Recent Challenges in Multifractal Analysis

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Motivation

Purpose of multifractal analysis : Introduce and study classification and model selection parameters for data (functions, measures, distributions, signals, images), which are based on global and local regularity

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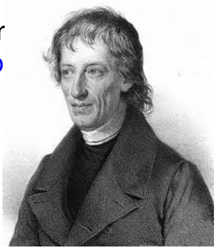
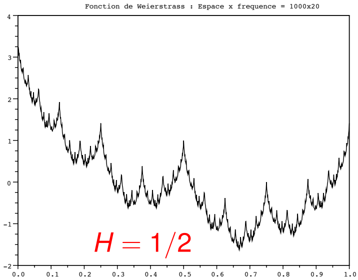
A key problem along the 19th century was to determine if a continuous function on $\mathbb{R}$ necessarily has points of differentiability

A first negative answer was obtained by B. Bolzano in 1830 but was unnoticed



A second counterexample due to K. Weierstrass settled the issue in 1872

Weierstrass functions



$$W_H(x) = \sum_{j=0}^{+\infty} 2^{-Hj} \cos(2^j x)$$

$$0 < H < 1$$

Weierstrass result was sharpened using a continuous scale of pointwise regularity indices

Pointwise regularity

Definition :
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally bounded function and $x_0 \in \mathbb{R}^d$;
$f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial P such that, for $|x - x_0|$ small enough,

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

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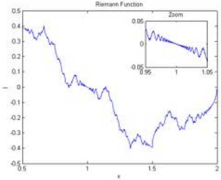
Theorem : (Hardy, 1916)
The Hölder exponent of W_H is constant and equal to H
(W_H is a mono-Hölder function)

Pointwise regularity : Example and open problems

Riemann's nondifferentiable function :

$$\mathcal{R}_2(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$$



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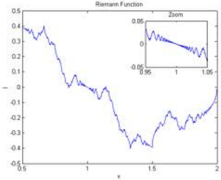

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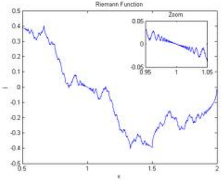

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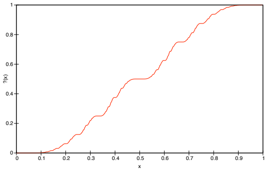
Minkowski's "question mark" function :

$?(x) : [0,1] \to [0,1]$

If $x = [0; a_1, \cdots a_n, \cdots]$ then

$$?(x) = 2 \sum \frac{(-1)^{n+1}}{2^{a_1 + \cdots + a_n}}$$



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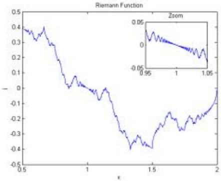

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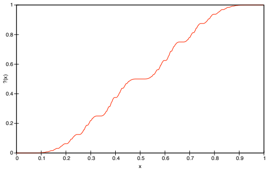
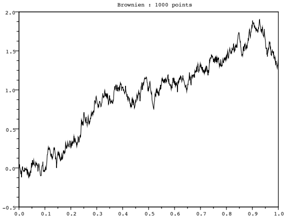


C. Hermite : "Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions qui n'ont pas de dérivée"

H. Poincaré called such functions "monsters"

Brownian motion

Economists (L. Bachelier) and physicists (A. Einstein) put into light the central role played by Brownian motion in modeling



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Brownian motion is the unique continuous process with independent and stationary increments

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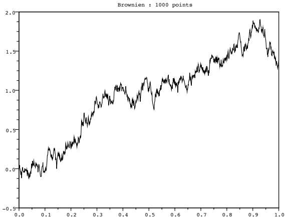

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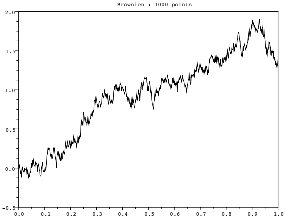

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In 1931, S. Banach proved that a "generic" continuous function on $\mathbb{R}$ is nowhere differentiable (in the sense of Baire categories)

Nowhere differentiable functions

Starting with the example of the surface of colloïds, and the coast of Brittany, J. Perrin, in his book, “Les atomes” published in 1913, insists that such examples, far from being exceptional, supply the right models for natural phenomena

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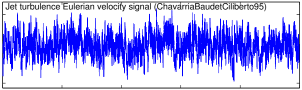
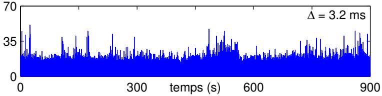
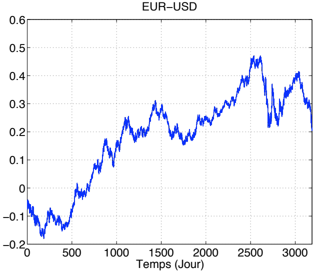


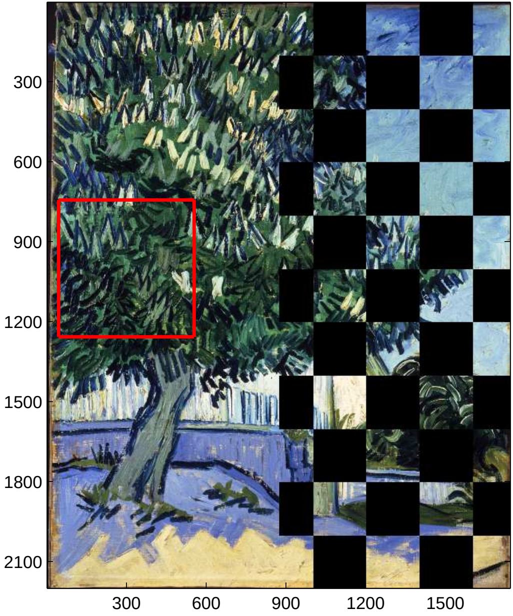
Fully developed turbulence



Internet Trafic



Euro vs Dollar (2001-2009)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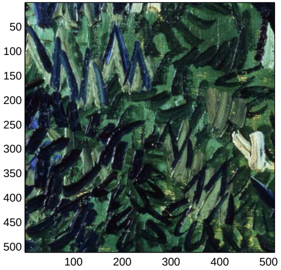
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Red channel

