrow \mathbf{G}_Y \leq \mathbf{G}_X$, If X contains $\frac{1}{d}\mathbb{Z} + c$ then $\mathbf{G}_X \geq d$.

Beurling's Gap Theorem

Definition

A sequence of disjoint intervals $\{I_n\}$ is *long* if

$$\sum \frac{|I_n|^2}{1 + \mathrm{dist}^2(0, I_n)} = \infty$$

and *short* otherwise.

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Theorem (Beurling's Gap Theorem)

If the complement of X is long then $\mathbf{G}_X = 0$.

A solution to the Gap Problem: Energy

Let $\Lambda = \{\lambda_1, ..., \lambda_n\}$ be a finite set of points on $\mathbb{R}$. Consider the quantity

$$L(\Lambda) = \sum_{k \neq l} \log |\lambda_k - \lambda_l|.$$

Physical interpretation: $L(\Lambda)$ is the energy of a system of electrons placed at the points of Λ (2D Coulomb gas).

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Key example:

Let $I \subset \mathbb{R}$ be an interval, $\Lambda = I \cap \frac{1}{D}\mathbb{Z}$. Then

$$\Delta = \#\Lambda = D|I| + o(|I|), \quad L(\Lambda) = \Delta^2 \log|I| + O(|I|^2)$$

as follows from Stirling's formula.

(Note: $\max L(\Lambda)$ is attained when the electrons are placed at the endpoints of the interval and at the roots of the Jacobi polynomial of degree $k - 2$.)

A solution to the Gap Problem: Short partitions

Let

$$\ldots < a_{-2} < a_{-1} < a_0 = 0 < a_1 < a_2 < \ldots$$

be a two-sided sequence of real points. We say that the intervals $I_n = (a_n, a_{n+1}]$ form a short partition of $\mathbb{R}$ if $|I_n| \to \infty$ as $|n| \to \infty$ and the sequence $\{I_n\}$ is short, i.e.

$$\sum \frac{|I_n|^2}{1 + \operatorname{dist}^2(I_n, 0)} < \infty.$$

A solution to the Gap Problem: D-uniform sequences

Let $\Lambda = \{\lambda_n\}$ be a sequence of distinct real points. We say that Λ is D-uniform if if there exists a short partition I_n such that

$$\Delta_n = D|I_n| + o(|I_n|) \quad \text{as} \quad n \to \pm\infty \quad \textit{(density condition)}$$

and

$$\sum_n \frac{\Delta_n^2 \log |I_n| - L_n}{1 + \operatorname{dist}^2(0, I_n)} < \infty \quad \textit{(energy condition)}$$

where

$$\Delta_n = \#(\Lambda \cap I_n) \quad \text{and} \quad L_n = L(\Lambda \cap I_n) = \sum_{\lambda_k, \lambda_l \in I_n,\ \lambda_k \neq \lambda_l} \log |\lambda_k - \lambda_l|.$$

A solution to the Gap Problem: The main theorem

Recall

$$\mathbf{G}_X = \sup \{ \ a \mid \exists \ \mu \not\equiv 0, \ \operatorname{supp} \mu \subset X, \quad \text{such that} \quad \hat{\mu} = 0 \quad \text{on} \quad [0, a] \}$$

Theorem

$$\mathbf{G}_X = \sup \{ \ D \mid X \ \textit{contains a } D\textit{-uniform sequence} \ \}.$$

Corollaries: separated sequences

Interior BM density of a discrete sequence:
$D_*(\Lambda) = \inf\{d \mid \exists \text{ long } \{I_n\} \text{ such that } \#(\Lambda \cap I_n) \leqslant d|I_n|, \ \forall n\}.$

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Corollary (M. Mitkovski, A.P.)

If $\Lambda \subset \mathbb{R}$ is a separated sequence then

$$\mathbf{G}_\Lambda = D_*(\Lambda).$$

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Example

Let $\Lambda = \{\lambda_n\}$ be a separated sequence such that $\lambda_n - n = O(n^{\frac{1}{2}-\varepsilon})$. Then

$$\mathbf{G}_\Lambda = 1.$$

The Type Problem

Let μ be a finite positive measure on the real line. For $a > 0$ denote by $\mathcal{E}_a$ the family of exponential functions

$$\mathcal{E}_a = \{e^{2\pi ist} |\ s \in [0, a]\}.$$

The exponential type of μ:

$$T_\mu = \inf\{a > 0 |\ \mathcal{E}_a \text{ is complete in } L^2(\mu)\}$$

if the set of such a is non-empty and infinity otherwise.

Problem

Find T_μ in terms of μ.

The Type Problem: History

This question first appears in the work of Wiener, Kolmogorov and Krein in the context of stationary Gaussian processes in 1930-40's. If μ is a spectral measure of a stationary Gaussian process, the property that $\mathcal{E}_a$ is complete in $L^2(\mu)$ is equivalent to the property that the process at any time can be predicted from the data for the time period from 0 to a.

The type problem can also be restated in terms of the Bernstein weighted approximation, see for instance Koosis' book. Important connections with spectral theory of second order differential operators were studied by Gelfand, Levitan and Krein.

Known results

A classical result by Krein (1945) says that if $d\mu = w(x)dx$ and $\log w(x)/(1+x^2)$ is summable then $T_\mu = \infty$. A partial inverse, proved by Levinson and McKean (1964), holds for even monotone w.