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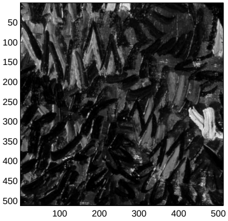

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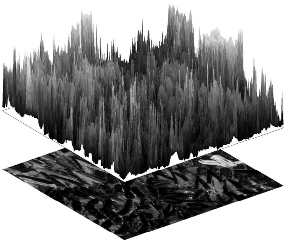

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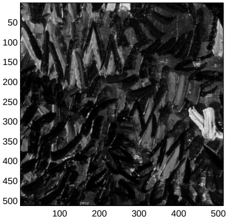

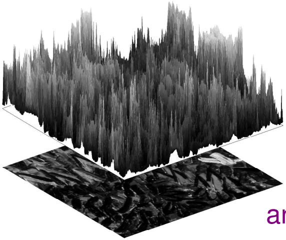

Nature,

Sciences,

and Arts

supply a large variety

of everywhere irregular

functions

Challenge :

Measure this irregularity

and use it for classification

and model selection

Fractional Brownian Motions

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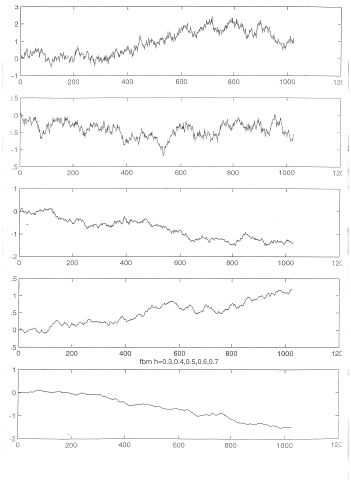
$H = 0.3$

$H = 0.4$

$H = 0.5$

$H = 0.6$

$H = 0.7$



Gaussian

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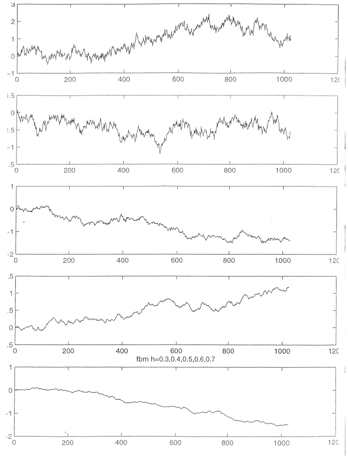

Gaussian processes with stationary increments

Theorem : (A.N. Kolmogorov)
The Hölder exponent of B_H is constant and equal to H

Challenge : Find a numerically stable way to decide if a real-life signal can be modeled by FBM

Kolmogorov scaling function (1941)

$$\int |f(x+\delta) - f(x)|^p dx \sim |\delta|^{\zeta_f(p)}$$

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Numerically : Regression on a log-log plot

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Numerically : Regression on a log-log plot

What is the scaling function of FBM ?

B_H is the unique centered Gaussian process such that

$$\forall x, \delta \geq 0, \qquad \mathbb{E}(|B_H(x+\delta) - B_H(x)|^2) = |\delta|^{2H}$$

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$$\int |f(x+\delta)-f(x)|^p dx \sim |\delta|^{\zeta_f(p)} \Longleftrightarrow \zeta_f(p) = \liminf_{j\to+\infty} \frac{\log\left(\int |f(x+\delta)-f(x)|^p dx\right)}{\log\delta}$$

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It follows that

$$|B_H(x+\delta)-B_H(x)| \sim |\delta|^H$$

$$\int |B_H(x+\delta)-B_H(x)|^p dx \sim |\delta|^{Hp}$$

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$$\int |B_H(x+\delta) - B_H(x)|^p dx \sim |\delta|^{Hp}$$

$\Longrightarrow$ Turbulence at small scale cannot be modeled by FBM (1950s)

Functional interpretation : Lipschitz spaces

$$\int |f(x+\delta) - f(x)|^p dx \sim |\delta|^{\zeta_f(p)}$$

Definition : Let $p \in [1, \infty)$; $f \in Lip\,(s, L^p(\mathbb{R}^d))$ if

$$\exists C > 0,\ \forall \delta > 0, \qquad \| f(\cdot + \delta) - f(\cdot) \|_p \leq C \cdot |\delta|^s$$

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$$\forall p \geq 1, \qquad \zeta_f(p) = p \cdot \sup \{ s : f \in Lip\ (s, L^p) \}$$

One can replace the spaces $Lip\ (s, L^p(\mathbb{R}^d))$ by Sobolev spaces

$$L^{p,s} = \{ f \in L^p :\ (-\Delta)^{s/2} f \in L^p \}$$

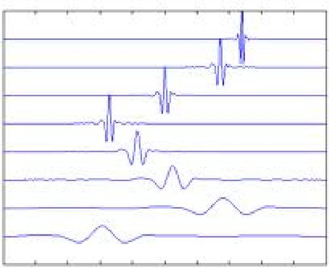
The scaling function yields a regularity index in the L^p norm

One-variable wavelet basis

A wavelet basis on $\mathbb{R}$ is generated by a smooth, well localized, oscillating function ψ such that the

$$\psi(2^j x - k), \qquad j, k \in \mathbb{Z}$$

form an orthogonal basis of $L^2(\mathbb{R})$



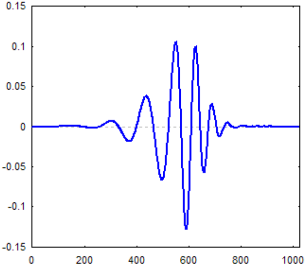
$\forall f \in L^2(\mathbb{R})$,

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\ \psi(2^j x - k)$$

where

$$c_{j,k} = 2^j \int f(x)\ \psi(2^j x - k)\ dx$$

Daubechies Wavelet



Credit to : http ://www.kfs.oeaw.ac.at/content/blogcategory/0/502/lang,8859-1/

Notations for wavelets on $\mathbb{R}$

Dyadic intervals

$$\lambda = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$$

Wavelets

$$\psi_\lambda(x) = \psi(2^j x - k)$$

Wavelet coefficients

$$c_\lambda = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)dx$$

Dyadic intervals at scale j

$$\Lambda_j = \{\lambda : \quad |\lambda| = 2^{-j}\}$$

Wavelet expansion of f

$$f(x) = \sum_j \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda(x)$$

Wavelets in 2 variables

In 2D, the wavelets used are tensor products :

$$\psi^1(x, y) = \psi(x)\varphi(y)$$

$$\psi^2(x, y) = \varphi(x)\psi(y)$$

$$\psi^3(x, y) = \psi(x)\psi(y)$$

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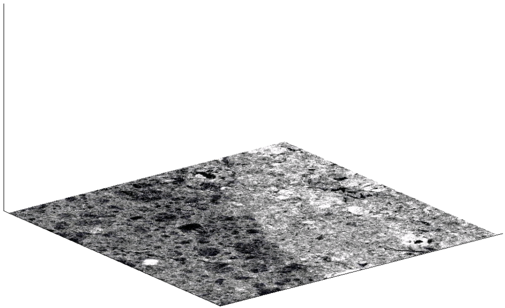
Notations

Dyadic squares : $\lambda = \left[\frac{k}{2^j}, \frac{(k+1)}{2^j}\right] \times \left[\frac{l}{2^j}, \frac{(l+1)}{2^j}\right[$

Wavelet coefficients

$$c_\lambda = 2^{2j} \int\int f(x,y)\, \psi^i\left(2^j x - k, 2^j y - l\right)\, dx\, dy$$

Computation of 2D wavelet coefficients



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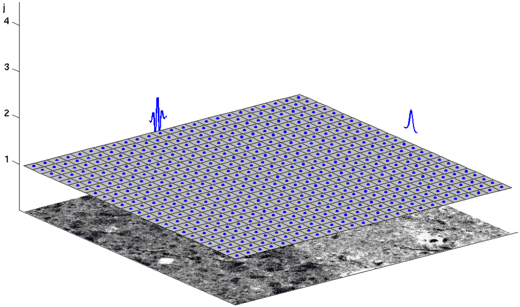

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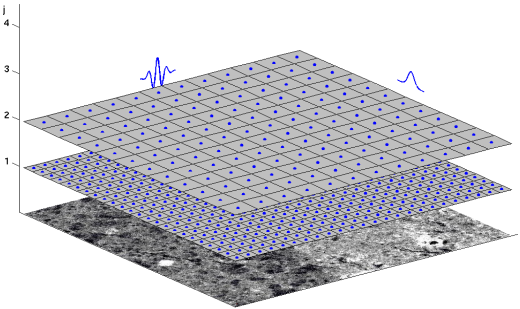

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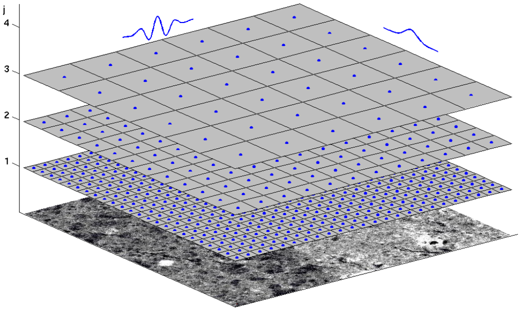

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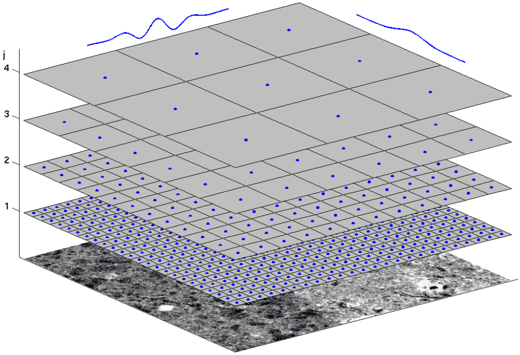

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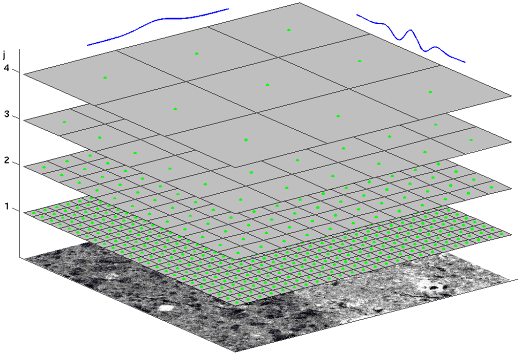

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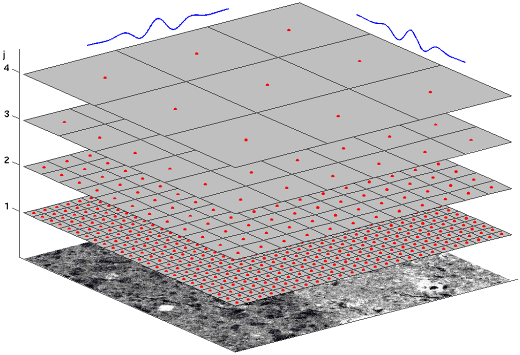