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A theorem by Duffin and Schaeffer (1945) says that if μ is a measure such that for any $x \in \mathbb{R}$

$$\mu([x-L, x+L]) > d$$

for some $L, d > 0$ then

$$T_\mu \geqslant L$$

(here μ is Poisson-finite, i.e. $\int d\mu(x)/(1+x^2) < \infty$).

Known results

For discrete measures, in the case $\operatorname{supp}\mu = \mathbb{Z}$, a deep result by Koosis shows an analogue of Krein's result: if $\mu = \sum w(n)\delta_n$ where

$$\sum \frac{\log w(n)}{1+n^2} > -\infty$$

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A recent result by Borichev and Sodin (2010) says that exponentially small perturbations of weight or support do not change the type of a measure.

Type Theorem

Let τ be a finite positive finite measure on the real line. We say that a function W is a τ-summable weight if W is lower semi-continuous, tends to ∞ at $\pm\infty$, $W \geq 1$ on $\mathbb{R}$ and $W \in L^1(\tau)$.

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Theorem

Let μ be a finite positive measure on the line. Then $T_\mu \geq a > 0$ if and only if for any μ-summable weight W and any $0 < D < a$ there exists a D-uniform sequence $\Lambda = \{\lambda_n\} \subset \operatorname{supp} \mu$ such that

$$\sum \frac{\log W(\lambda_n)}{1 + {\lambda_n}^2} < \infty.$$

Corollaries: discrete measures

Extension of Kousis' result:

Corollary

Let $A = \{a_n\} \subset \mathbb{R}$ *be a separated sequence and let* $\mu = \sum w(n)\delta_{a_n}$ *be a positive finite measure on* A*. Consider the set* S *of all subsequences* $\{a_{n_k}\}$ *of* A *satisfying*

$$\sum_k \frac{\log w(n_k)}{1 + {n_k}^2} > -\infty.$$

Then

$$T_\mu = \sup_{B \in S} D_*(B).$$

Toeplitz approach (N. Makarov, A. P.). An example.

Let μ be a positive finite singular measure on $\mathbb{R}$. Suppose that there exists $f \in L^2(\mu), f \perp e^{2\pi ist}$ for all $s \in [0, a]$. Then $Kf\mu$ is divisible by $S^a = e^{2\pi iaz}$.

Consider an inner function $\theta(z)$ in the upper halfplane satisfying

$$K\mu(z) = \int \frac{1}{t-z} d\mu(t) = i\frac{1-\theta}{1+\theta}.$$

By Clark theory the function $F(z) = (1-\theta)Kf\mu$ belongs to the model space $H^2 \ominus \theta H^2$ in the upper halfplane. If $F = S^aG$, then G belongs to the kernel of the Toeplitz operator $T_{\bar\theta S^a}$. Thus the type of μ is greater than a iff $kerT_{\bar\theta S^a} \neq \emptyset$.

Oscillations of Fourier Integrals

Theorem (Sturm, 1836; Hurwitz)

Let

$$f(x) = \sum_{n \geq m} (c_n e^{2\pi i n x} + \bar{c}_n e^{-2\pi i n x})$$

be a smooth function. Then f has at least $2m$ sign changes on $[-1, 1]$.

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Generalizations?

High-Pass Signals

Let $f \in L^1(\mathbb{R})$ be such that $\hat{f} = 0$ on $[0, a]$ for some $a > 0$.

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Such functions are important in Electrical Engineering: they correspond to so-called high-pass signals.

Exercise: prove that a high-pass signal cannot be positive.

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Exercise: prove that a high-pass signal cannot be positive.

Problem (Grinevich, 1965; included in Arnold Problems, 2000)

How fast should a function with a spectral gap (a high-pass signal) oscillate?

A Theorem by Eremenko and Novikov

If $f \in L^1(\mathbb{R})$, denote by $s(f, r)$ the number of sign changes of f on $[0, r]$ (in any reasonable sense).

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Theorem (Eremenko-Novikov, 2003)

Suppose f has a spectral gap, that is $\hat{f}$ vanishes on $[0, a]$ for some $a > 0$. Then

$$\liminf_{r \to \infty} \frac{s(f, r)}{r} \geq a.$$

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Suppose f has a spectral gap, that is $\hat{f}$ vanishes on $[0, a]$ for some $a > 0$. Then

$$\liminf_{r \to \infty} \frac{s(f, r)}{r} \geq a.$$

The theorem proves a conjecture by Grinevich (1965). Extends results by Krein, Levin, Ostrovski and Ulanovski.

Oscillations of Fourier Integrals

Let $X, Y \subset \mathbb{R}$ be closed sets. Denote

$$\mathbf{G}(X, Y) = \sup\{\ a \mid \exists\ \mu,\ \mu > 0 \text{ on } X,\ \mu < 0 \text{ on } Y,\ \hat{\mu} = 0 \ \text{ on } \ [0, a]\}$$

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Theorem (M. Mitkovski, A.P.)

$$\mathbf{G}(X,Y) = \sup\{D\ |\exists D\text{-}uniform\ \{\lambda_n\}, \{\lambda_{2n}\} \subset X, \{\lambda_{2n+1}\} \subset Y\}$$

(Note: D-uniform sequences are enumerated in increasing order.)

Oscillations of Fourier Integrals

We say that a finite measure μ on $\mathbb{R}$ changes signs on (a, b) if there exist sets $A, B \subset (a, b)$ such that $\mu(A) > 0$ and $\mu(B) < 0$.

Corollary

If μ has a spectral gap of the size D then there exists a D-uniform sequence $\{\lambda_n\}$ such that μ changes signs on each $(\lambda_n, \lambda_{n+1})$.