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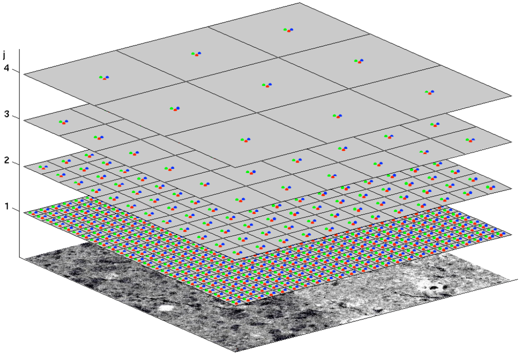

The wavelet scaling function

Let $f : \mathbb{R}^d \to \mathbb{R}$; its wavelet scaling function is defined $\forall p > 0$ by

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p) j} \quad \text{i.e.} \quad \zeta_f(p) = \liminf_{j \to +\infty} \frac{\log \left(2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \right)}{\log(2^{-j})}$$

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Besov spaces : Let $p > 0$; $f \in B_p^{s,\infty}(\mathbb{R}^d)$ if

$$\exists C, \forall j : \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \le C \cdot 2^{-spj}$$

The wavelet scaling function

Let $f : \mathbb{R}^d \to \mathbb{R}$; its wavelet scaling function is defined $\forall p > 0$ by

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p) j} \quad \text{i.e.} \quad \zeta_f(p) = \liminf_{j \to +\infty} \frac{\log \left(2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \right)}{\log(2^{-j})}$$

Besov spaces : Let $p > 0$; $f \in B^{s,\infty}_p(\mathbb{R}^d)$ if

$$\exists C, \forall j : \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \leq C \cdot 2^{-spj}$$

$$\zeta_f(p) = \liminf_{j \to +\infty} \frac{\log \left(2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \right)}{\log(2^{-j})} = p \cdot \sup \left\{ s : \quad f \in B^{s,\infty}_p(\mathbb{R}^d) \right\}$$

Embeddings between Lipschitz and Besov spaces imply that, when $p \geq 1$, the wavelet scaling function coincides with Kolmogorov's scaling function

The role of the wavelet scaling function

$$\forall p > 0, \qquad \zeta_f(p) = p \cdot \sup \left\{ s : f \in B_p^{s,\infty} \right\}$$

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It is defined by regression on log-log plots

- If $\zeta_f(1) > 1$, then $f \in BV$
- If *f* is a measure, then $\zeta_f(1) \geq 0$
- If $\zeta_f(1) > 0$, f then belongs to L^1
- If $\zeta_f(2) > 0$, then $f \in L^2$

Motivations :

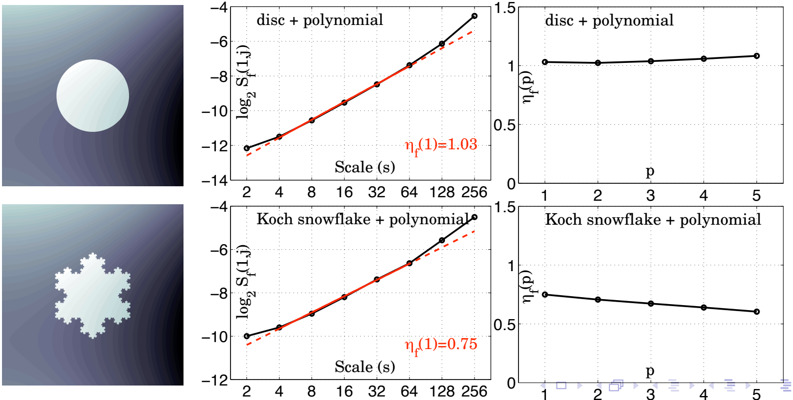
- Y. Gousseau, J.-M. Morel : *Are natural images of bounded variation ?* (SIAM J. Math. Anal., Vol. 3, 2001)
- Jump models and finite quadratic variation assumption in finance

Wavelet scaling functions of synthetic images

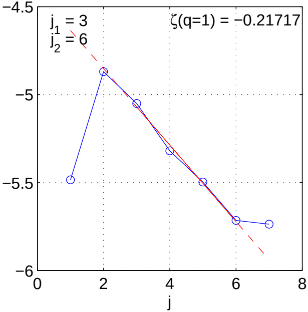
Wavelet scaling function $\zeta_f(p)$:

$$2^{-2j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p)\, j}$$

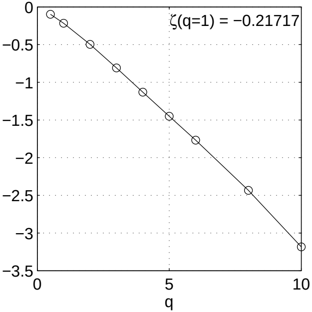
Disk : $\zeta_f(p) = 1$ Van Koch snowflake : $\zeta_f(p) = 2 - \frac{\log 4}{\log 3} \sim 0.74$

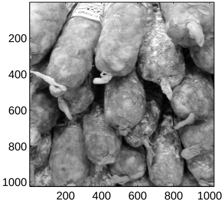


Natural images



DWT fonction d'echelle ζ(q)

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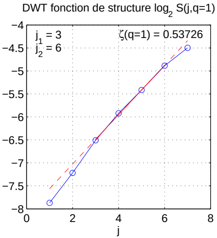
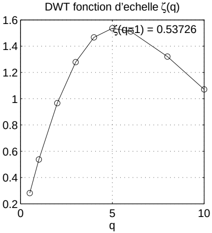

DWT fonction de structure log2 S(j,q=1)
j1 = 3
j2 = 6
ζ(q=1) = 0.53726
−4
−4.5
−5
−5.5
−6
−6.5
−7
−7.5
−8
0
2
4
6
8
j


DWT fonction d'echelle ζ(q)
ζ(q=1) = 0.53726
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
5
10
q

Uniform Hölder regularity

Hölder spaces : Let $\alpha \in (0,1)$; $f \in C^\alpha(\mathbb{R}^d)$ if

$$\exists C, \quad \forall x, y : \qquad |f(x) - f(y)| \leq C \cdot |x - y|^\alpha$$

$$\forall \alpha \in \mathbb{R}, \qquad C^\alpha(\mathbb{R}^d) = B^\alpha_\infty(\mathbb{R}^d)$$

The uniform Hölder exponent of f is

$$H_f^{min} = \sup\{\alpha : \; f \in C^\alpha(\mathbb{R}^d)\}$$

Numerical computation

Let $\omega_j = \sup_{\lambda \in \Lambda_j} |c_\lambda|$ then $H_f^{min} = \liminf_{j \to +\infty} \dfrac{\log(\omega_j)}{\log(2^{-j})}$

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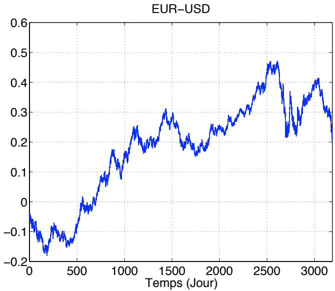
$$H_f^{min} > 0 \quad \Longrightarrow \quad f \text{ is continuous}$$

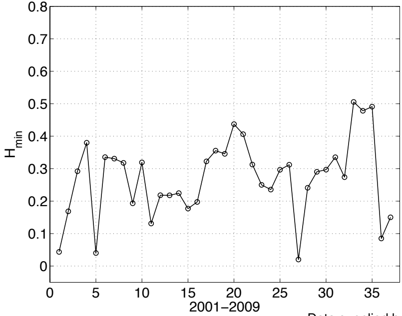
$$H_f^{min} < 0 \quad \Longrightarrow \quad f \text{ is not locally bounded}$$

Validity of jump models in finance

Euro vs. USD
2001-2009



H_f^{min}

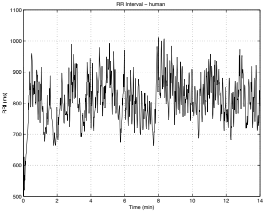


Data supplied by Vivienne Investissement

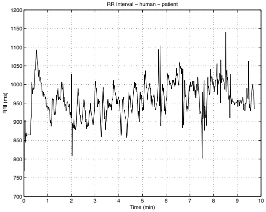
Classification based on the uniform Hölder exponent

Heartbeat intervals

Healthy



Heartbeat failure



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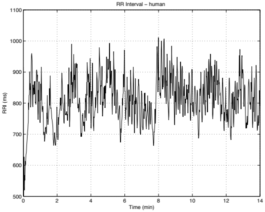
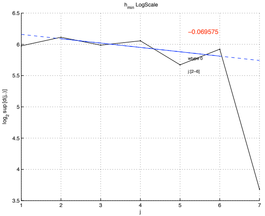
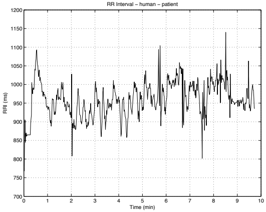
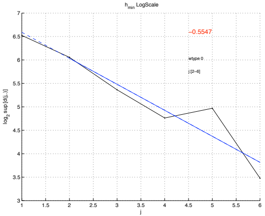




$$H_f^{min} = -0.06$$

Heartbeat failure





$$H_f^{min} = -0.55$$

Function space regularity :

Validity of stochastic integration tools in finance

Definition : A function $f : \mathbb{R} \to \mathbb{R}$ has finite quadratic variation if

$$\exists C, \ \ \forall a, h \in (0,1], \qquad \sum_n |f((n+1)h - a) - f(nh - a)|^2 \leq C$$

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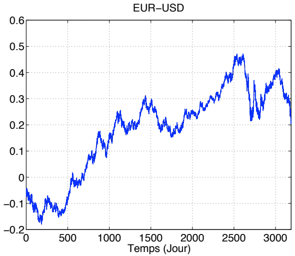
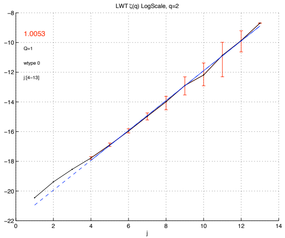


Euro vs. USD (2001-2009)



$\zeta_f(2) = 1.0053$

Limitations

Classification only based on the wavelet scaling function or on the uniform Hölder exponent proved insufficient in several occurrences (turbulence, data mining, ...)

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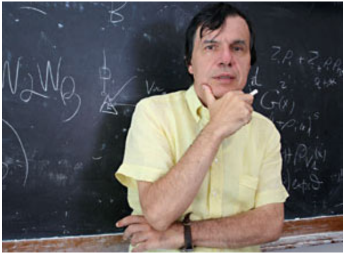
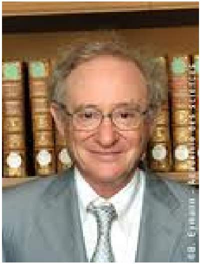
Further developments were based on seminal ideas introduced by U. Frisch and G. Parisi, and paved the way to the construction of a new scaling function



Giorgio Parisi



Uriel Frisch

Multifractal spectrum (Parisi and Frisch, 1985)

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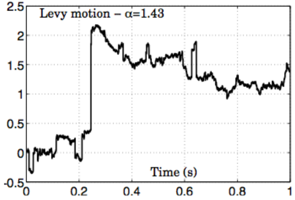
Parisi and Frisch's fundamental idea was that the nonlinearity of the scaling function reflects the presence of a whole range of fractal sets E_H, and that the scaling function yields information on the "sizes" of these sets

Two results showed that multifractal analysis does not only concern “strange examples” :

I : Probabilistic result :

Definition : A Lévy process is a stochastic process with independent and stationary increments, i.e. :

$X_{t+s} - X_t$ is independent of the $\{X_u,\ u \leq t\}$ and has the same law as X_s



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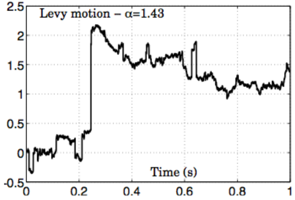

Theorem : (S.J.) "Most" Lévy processes have multifractal sample paths, with a linear multifractal spectrum :

$$D_f(H) = \begin{cases} RH & \text{if} \quad H < 1/R \\ -\infty & \text{else.} \end{cases}$$

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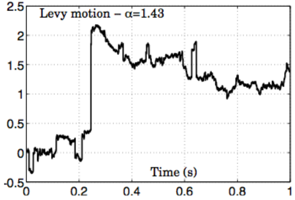

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In each case, the sets E_H are everywhere dense

$$\Longrightarrow$$

The numerical determination of the Hölder exponent is hopeless

II : Generic results :

Definition : Let E be a metric Banach space. A Borel set $A \subset E$ is Haar null if there exists a compactly supported probability measure μ on E such that

$$\forall x \in E, \qquad \mu(A + x) = 0$$

A set is prevalent if its complement is Haar null

If a property holds on a prevalent set, it is said to hold almost everywhere

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Theorem : (A. Fraysse and S. J.) Let $s > d/p$; then quasi-every and almost every function f of $L^{p,s}(\mathbb{R}^d)$ is multifractal, and its spectrum is given by

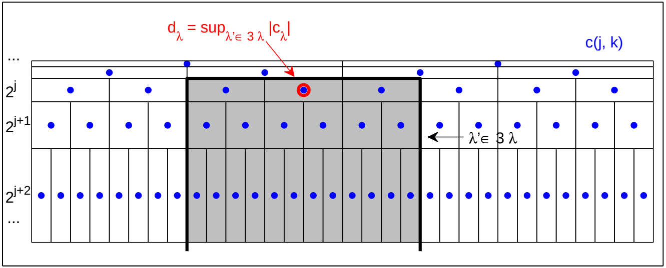
$$D_f(H) = \begin{cases} p(H-s)+d & \text{if } \quad H \in \left[s - \frac{d}{p}, s\right] \\ -\infty & \text{else} \end{cases}$$

Wavelet leaders

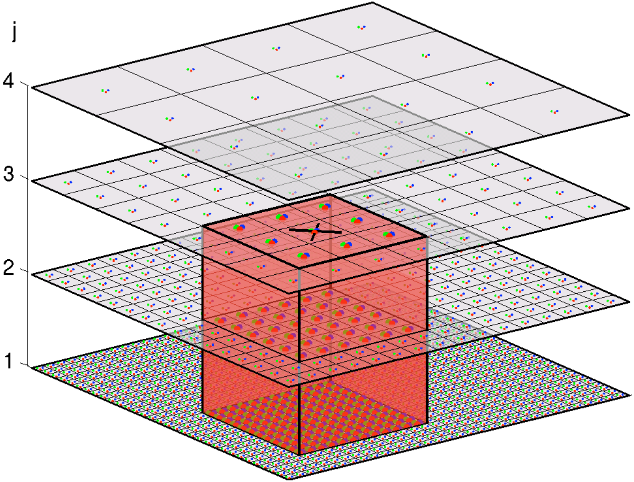
Let λ be a dyadic cube ; 3λ denotes the cube of same center and three times wider

Let f be a locally bounded function ; the wavelet leaders of f are

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$



Computation of 2D wavelet leaders



Wavelet leaders allow to estimate the pointwise Hölder exponent

Leader scaling function

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Oscillation spaces : Let $p > 0$; $f \in O^s_p(\mathbb{R}^d)$ if

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Similar to Wiener Amalgam Spaces (H. Feichtinger, K. Gröchenig)

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Advantages :

- It is "well defined" for all $p \in \mathbb{R}$
- For p large enough, $\zeta_f(p) = \eta_f(p)$
- If $\psi^i \in \mathcal{S}(\mathbb{R}^d)$, then $\forall p \in \mathbb{R}$, $\eta_f(p)$ is independent of the wavelet basis
- η_f is invariant under "smooth perturbations " of f

Multifractal formalism

Since η_f is a concave function, there is no loss of information in rather considering its Legendre transform :

The Legendre Spectrum of f is

$$L_f(H) = \inf_{p \in \mathbb{R}} \left(d + Hp - \eta_f(p) \right)$$

Theorem : Let $D_f(H)$ denote the Hausdorff dimension of the set of points where $h_f(x) = H$. If $f \in C^\varepsilon(\mathbb{R}^d)$ for an $\varepsilon > 0$ then

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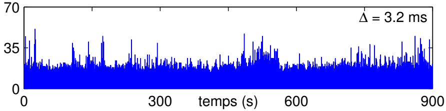
The multifractal formalism is satisfied when equality holds

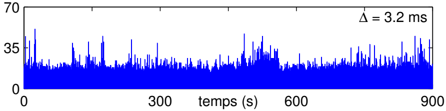
Open problem : Find “reasonably weak” general hypotheses implying the validity of the multifractal formalism

Monohölder vs. Multifractality



Internet traffic

Monohölder vs. Multifractality



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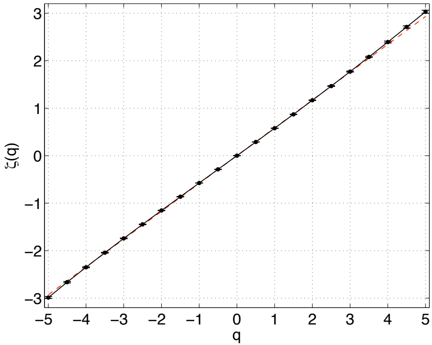
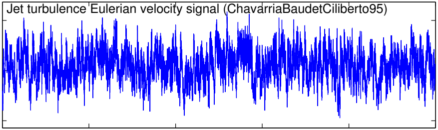


Wavelet leader scaling function

Data from http ://mawi.wide.ad.jp/mawi/

Model refutation : Fully developed turbulence

(joint work with Bruno Lashermes)



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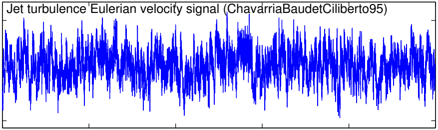
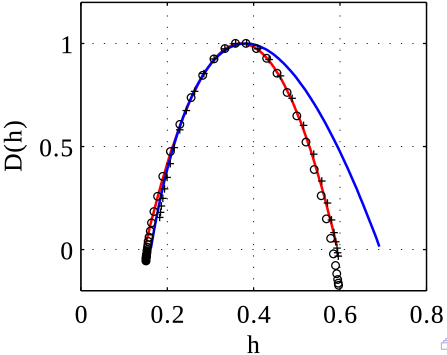


Log-normal vs. Log-Poisson model

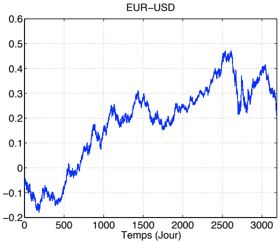


Evidence of time-evolution : Finance

Multifractal analysis

of the USD-Euro

change rate



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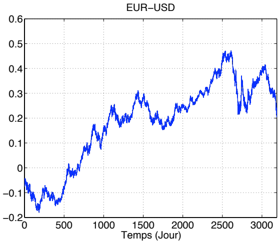
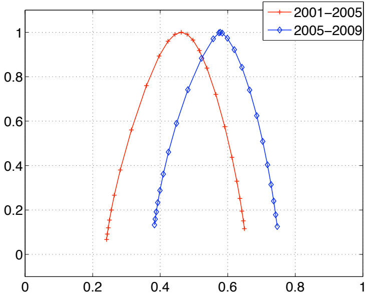


Legendre

spectra

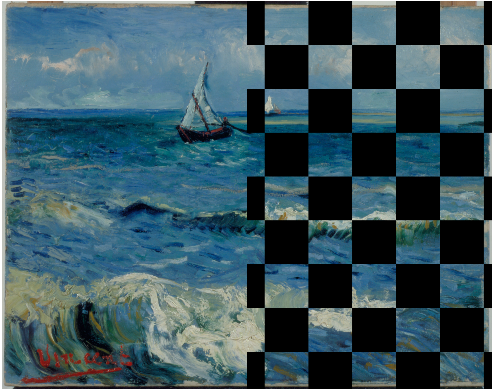


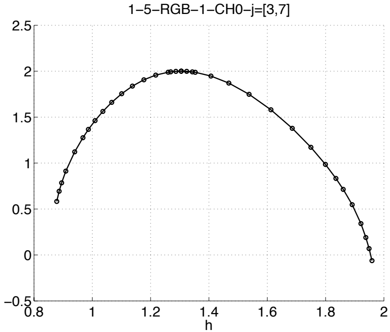
Multifractal analysis of paintings : Van Gogh challenge

Initiated by the Van Gogh Museum (Amsterdam)

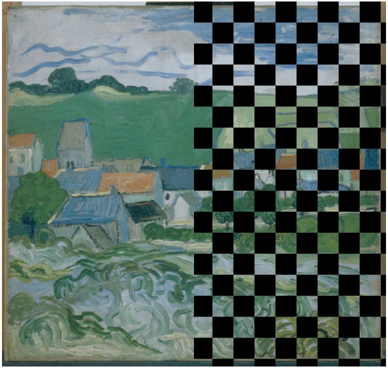
Coordinated by I. Daubechies and R. Johnson

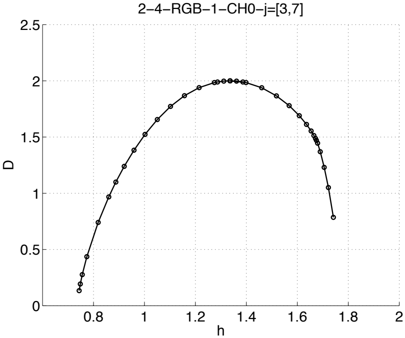
(joint work with D. Rockmore)





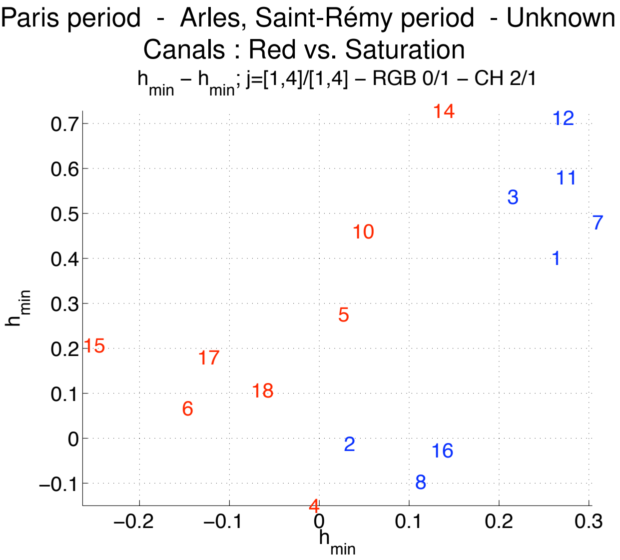
Van Gogh : Arles and Saint-Rémy period





Van Gogh : Paris period

Challenge : Date



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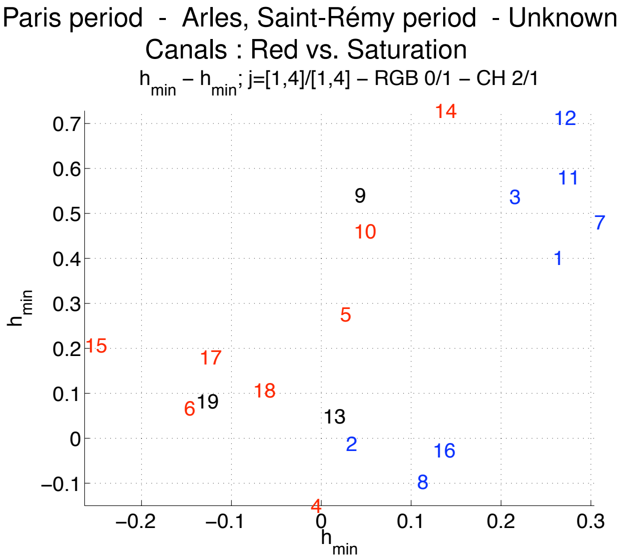
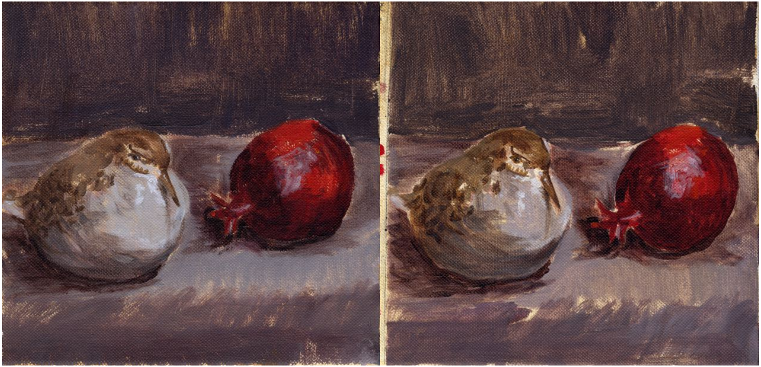


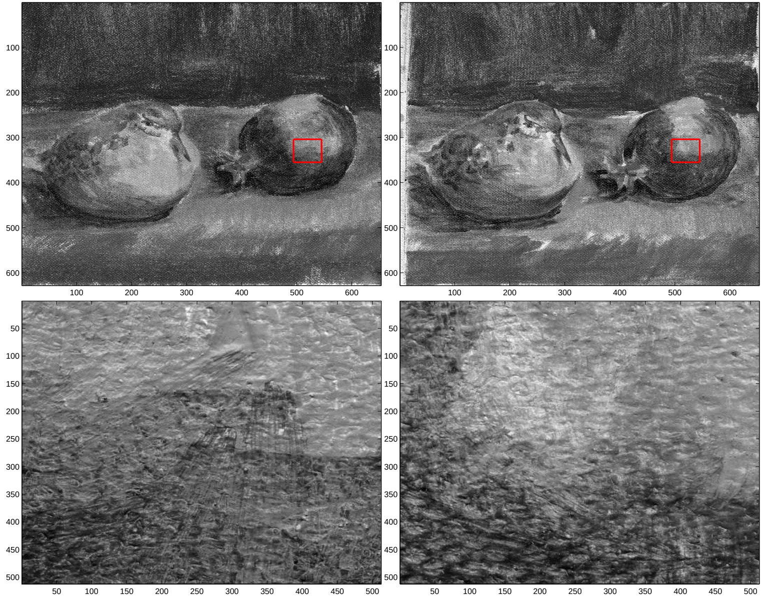
Original and copy : Stylometry issues

Experiment initiated by I. Daubechies

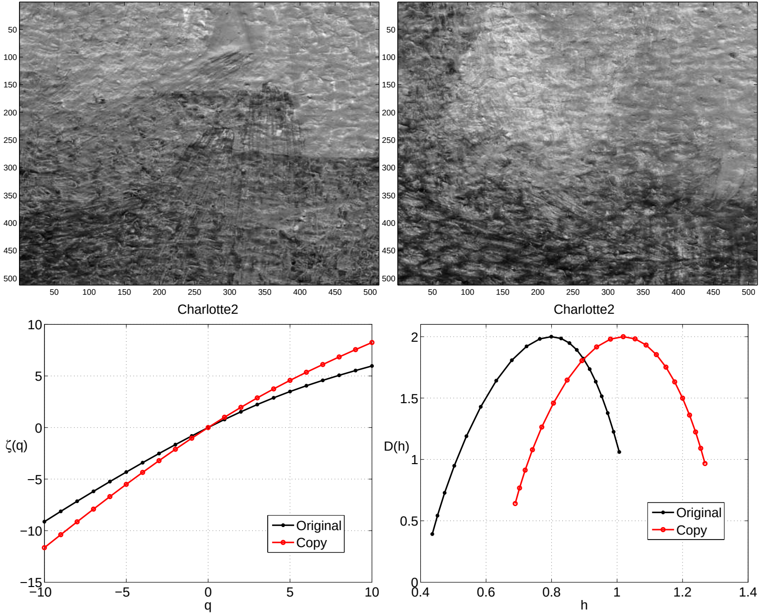


Original paintings and copies by Charlotte Caspers

Original and copy : Charlotte Caspers



Original and copy : Charlotte Caspers



Open problems / Directions of research

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- Perform the multifractal analysis of invariant measures of dynamical systems